Witten's Problems, Set Three — N°. 1 (solution written by P. Deligne & D. Freed)

Preface. It should have been added to the assumptions on $(W^{2,2}, \Lambda_+, \Lambda_-)$ that the Frobenius pairing

$$(1) \qquad [\quad , \quad]\colon \Lambda_+ \otimes \Lambda_- \to T/(\Lambda_+ \oplus \Lambda_-)$$

vanishes. If this condition holds, there are local coordinates $(u, v, \theta_+, \theta_-)$ in which Λ_+ and Λ_- are generated by D_+, D_- as in the given formuli, with the "i" omitted. We hope to explain elsewhere why "i" was there. A down to earth explanation is that the formuli describe Λ_+ in Λ_- in a complex coordinate system $(u, v, \theta_+, \theta_-)$, with u, v, $\iota^{1/2}\theta_+$ and $\iota^{1/2}\theta_-$ real. In what follows, the "i" will be suppressed.

Proof. If D_+ generates Λ_+, D_+ and D_+^2 generates an integrable distribution, as results from the identity $[D_+, D_+^2] = 0$. The distributions $\langle D_+, D_+^2\rangle$ and $\langle D_-, D_-^2\rangle$ are integrable and, by assumption, transversal. They correspond locally to a decomposition

$$W^{2,2} = W_+^{1,1} \times W_-^{1,1} \ .$$

The assumption that $[\Lambda_+, \Lambda_-]$ be zero modulo $\langle D_+, D_+^2, D_-\rangle$ amounts to saying that, locally, Λ_- comes from a distribution on the factor $W_-^{1,1}$. The assumption (1) hence assume that Λ_+ and Λ_- come from distribution, still noted Λ_+ and Λ_-, on $W_+^{1,1}$ and $W_-^{1,1}$.

To find local coordinates in which Λ_+ and Λ_- are as given, one can consider separately $(W_+^{1,1}, \Lambda_+)$ and $(W_-^{1,1}, \Lambda_-)$. Write (W, Λ) for $(W_+^{1,1}, \Lambda_+)$ or $(W_-^{1,1}, \Lambda_-)$. It remains to choose an odd vector field D generating Λ and to apply the

Lemma. *Consider the systems (W, D): W of dimension $(1,1)$, D an odd vector field such that D and D^2 span the tangent bundle T. They are all locally isomorphic.*

Proof. The Lie algebra F freely generated by an odd generator x is spanned by x and x^2, as is shown by the identity $[x, x^2] = 0$. Let G be the corresponding simply connected group. The Lie algebra of infinitesimal diffeomorphisms of W is the Lie algebra of vector fields, with the opposite of the usual bracket. Let ρ be the morphism $F \to \operatorname{Lie}\operatorname{Diff}(W)$

for which $\rho(x) = D$. it integrates to a local action of G on W. For w a point of W, the map $g \mapsto gw$ is an isomorphism from a neighborhood of e in G to one of w in W. By construction, D corresponds to the right invariant vector field x on G.

Remark. The formuli given, in local coordinates, for D_+ and D_- show that D_+ and D_- commute: the distributions Λ_+ and Λ_- are locally generted by commuting vector fields D_+ and D_-. Other commuting vector fields D_+, D_- generating Λ_+ and Λ_- are of the form f_+D_+, f_-D_-, with $D_-f_+ = D_+f_- = 0$.

Remark. The discussion up to now works as well in a relative setting, over a basis B. What comes next does not: as S is of dimension $(*, 1)$, we have a canonical projection $W \to W_{\text{red}}$: the quotient $\mathcal{O}_{\text{red}}$ of $\mathcal{O}_W$ is an isomorphic image of the subsheaf $\mathcal{O}^+$ of $\mathcal{O}$. For $W^{2,2} = W^{1,1} \times W^{1,1}$, this gives a canonical retraction

$$p\colon W^{2,2} \to W^{2,2}_{\text{red}} \ .$$

By integration along the fibers of p, a density on $W^{2,2}$ projects to a density on $W^{2,2}_{\text{red}}$. In local coordinates, it is given by

$$p_*(f.du\,dv(d\theta_+ d\theta_-)^{-1}) = (D_-D_+f)du\,dv$$

where $D_-\ D_+\ f$ is to be restricted to $W^{2,2}_{\text{red}}$ (coordinates u, v), defined in $W^{2,2}$ by the vanishing of θ_+ and θ_-.

Of course, once we have p, we keep having it after change of basis: we have a morphism induced by p from $W^{2,2} \times B$ to $W^{2,2}_{\text{red}} \times B$. The formula for p_* remains valid.

