Witten's Problems, Set Four — N$^{\text{o}}$. 2

(solution written by D. Freed)

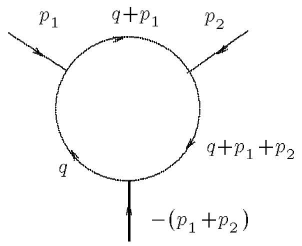
For practice, we solve this problem using dimensional regularization. The dimensions of the fields and parameters are

$$[\phi] = \tfrac{n-2}{2}$$
$$[m] = 1$$
$$[g] = \tfrac{6-n}{2}$$

Introduce a parameter μ with the dimension of mass and a dimensionless parameter λ. Then the lagrangian density is

$$\mathcal{L} = \left\{ \tfrac{1}{2}\left(|d\phi|^2 + m^2\phi^2\right) + \lambda\mu^{\frac{6-n}{2}} \tfrac{\phi^3}{3!} \right\} d^n x\ .$$

To order λ^3 the 3-point function is computed by the diagram shown. Here we are working in



momentum space and label the edges of the diagram with momenta. Applying the Feynman rules for Euclidean space, we evaluate the diagram as

$$\Gamma = -\lambda^3\mu^{\frac{3}{2}(6-n)} \int \tfrac{d^n q}{(2\pi)^n} \tfrac{1}{(q^2+m^2)((q+p_1)^2+m^2)((q+p_1+p_2)^2+m^2)}$$

Now use Feynman parameters to rewrite the integral as

$$
\begin{aligned}
(1)\qquad \Gamma_n &= -\lambda^3\mu^{\frac{3}{2}(6-n)}\int \tfrac{d^nq}{(2\pi)^n}\int_0^1 d\alpha\int_0^{1-\alpha} d\beta \tfrac{1}{[(1-\alpha-\beta)q^2+\alpha(q+p_1)^2+\beta(q+p_1+p_2)^2+m^2]^3}\\
&= -\lambda^3\mu^{\frac{3}{2}(6-n)}\int_0^1 d\alpha\int_0^{1-\alpha} d\beta\int \tfrac{d^nq}{(2\pi)^n}\tfrac{1}{[q^2+A^2]^3},
\end{aligned}
$$

where

$$
(2)\qquad A^2 = m^2+\alpha(1-\alpha)p_1^2+\beta(1-\beta)(p_1+p_2)^2-2\alpha\beta p_1(p_1+p_2).
$$

the interchange of integrals is justified for n sufficiently small (here $n<6$). Note that we made a shift in the integration variable q. Now the rules for integrals in n dimensions (n complex!) give

$$
(3)\qquad
\begin{aligned}
\int \tfrac{d^nq}{(2\pi)^n}\tfrac{1}{[q^2+a^2]^3} &= \tfrac{1}{(2\pi)^n}\tfrac{\pi^{n/2}}{\frac{1}{2}\Gamma(n/2)}\int_0^\infty dr\, r^{n-1}\tfrac{1}{[r^2+A^2]^3}\\
&= \tfrac{1}{2^{n-1}\pi^{n/2}\Gamma(n/2)}\cdot\tfrac{1}{2}\tfrac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(3-\frac{n}{2}\right)}{\Gamma(3)}A^{n-6}\\
&= \tfrac{\Gamma\left(3-\frac{n}{2}\right)}{2^{n+1}\pi^{n/2}}A^{n-6}.
\end{aligned}
$$

Here $\frac{\pi^{n/2}}{\frac{1}{2}\Gamma(n/2)}$ is the volume of the unit sphere in n dimensions, and for any N,

$$
\begin{aligned}
\int_0^\infty dr\, r^{n-1}\tfrac{1}{[r^2+A^2]^N} &= \int_0^\infty dr\, r^{n-1}\tfrac{1}{\Gamma(N)}\int_0^\infty dt\, e^{-(r^2+A^2)t}t^{N-1}\\
&= \tfrac{A^{n-2N}}{\Gamma(N)}\int_0^\infty \tfrac{dt}{t}\int_0^\infty dr\, e^{-r^2t}r^{n-l}e^{-t}t^N\\
&= \tfrac{A^{n-2N}}{2\Gamma(N)}\int_0^\infty \tfrac{dt}{t}\int_0^\infty \tfrac{du}{u}e^{-u}u^{\frac{n}{2}}e^{-t}t^{N-n/2}\\
&= \tfrac{1}{2}\tfrac{\Gamma\left(\frac{n}{2}\right)\Gamma(N-n/2)}{\Gamma(N)}A^{n-2N}\ .
\end{aligned}
$$

The interchange of integrations is valid for n small. Now (3) is finite for $n<6$ and has a simple pole at $n=6$. Collecting (1) and (3), we find

$$
\begin{aligned}
\Gamma_n &= -\tfrac{\lambda^3}{2^{n+1}\pi^{n/2}}\Gamma\left(\tfrac{6-n}{2}\right)\mu^{\frac{3}{2}(6-n)}\int_0^1 d\alpha\int_0^{1-\alpha} d\beta A^{n-6}\\
&= -\tfrac{\lambda^3B_n}{2^{n+1}\pi^{n/2}}\mu^{\frac{3}{2}(6-n)}\Gamma\left(\tfrac{6-n}{2}\right),
\end{aligned}
$$

where

$$
B_n = \int_0^1 d\alpha\int_0^{1-\alpha} d\beta\left[m^2+\alpha(1-\alpha)p_1^2+\beta(1-\beta)(p_1+p_2)^2-2\alpha\beta p_1(p_1+p_2)\right]^{\frac{n-6}{2}}.
$$

Note $B_6 = 1/2$ and B_n is a holomorphic function of n near $n = 6$. Introduce $\varepsilon = 3 - n/2$ and write

$$\Gamma_n = -\lambda^3 \mu^\varepsilon \left\{ \tfrac{1}{2(4\pi)^3} \Gamma(\varepsilon)(4\pi\mu^2)^\varepsilon B_n \right\}$$

and then expand around $\varepsilon = 0$:

$$\Gamma_n =$$

$$-\lambda^3 \mu^\varepsilon . \tfrac{1}{2(4\pi)^3} \left\{ \tfrac{1}{2\varepsilon} - \tfrac{\gamma}{2} - \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \ln \left(\tfrac{m^2 + \alpha(1-\alpha)p_1^2 + \beta(1-\beta)(p_1+p_2)^2 - 2\alpha\beta p_1(p_1+p_2)}{4\pi\mu^2} \right) \right\} + O(\varepsilon),$$

where γ is Euler's constant.

To renormalize for $n = 6$, we introduce the new lagrangian density

$$\mathcal{L}' = \left\{ \tfrac{1}{2} \left(|d\phi|^2 + m^2\phi^2 \right) + \lambda \left(1 + \tfrac{\lambda^2}{2(4\pi)^3} \tfrac{1}{6-n} \right) \mu^{\frac{6-n}{2}} \tfrac{\phi^3}{3!} \right\} d^n x \ .$$

This introduces a diagram which cancels the pole in Γ_n at $n = 6$, and the new 3-point function to one loop is the finite part of Γ_n at $n = 6$:

$$\tfrac{\lambda^3}{2(4\pi)^3} \left\{ \tfrac{\gamma}{2} + \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \ln \left(\tfrac{m^2 + \alpha(1-\alpha)p_1^2 + \beta(1-\beta)(p_1+p_2)^2 - 2\alpha\beta p_1(p_1+p_2)}{4\pi\mu^2} \right) \right\} \ .$$