"Problem Set Three"

(1) Here we will study the theory related to "differential forms on the loop space of a manifold."

We let $W^{2,2}$ be a supermanifold of the indicated dimension with one-dimensional odd distributions Λ_+, Λ_- generated by vector fields D_+, D_- such that D_+, D_-, D_+^2, D_-^2 generate the tangent bundle of $W^{2,2}$. The flat model is $\mathbf{R}^{2,2}$ with even coordinates u, v, and odd coordinates θ_-, θ_+, with

$$\begin{aligned} D_+ &= \frac{\partial}{\partial\theta_+} - i\theta_+\frac{\partial}{\partial v} \\ D_- &= \frac{\partial}{\partial\theta_-} - i\theta_-\frac{\partial}{\partial u}. \end{aligned} \tag{1}$$

$\mathbf{R}^{2,2}$ has an infinite dimensional supergroup of automorphisms, roughly $G \times G$, where G is the symmetry group of the supermanifold $\mathbf{R}^{1,1}$ (with even and odd coordinates t, θ and distribution generated by $\partial_\theta + i\theta\partial_t$). For instance, generators are

$$\begin{aligned} Q_+ &= \frac{\partial}{\partial\theta_+} + i\theta_+\frac{\partial}{\partial v} \\ Q_- &= \frac{\partial}{\partial\theta_-} + i\theta_-\frac{\partial}{\partial u}. \end{aligned} \tag{2}$$

Let M be a fixed Riemannian manifold with metric g. Let $\Phi : W^{2,2} \to M$ be a map, and let

$$L = \int_{W^{2,2}} g_{IJ} D_+\Phi^I D_-\Phi^J. \tag{3}$$

(a) By now it should be routine to interpret the integrand as a section of the sheaf of volume forms and thereby show that the integral makes sense and that L is invariant under all of the symmetries of $W^{2,2}$.

Show that this has the following generalization: if B is a two-form on M, we can add another term

$$L' = \int_{W^{2,2}} B_{IJ} D_+\Phi^I D_-\Phi^J \tag{4}$$

which still preserves the symmetries of $W^{2,2}$. (Of course we can write a lot of other terms, some of which will be discussed later, if we pick a volume form on $W^{2,2}$.) Adding L' will give a theory that involves not ordinary differential forms on the path or loop space of M, but differential forms with values in a certain line bundle. Show that (4) is unchanged (up to a total derivative) if one replaces B by $B' = B + d\Lambda$.

(b) To understand more concretely what this theory describes, compute in the flat $\mathbf{R}^{2,2}$ and set

$$\begin{aligned} X &= \Phi|_{\theta_\pm=0} \\ \psi_+ &= D_+\Phi|_{\theta_\pm=0} \\ \psi_- &= D_-\Phi|_{\theta_\pm=0} \\ F &= -iD_+D_-\Phi|_{\theta_\pm=0}. \end{aligned} \tag{5}$$

(So X is a map $X : W_{red} \to M$, while $\psi_\pm$ and F take values in the obvious line bundles on W_{red} tensored with $X^*(TM)$, and the definition of F uses the connection on $X^*(TM)$ as well as the explicit $D_\pm$.)

Calculate $L = \int du\, dv\, d\theta_+\, d\theta_- S$ (with S as in (3) above – omit the generalization using the two-form) by writing it as $L = \int du\, dv\, D_+D_-S|_{\theta_\pm=0}$.

Don't be too careless in differentiating the metric – note the appearance of the Riemann tensor!

You should see that ψ_+ are right-moving fermions, with values in $X^*(TM)$, and ψ_- are left-movers, again with values in $X^*(TM)$. So as in last week's problem set, quantization of ψ_+ and ψ_- will each give spin bundles on the loop or path space $\mathcal{L}M$ of M. Call these spin bundles $S_L(\mathcal{L}M)$ and $S_R(\mathcal{L}M)$. S_L and S_R are not isomorphic, unlike the finite-dimensional case, because the positive energy condition gives opposite polarizations for left and right movers.