To give $\phi\colon W^{2,2} \to M$ amounts to giving its restriction X to $W^{1,1}_{\text{red}}$, $\psi_\pm = D_p m X$: odd sections of the pull back X^*TM of the tangent bundle of M, and $F = \nabla_{D+} D_- X = -D_- \nabla_{D+} X$, an even section of this pull back. For the abuse of language used, see Set One, N° 2. Here D_+ and D_- are commuting generators of C_+ and D_-. if $\Lambda^0_\pm$ is the restriction of $\Lambda_\pm$ to $W^{1,1}_{\text{red}}$, indepedently of a choice of $D_\pm$, one can see $\psi_\pm$ and F as even sections of $(\Lambda_\pm)^\vee \otimes X^*TM$ and $(\Lambda_+ \otimes \Lambda_-)^\vee \otimes X^*TM$. Using X, $\psi_\pm$ and F instead of ϕ, and $p_*\mathcal{L}$ instead of $\mathcal{L}$, is working "in components".

Remark (to (b)). If to a langrangian density $\mathcal{L}(\phi)$ is added a density of the form $d\mathcal{M}(\phi)$, with $\mathcal{M}(\phi)$ a codimension 1 integral form attached locally to ϕ, the space of stationary points does not change: we are to consider deformations with compact support ϕ_u of ϕ, and

$$\int d\mathcal{M}(\phi_u) - d\mathcal{M}(\phi) = \int d(\mathcal{M}(\phi_u) - \mathcal{M}(\phi)) = 0$$

as $\mathcal{M}(\phi_u) - \mathcal{M}(\phi)$ has compact support.

The 1-form attached to a space like hypersurface Γ changes, but the 2-form on the space of stationary points does not: the 1-form changes by

$$\delta\phi \longmapsto \int_\Gamma \partial_u \mathcal{M}(\phi + \delta\phi) \ ,$$

the differential of $\phi \mapsto \int_\Gamma \mathcal{M}(\phi)$.

Solution. (a) The bundle of volume forms is canonically isomorphic to $\Lambda_+^{-1} \otimes \Lambda_-^{-1}$. To see this, note that the Frobenius pairing is an isomorphism

$$\text{Tr: } \Lambda_+ \otimes \Lambda_+ \oplus \Lambda_- \otimes \Lambda_- \longrightarrow TW/\Lambda_+ \oplus \Lambda_- \ ,$$

and from the exact sequence

$$0 \to \Lambda_+ \oplus \Lambda_- \to TW \to TW/\Lambda_+ \oplus \Lambda_- \to 0$$

we conclude

$$\begin{aligned}
\text{Vol}(TW) &\simeq \text{Vol}(\Lambda_\otimes \otimes \Lambda_-) \otimes \text{Vol}(TW/\Lambda_+ \oplus \Lambda_-) \\
&\simeq (\Lambda_+ \otimes \Lambda_-) \otimes \text{Vol}(\Lambda_+^{\otimes 2} \oplus \Lambda_-^{\otimes 2}) \\
&\simeq (\Lambda_+ \otimes \Lambda_-) \otimes (\Lambda_+^{\otimes -2} \otimes \Lambda_-^{\otimes -2}) \\
&\simeq \Lambda_+^{-1} \otimes \Lambda_-^{-1}
\end{aligned}$$

The lagrangian density is

$$\mathcal{L} = (D_+\Phi, D_-\Phi) D_+^{-1} \otimes D_-^{-1} \ .$$

The same argument shows that for a 2-form B,

$$\mathcal{L}'_B = B(D_+\Phi, D_-\Phi)D_+^{-1} \otimes D_-^{-1}$$

is well defined.

For any vector field ξ we have $\mathcal{L}_\xi = [d, \iota_\xi] = d\iota_\xi + i_\xi d$, relating the Lie derivative and the contraction, so if λ is an even 1-form, then*

$$\begin{aligned} d\lambda(\xi_1, \xi_2) &= \pm\iota_{\xi_2}\iota_{\xi_1} d\lambda = \pm\iota_{\xi_2}\left(\mathcal{L}_{\xi_1}\lambda - d\iota_{\xi_1}\lambda\right) \\ &= \left[\mathcal{L}_{\xi_1}\iota_{\xi_2} - \iota_{[\xi_1,\xi_2]}\lambda\right] \mp \mathcal{L}_{\xi_2}\iota_{\xi_1}\lambda \end{aligned} \tag{1}$$

