*"Problem Set Two"*

(1) Consider the free electromagnetic field on $\mathbf{R}^n$ (with a metric of signature $(+--\ldots-)$). Consider the free electromagnetic field – a connection $A$ with curvature (or field strength $F = dA$) and Lagrangian

$$I = -\frac{1}{4e^2}\int_{\mathbf{R}^n} F \wedge *F \tag{1}$$

regarded as a functional of $A$. ($e$ is a constant, related to the charge of the electron; it has been written here in its usual place but in this classical exercise you can set it to 1 by rescaling $A$.)

(a) Derive the classical field equations and verify that they are second order differential equations for $A$.

(b) Compute the closed two-form $\omega_0$ on the space $M_0$ of solutions of the classical field equations.

(c) Let $G$ be the group of gauge transformations, which act by $A \to A + d\phi$, where $\phi$ is a function on $\mathbf{R}^n$.

Let $M = M_0/G$ be the space of solutions of the equations modulo gauge transformations.

Show that $\omega_0$ is the pullback to $M_0$ of a closed two-form $\omega$ on $M$, and show that $\omega$ is non-degenerate.

(2) In this problem, we will consider the free electromagnetic field in three dimensions, say on $\mathbf{R}^3$ with flat metric of signature $(+--)$. The interest in three dimensions is that it is possible to add an extra term to the Lagrangian, which becomes

$$I = -\frac{1}{4e^2}\int F \wedge *F + \mu \int A \wedge F. \tag{2}$$

(If you prefer, set $e$ to 1 and $\mu$ to $\pm 1$ by rescaling of $A$ and dilation of $\mathbf{R}^3$.) Note that the second term is invariant under $A \to A + d\phi$ where $\phi$ has compact support.

(a) Write the equations of motion in Fourier space and show that the space of solutions mod gauge transformations is supported not on a cone $p^2 = 0$ but on a certain hyperboloid $H$ of the form $p^2 = \text{constant}$.

(b) The space of solutions of the equations mod gauge transformations is a line bundle over $H$ to which the action of $SO(2,1)$ lifts. Describe this line bundle. What happens if you change the sign of $\mu$?

(3) In the next few exercises, we develop facts and supersymmetric models that are important in string theory.

(a) Consider in two dimensions $\mathbf{R}^2$ with coordinates $t, x$ and metric $dt^2 - dx^2$ the Lagrangian of a free massless boson $\phi$

$$L = -\frac{1}{2}\int_{\mathbf{R}^2} d\phi \wedge *d\phi. \tag{3}$$

Show that the general solution of the equations of motion is of the form $\phi(t,x) = \phi_R(t-x) + \phi_L(t+x)$ where the decomposition is unique up to adding constants to $\phi_R$ and $\phi_L$. We call them right- and left-moving waves. Show that the decomposition in right and left- moving waves is compatible with the symplectic structure. (This means that the symplectic structure paired with one left-moving variation of $\phi$ and one right-moving one gives zero.)

(b) Consider the Dirac equation in two dimensions

$$\Gamma \cdot \partial\psi = 0 \tag{4}$$

where $\psi$ is a section of the spin bundle $S$ of $\mathbf{R}^2$. Show that, under the decomposition $S = S_+ \oplus S_-$, $\psi$ decomposes as a sum of a right-moving wave with values in $S_+$ and a left-moving wave with values in $S_-$.

(4) Now we get closer to physics. First we construct a supersymmetric model in two dimensions in which all odd variables are right-moving.

We start with a $(2,1)$ supermanifold $W$ with a one-dimensional even distribution $B_-$ generated by a vector field $\partial_-$ and a one-dimensional odd distribution generated by a vector field $D$; we require that $TX$ is generated by $\partial_-$, $D$, and $\partial_+ = -D^2$.