Nonetheless, the tensor product $S_L \otimes S_R$ would in finite dimensions be isomorphic to the de Rham complex $\Omega^*(\mathcal{L}M)$, so this theory gives a version of Hodge - de Rham theory of $\mathcal{L}M$.

Show that this theory is $\mathbf{Z}_2 \times \mathbf{Z}_2$ graded, where one $\mathbf{Z}_2$ acts by $\theta_+ \to -\theta_+$ and the other by $\theta_- \to -\theta_-$. The two $\mathbf{Z}_2$'s would correspond for a finite-dimensional manifold to the Hodge $*$ and the operator that acts on p-forms by $(-1)^p$. The $\mathbf{Z}$-valued grading of differential forms is lost in this (good!) version of Hodge theory of loop space.

(c) Let $u = t-x$, $v = t+x$, and consider only Φ's that are invariant under translations of x. The theory thus reduces to a theory on $\mathbf{R}^{1,2}$ with coordinates t, θ_+, θ_-, and

$$L = \int dt d\theta_+ d\theta_- g_{IJ} D_+\Phi^I D_-\Phi^J. \tag{6}$$

Q_+ and Q_- in this theory correspond to $d + *d*$ and $i(d - *d*)$ acting on differential forms.

Note that in reducing from $\mathbf{R}^{2,2}$ to $\mathbf{R}^{1,2}$ we have lost the infinite-dimensional symmetry algebra of $\mathbf{R}^{2,2}$. Show however, that one new symmetry appeared in the reduction – an

$SO(2)$ that rotates θ_+, θ_-. Show that this corresponds in the quantum theory to the $\mathbf{Z}$-grading of differential forms. (A $\mathbf{Z}_2$ -subgroup of $SO(2)$, which acts by sign change of both θ_+ and θ_-, is present in the two-dimensional version, as seen in (b).)

(d) Now we want to compare this theory to a theory on $\mathbf{R}^{2,1}$ which is obtained from $\mathbf{R}^{2,2}$ by setting $\theta_- = 0$.

We set $\Phi' = \Phi|_{\theta_-=0}$, and $\Lambda_- = D_-\Phi|_{\theta_-=0}$. Write $\int du\, dv\, d\theta_+\, d\theta_-\; S = \int du\, dv\, d\theta_+\, D_-S|_{\theta_-=0}$ to convert the Lagrangian on $\mathbf{R}^{2,2}$ to an equivalent Lagrangian on $\mathbf{R}^{2,1}$.

Compare with the theory of "spinors on $\mathcal{L}M$ with values in a vector bundle" that was in problem 5 of problem set 2, and show that – as befits "differential forms on $\mathcal{L}M$" – the theory discussed here is a special case of "spinors on $\mathcal{L}M$ with values in a vector bundle."

(e) So far we have seen that the theories of Dirac and Hodge-de Rham operators on a manifold M as customarily studied are the one-dimensional versions of two-dimensional theories. (The physics that goes with this is of course string theory.) For Dirac, that is as far as we can go – the chain that leads in this way to the Dirac operator on a manifold begins in two dimensions. But I want to point out that the Hodge-de Rham theory can come by reduction of a three-dimensional theory (generally less important in physics, so far at least, as it is less well-behaved quantum mechanically).

Here we start with a supermanifold $W^{3,2}$ of the indicated dimension, with a two-dimensional odd distribution A which is generic in the sense that A and $\{A, A\}$ generate the tangent bundle of $W^{3,2}$. A sharp difference from the two-dimensional case is that we need a volume form ϵ on A. Letting $\Phi : W^{3,2} \to M$ be a map as before to a fixed Riemannian manifold M, and D_1, D_2 a basis of A such that $\epsilon(D_1, D_2) = 1$, the Lagrangian is

$$L = \int g_{IJ} D_1\Phi^I\, D_2\Phi^J. \tag{7}$$

Show that this is well-defined, and invariant under all symmetries that preserve the distribution and volume form.