Here the sign $\pm$ is $-$ if both ξ_1, ξ_2 are odd and $+$ otherwise. We apply this to $\mathcal{L}'_B$ for $B = d\Lambda$

$$\begin{aligned} \mathcal{L}'_{d\Lambda} &= d\Lambda(D_+\Phi, D_-\Phi)\Delta \\ &= -\iota_{D_-}\iota_{D_+}\Phi^*\Lambda \cdot \Delta \\ &= \left\{\mathcal{L}_{D_+}\left(\iota_{D_-}\Phi^*\Lambda\right) + \mathcal{L}_{D_-}\left(\iota_{D_+}\Phi^*\Lambda\right)\right\}\Delta \\ &= \mathcal{L}_{D_+}\left(\iota_{D_-}\Phi^*\Lambda \cdot \Delta\right) + \mathcal{L}_{D_-}\left(\iota_{D_+}\Phi^*\Lambda\Delta\right) \\ &= -d\left(\iota_{D_-}\Phi^*\Lambda \cdot \iota_{D_+}\Delta + \iota_{D_+}\Phi^*\Lambda \cdot \iota_{D_-}\Delta\right) \ . \end{aligned} \tag{2}$$

In this computation $\Delta = D_+^{-1} \otimes D_-^{-1}$, and we made use of the fact that $\mathcal{L}_{D_\pm}(\Delta) = 0$.

(b) Let $i\colon \mathbb{R}^2 \hookrightarrow \mathbb{R}^{2,2}$ be the inclusion. One has $\iota^*(\theta^\pm) = 0$. If p is the projection $(u, v, \theta_+, \theta_-) \to (u, v)$ of $\mathbb{R}^{2,2}$ to $\mathbb{R}^2$, and Δ the density $du\,dv(d\theta_+ d\theta_-)^{-1}$, we have $p_*(f\Delta) = \iota^*(D_+D_-f)du\,dv$. For $\mathcal{L}$ we have, omitting $du\,dv$ for convenience:

$$\begin{aligned} p_*(\mathcal{L}) &= i^*D_+D_-\mathcal{L} = i^*D_+D_-(D_+\Phi, D_-\Phi) \\ &= i^*D_+\left\{(\nabla_-D_+\Phi, D_-\Phi) - (D_+\Phi, \nabla_-D_-\Phi)\right\} \\ &= i^*\{(\nabla_+\nabla_-D_+\Phi, D_-\Phi) + (\nabla_-D_+\Phi, \nabla_+D_-\Phi) - (\nabla_+D_+\Phi, \nabla_-D_-\Phi) + \\ &\qquad + (D_+\Phi, \nabla_+\nabla_-\Phi)\} \\ &= i^*\{(R(D_+\Phi, D_-\Phi)D_+\Phi, D_-\Phi) - (\nabla_-\nabla_+D_+\Phi, D_-\Phi) + \\ &\qquad + (\nabla_+D_-\Phi, \nabla_+D_-\Phi) - (\nabla_+D_+\Phi, \nabla_-D_-\Phi) + (D_+\Phi, \nabla_+\nabla_-D_-\Phi)\} \\ &= (R(\psi_+, \psi_-)\psi_+, \psi_-) + (i^*\nabla_-\partial_v\Phi, \psi_-) + (F, F) - (\partial_vX, \partial_uX) - (\psi_+, i^*\nabla_+\partial_u\Phi) \\ &= (R(\psi_+, \psi_-)\psi_+, \psi_-) + (\nabla_v\psi_-\psi_-) + (F, F) - (\partial_vX, \partial_uX) - (\psi_+, \partial_u\psi_+) \end{aligned}$$

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*See forthcoming Deligne's appendix to Bernstein's lectures for details about the sign conventions.

Here we modified the notation of the problem to remove the factors of $\sqrt{-1}$:

$$
\begin{aligned}
D_+ &= \tfrac{\partial}{\partial \theta_+} - \theta_+ \tfrac{\partial}{\partial v} \\
D_- &= \tfrac{\partial}{\partial \theta_-} - \theta_- \tfrac{\partial}{\partial u}
\end{aligned}
$$

and we expressed ϕ in components:

$$
\begin{aligned}
X &= i^* \Phi \\
\psi_+ &= i^* D_+ \Phi \\
\psi_- &= i^* D_- \Phi \\
F &= i^* \nabla_+ D_- \Phi \ .
\end{aligned}
$$

Also, $\nabla_\pm$ denotes the covariant derivative with respect to $D_\pm$.