A "flat" model is $\mathbf{R}^{2,1}$ with even coordinates $u, v$, odd coordinate $\theta$, and

$$\begin{aligned} \partial_- &= \frac{\partial}{\partial u} \\ D &= \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial v} \\ \partial_+ &= \frac{\partial}{\partial v}. \end{aligned} \tag{5}$$

Notice that in this problem (unlike some previous exercises) one is given only the distributions, and not specific vector fields generating them.

Let $M$ be a Riemannian manifold with metric $g$ and $X : W \to M$ a map. The Lagrangian we want

$$L_X = \int_W g_{IJ} \partial_- X^I D X^J. \tag{6}$$

(a) Show that the Lagrangian is well-defined (and invariant under all automorphisms of $W$ that preserve the distributions) by interpreting the integrand as a section of the Berezinian.

(b) By performing the theta integral, reduce this to an ordinary Lagrangian. Show that (i) the reduction of the Lagrangian (throwing away odd variables) is the Lagrangian for a harmonic map to $M$, (ii) the equations of motion are such that the odd variables are right-moving, that is functions of $v$ and not $u$.

(c) Thus the phase space (=space of classical solutions) $Y$ is a bundle over $Y_{red}$, where $Y_{red}$ is the space of harmonic maps and the normal bundle to $Y_{red}$ in $Y$ is $\Pi V$, where $V$ is some vector bundle over $Y_{red}$. From (b) you should know that $Y_{red}$ is the cotangent bundle of $Hom(\mathbf{R}, M)$, where we identify a classical solution with the initial conditions on the spacelike hypersurface $u = -v$. Show that $V$ is the tangent bundle of $Hom(\mathbf{R}, M)$. Thus, a quantization of this system gives (formally) a spin bundle of $Hom(\mathbf{R}, M)$, together with a Dirac operator. To be precise, the Dirac operator is the one that arises upon quantization of the Hamiltonian function associated with the motion on the phase space that comes from the vector field on $W$

$$\frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial v}. \tag{7}$$

(d) Consider the flat model with $W = \mathbf{R}^{2,1}$ as above. Let $u = t+x$, $v = t-x$, Restrict the Lagrangian to maps $X$ that are invariant under translations in $x$. The quotient of $\mathbf{R}^{2,1}$ by such translations is $\mathbf{R}^{1,1}$. Show that for these translation-invariant maps, $L$ reduces to

$$L' = \int dt\, d\theta g_{IJ} \frac{dX^I}{dt} D X^J. \tag{8}$$

Recall from a previous problem set that quantization of $L'$ gives the spin bundle and Dirac operator of $M$. Our previous result is thus essentially a consequence of the present more general one upon specializing to the translation-invariant maps.

(5) Now we will consider the Dirac operator on $Hom(\mathbf{R}, M)$ with values in a certain kind of vector bundle.

Let $Q$ be a finite dimensional rank $n$ real vector bundle over $M$ with structure group $SO(n)$, and a choice of metric.

Let $B_-^{1/2}$ be a square root of the even distribution $B_-$ of the previous exercise. Let $\Lambda$ be an odd section of $B_- \otimes X^*(Q)$. The Lagrangian for $\Lambda$ is

$$L_\Lambda = \int_W (\Lambda, D\Lambda). \tag{9}$$

For the coupled system of the map $X$ and $\Lambda$, one simply adds the two Lagrangians to get $L = L_X + L_\Lambda$.

(a) Show that this is well-defined (and invariant under all of the diffeomorphisms of $W$ that preserve the distributions).

(b) Writing $\Lambda = \lambda + \theta G$, perform the $\theta$ integral, to show that $G$ is a non-progagating "auxiliary field" (whose field equations have a unique solution in terms of the other variables) while the odd variables $\lambda$ are "right-moving." Keeping $X$ fixed, show that the space of critical points of $L_\Lambda$ is a vector bundle $P = \Gamma(\mathbf{R}, X^*(Q))$ over $Hom(\mathbf{R}, M)$. Thus, quantization of this system will give a spin bundle derived from $P$, and the operator that in the quantum theory is derived from (7) is a Dirac operator on $Hom(\mathbf{R}, M)$ with values in $P$.