The flat model is $\mathbf{R}^{3,2}$ with odd coordinates θ^A, $A = 1, 2$ and even coordinates y^{AB} (with $y^{AB} = y^{BA}$ so there are three y's) and

$$D_A = \frac{\partial}{\partial\theta^A} - i\theta^B \frac{\partial}{\partial y^{AB}}. \tag{8}$$

Given a real structure and volume form on the space of the θ^A's the group that can act linearly on the θ^A's is $SL(2, \mathbf{R})$. This is the double cover of $SO(2, 1)$, which acts on the y's, preserving a metric of signature $+--$. The model lacks the infinite-dimensional symmetry group of the two-dimensional case, but has the following finite-dimensional group of symmetries. Let

$$Q_A = \frac{\partial}{\partial \theta^A} + i\theta^B \frac{\partial}{\partial y^{AB}}. \tag{9}$$

Show that the Q_A preserve the distribution, and that together with the $\{Q_A, Q_B\}$ and the $SL(2, \mathbf{R})$, they generate three-dimensional super-Poincaré. Together with some orientation and time-orientation reversing symmetries, this is the full symmetry group of $\mathbf{R}^{3,2}$ with the volume form.

Show that, if we impose invariance under translation of one of the y's, the model reduces to the two-dimensional model introduced above.

If one imposes invariance under translations of *two* of the y's – in fact, pick coordinates in which the metric is $dt^2 - dx_1^2 - dx_2^2$ and impose invariance under translation of x_1, x_2 – then we get the theory on $\mathbf{R}^{1,2}$ that is related upon quantization to ordinary differential forms on M with the d and d^* operators. This theory therefore has an $SO(2)$ symmetry of rotation of the $x_1 - x_2$ plane – it is the $SO(2)$ noted above that rotates θ_+ and θ_- in the original presentation; the generator acts on p-forms by multiplication by p (or $p - \frac{1}{2}\dim M$).

(2) On differential forms on M there acts the $SO(2)$ just noted, commuting with the Laplacian. It comes by dimensional reduction from a $1 + 2 = 3$ dimensional theory. Differential forms on a generic M cannot come from reduction of a four-dimensional theory, as the $SO(2)$ - which is better called $Spin(2)$ - would then have to extend to a $Spin(3)$.

Likewise, the Dirac equation on a general M cannot come by reduction from a theory above two dimensions. If it came from a three-dimensional theory, we would need to find a $Spin(2)$ action on the spin bundle, commuting with the Laplacian and acting non-trivially on the Dirac operator. So $d = 2$ is the "starting point" in this sense for the Dirac equation.

But suppose M is a Kahler manifold. Then the $Spin(2)$ action on differential forms *does* extend to a $Spin(3) = SU(2)$ action (sometimes called $SL(2)$, but $SU(2)$ is the group that acts on the *Hilbert space* of differential forms preserving the structure). d and d^* can be decomposed into four operators ∂, $\bar{\partial}$, ∂^*, $\bar{\partial}^*$. These are (complex) linear combinations of four self-adjoint operators that transform as the four dimensional real spinor representation of $Spin(3)$. This suggests that differential forms on a Kahler manifold M can come from a theory in $1 + 3 = 4$ dimensions.

Likewise if M is hyper-Kahler then the $Spin(3)$ extends further to a $Spin(5)$ with d and d^* generating eight self-adjoint operators that transform as the eight dimensional real spinor of $Spin(5)$. This suggests that differential forms on M can come from a theory in $1 + 5 = 6$ dimensions.