We compute how $\mathcal{L}$ changes by an *even* deformation of ϕ:

$$
\begin{aligned}
\delta \mathcal{L} = \delta (D_+ \Phi, D_- \Phi) &= (\nabla_\delta D_+ \Phi, D_- \Phi) + (D_+ \Phi, \nabla_\delta D_- \Phi) \\
&= (\nabla_+ \delta \Phi, D_- \Phi) + (D_+ \Phi, \nabla_- \delta \Phi) \\
&= D_+ (\delta \Phi, D_- \Phi) - D_- (D_+ \Phi, \delta \Phi) - 2 (\nabla_+ D_- \Phi, \delta \Phi)
\end{aligned}
$$

The first two terms are exact, and the equation of motion is

$$\nabla_+ D_- \Phi = 0 \ . \tag{3}$$

We evaluate the components of (3) by applying i^*, $i^* \nabla_+$, $i^* \nabla_-$, and $i^* \nabla_+ \nabla_-$. The results are:

$$
\begin{aligned}
i^* (\nabla_+ D_- \Phi) &= F \ . \\
i^* \nabla_+ (\nabla_+ D_- \Phi) &= R(\psi_+, \psi_-) \psi_+ + \nabla_v \psi_- \\
i^* \nabla_- (\nabla_+ D_- \Phi) &= R(\psi_-, \psi_+) \psi_- + \nabla_u \psi_+
\end{aligned}
$$

We give details for the last of these computations:

$$
\begin{aligned}
i^*\nabla_+\nabla_-(\nabla_+ D_-\Phi) &= i^*\nabla_+\left\{R(D_-\Phi, D_+\Phi)D_-\Phi - \nabla_+\nabla_- D_-\Phi\right\} \\
&= i^*\nabla_+\left\{-\tfrac{1}{2}\,R(D_-\Phi, D_-\Phi)D_+\Phi - \nabla_+\nabla_- D_-\Phi\right\} \\
&= i^*\{(\nabla_+ R)(D_-\Phi, D_+\Phi)D_+\Phi - R(\nabla_+ D_-\Phi, D_-\Phi)D_+\Phi - \\
&\qquad - \tfrac{1}{2}\,R(D_-\Phi, D_-\Phi)\nabla_+ D_+\Phi + \nabla_u\partial_v\Phi - \tfrac{1}{2}\,R(D_+\Phi, D_+\Phi)\partial_v\Phi\} \\
&= (\nabla_+ R)(\psi_-, \psi_-)\psi_+ - R(F, \psi_-)\psi_+ - \tfrac{1}{2}\,R(\psi_-, \psi_-)\partial_v X + \nabla_u\partial_v X - \\
&\qquad - \tfrac{1}{2}\,R(\psi_+, \psi_+)\partial_v X\ .
\end{aligned}
$$

In the second line we use the Bianchi identity and in the third we use

$$
\nabla_+\nabla_+ = -\nabla_u + \tfrac{1}{2}\,R(D_+, D_+)\ .
$$

Of course, we can set $F = 0$ in the final expression because of the first equation of motion. Collating the results, we obtain the equations of motion in components:

$$
\begin{aligned}
F &= 0 \\
\nabla_v\psi_- &= -R(\psi_+, \psi_-)\psi_+ \\
\nabla_u\psi_+ &= -R(\psi_-, \psi_+)\psi_- \\
\nabla_u\partial_v X &= \tfrac{1}{2}\left\{R(\psi_-, \psi_-)\partial_v X + R(\psi_+, \psi_+)\partial_u X\right\} - (\nabla_+ R)(\psi_-, \psi_-)\psi_+
\end{aligned}
$$

(d) Let $j\colon \mathbb{R}^{2,1} \hookrightarrow \mathbb{R}^{1,1}$ so that j^* sets $\theta_- = 0$. Then

$$
\begin{aligned}
j^* D_{\mathfrak{L}} &= j^* D_-(D_+\Phi, D_-\Phi) \\
&= j^*\left\{(\nabla_- D_+\Phi, D_-\Phi) - (D_+\Phi, \nabla_- D_-\Phi)\right\} \\
&= j^*\left\{(\nabla_+ D_-\Phi, D_-\Phi) + (D_+\Phi, \partial_u\Phi)\right\} \\
&= (\nabla_+\Lambda, \Lambda) + (D_+\Phi', \partial_u\Phi')
\end{aligned}
$$

where

$$
\begin{aligned}
j^*\Phi &= \Phi' \\
j^* D_-\Phi &= \Lambda\ .
\end{aligned}
$$