(c) Show that in the special case that $Q = TM$ is the tangent bundle to $M$, $P$ is the tangent bundle of $Hom(\mathbf{R}, M)$, so quantization of $L_X + L_\Lambda$ will give for the quantum Hilbert space a spin bundle $S$ of $Hom(\mathbf{R}, M)$ with values in another spin bundle $S'$. (Unlike the finite-dimensional case, $S$ and $S'$ are not isomorphic as they are constructed using different polarizations: one set of odd variables is left-moving and one is right-moving.)

In finite dimensions, $S \otimes S'$ is isomorphic to the de Rham complex. (An explanation of that is in exercise 6(b) below.) Thus, this theory in the special case $V = TM$ is formally related to some sort of differential forms on $Hom(\mathbf{R}, M)$. One might suspect that this special case has additional symmetry, if one takes the metric on $V$ to be the Riemannian metric. That is true and will be pursued later.

(6) This is really more an explanation than an exercise.

(a) Let $M$ be an $n$-dimensional differentiable manifold (no metric). The rank $2n$ bundle $W = TM \oplus T^*M$ has a natural quadratic form (in which $TM$ and $T^*M$ are isotropic and paired with each other) which can be interpreted as a symplectic form on $\Pi W$.

Quantize $\Pi W$ to get the associated bundle $S(W)$ of spinors of $W$. Identify it with the de Rham complex $\Omega^*(M)$.

Therefore, $TM \oplus T^*M$ must act on $\Omega^*(M)$. Show that a section of $TM \oplus T^*M$ of the form $v \oplus \alpha$ (with $v$ and $\alpha$ a vector field and one-form respectively) acts on a differential form $\theta$ by $\theta \to i_v\theta + \alpha \wedge \theta$. Deduce the classical formula $(i_v\alpha + \alpha i_v) = \alpha(v)$.

(b) If a metric is introduced on $M$, then $\Pi TM$ (or equivalently $\Pi T^*M$) can be quantized to get the spin bundle $S(M)$. (For present purposes ignore possible global obstructions to this.) Suppose for simplicity that $M$ is even-dimensional. Explain from the present point of view the standard relation between the de Rham complex and the spin bundle: $\Omega^*(M) \cong S(M) \otimes S(M)$ (over $\mathbf{C}$).

(c) Consider a supermanifold $N$. To keep things simple, take $N = \mathbf{R}^{0,n}$, with odd coordinates $\theta_i$, $i = 1, \ldots, n$.

$X = \Pi TN \oplus \Pi T^*N$ is a $2n$-dimensional vector space with a natural symplectic form, in which $\Pi TN$ and $\Pi T^*N$ are isotropic and paired with each other.

As in (a) above, one can try to quantize $X$ to get "the de Rham complex of $N$." The difference from (a) is such a quantization will be infinite dimensional. Moreover, the only natural ($Sp(2n, \mathbf{R})$-invariant) quantization of such a symplectic vector space gives a Hilbert space of certain $\mathbf{L}^2$ functions.

However, an algebraic quantization can be obtained if one picks a polarization, that is a maximal isotropic subspace $Y$ of $X$, and asks that there should be a vector annihilated by $Y$. The quantization of $X$ – that is, the "differential forms on $N$" – is then the space of polynomials on $X/Y$. So the Clifford algebra of $X$ (and in particular the subalgebra $\Lambda^*\Pi T^*N$) acts on $S^*(X/Y)$ for any $Y$. (But there is no way to identify $S^*(X/Y)$ for different $Y$.)

Show that the case $Y = \Pi TN$ corresponds to what Berstein described as the "differential forms of finite order" on $N$ while the case $Y = \Pi T^*N$ corresponds to the "differential forms of finite codimension."