Recall that, according to Nahm's theorem, six is the maximum dimension for supersymmetric sigma models. The supersymmetric theory in six dimensions of maps to a hyper-Kahler manifold (plus odd variables) does exist – but like all the limiting cases allowed by Nahm's theorem (the others being gauge theory in ten dimensions and gravity in eleven dimensions) there is no known construction via supermanifolds.

For the four-dimensional case, however, with M a Kahler manifold, a superspace construction does exist. For the time being, we will only describe it locally. (It takes a few more steps to describe a more global version.)

We consider a (4,4) supermanifold W with the structure described in Bernstein's lectures – two complex conjugate integrable distributions A_+ and A_- of dimension $(0,2)$ such that $\{A_+, A_-\}$ generates $TM/(A_+ \oplus A_-)$, A_+ and A_- are endowed with volume forms. A chiral superfield is a field ϕ that is a function on W/A_-. If M is a complex manifold, a chiral map $\Phi : W \to M$ is one such that, for f a local holomorphic function on M, $\Phi^*(f)$ is a chiral function on W. In that case, $\Phi^*(\bar{f})$ is antichiral, that is, it is a function on W/A_+.

Let K be a function on W and consider the Lagrangian

$$L = \int_W d^4y d^4\theta \, \Phi^*(K). \tag{10}$$

$d^4y\, d^4\theta$ is the section of the Berezinian coming from the volume forms on $A_\pm$.

(a) Show that L is invariant under $K \to K + f + \bar{f}$, where f is a local holomorphic function on W.

(b) Recall that any K determines the two-form $\omega = -i\bar{\partial}\partial K$ which for a suitable class of K's is the Kahler class of a Kahler metric g on M. Of course, ω is invariant under the transformation considered in (a).

Show that L can be written just in terms of ω (or equivalently g) and that the "reduced" part of L is the harmonic map action, specialized to maps to a Kahler manifold:

$$L = \int_{W_{red}} d^4y g_{i\bar{j}}(dX^i, dX^{\bar{j}}) \tag{11}$$

where X^i are local holomorphic coordinates on M.

(c) Specialize now to the flat model with $W = \mathbf{R}^{4,4}$. A_+ is generated by

$$D_A = \frac{\partial}{\partial\theta^A} - i\sigma^m_{A\dot{A}}\bar{\theta}^{\dot{A}}\frac{\partial}{\partial y^m} \tag{12}$$

where θ^A are odd coordinates transforming as "positive" spinors of $Spin(1,3) = SL(2,\mathbf{C})$, $\bar{\theta}^{\dot{A}}$ are their complex conjugates, and y^m are even coordinates. σ is the isomorphism $S_+ \otimes S_- = V$. A_- is generated by

$$D_{\dot{A}} = \frac{\partial}{\partial \theta^{\dot{A}}} - i\sigma^m_{A\dot{A}}\bar{\theta}^A \frac{\partial}{\partial y^m}. \tag{13}$$

Impose invariance under a space-like translation of W, say a translation of y^3 (the metric being $(dy^0)^2 - \sum_{i=1}^3 (dy^i)^2$ to get a model on $\mathbf{R}^{3,4}$. Show that this is equivalent to the $\mathbf{R}^{3,2}$ σ model described in problem 1(e) (but specialized to the case that M is Kahler). Here is an attempt to explain the strategy: under $Spin(1,2)$, the two spinor representations of $Spin(1,3)$ become isomorphic. So after the reduction, A_+ and A_- are naturally isomorphic. One can in an $SO(1,2)$-invariant way take the "real" combination $\chi^A = \theta^A + \bar{\theta}^A$ of the θ's and $\bar{\theta}$'s. The $\mathbf{R}^{3,2}$ we want has even coordinates y^i, $i = 0,1,2,$ and odd coordinates χ^A. By "integrating over the fibers" of a map from $\mathbf{R}^{3,4}$ to $\mathbf{R}^{3,2}$, one reduces (11) (specialized to y^3-independent maps) to the $\mathbf{R}^{3,2}$ Lagrangian of problem 1(e).