"Problem Set Four"

(1) Consider a simple harmonic oscillator in one dimension. The Lagrangian is

$$L = \int dt \left(\frac{m}{2} \left(\frac{dx}{dt} \right)^2 - \frac{1}{2} m \omega^2 x^2 \right). \tag{1}$$

The Hamiltonian is accordingly

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 \tag{2}$$

where

$$p = -i \frac{d}{dx} \tag{3}$$

acting on $\mathbf{L}^2(\mathbf{R})$.

Calculate the kernel of the operator e^{-itH}, that is the matrix element

$$\langle b | e^{-iHt} | a \rangle, \tag{4}$$

using path integrals.

That is, find the classical path $x_0(t)$ that starts at a at time 0 and ends at b at time t. Write $x(t) = x_0(t) + \tilde{x}(t)$. By substituting $x(t)$ in L, write the action as a quadratic function of $\tilde{x}(t)$ plus a known function of a, b and t. Do the path integral over $\tilde{x}(t)$ by interpreting it as a determinant which you must study somehow.

In fact after dividing out formally by an infinite but t-independent constant - or if you prefer, upon using something like a zeta function definition of the determinant - you should be able to reduce to an infinite sum that will look something like

$$\prod_{n=1}^{\infty} \left(1 + \frac{p^2}{n^2} \right) \tag{5}$$

for some p. This product in fact defines a certain elementary function of p.

(2) Consider the scalar field theory with Lagrangian

$$L = \int d^n x \left(\frac{1}{2} |d\phi|^2 + \frac{1}{2} m^2 \phi^2 + \frac{g}{3!} \phi^3 \right) \tag{6}$$

in n-dimensional Euclidean space.

Calculate the contribution of order g^3 to the three-point function $\langle \phi(x) \phi(y) \phi(z) \rangle$. (It is best to work in momentum space, that is to set $z = 0$ and take the Fourier transform with respect to x and y.)

(a) Demonstate that the result is finite in $n \leq 6$, and for $n = 6$, describe the precise form of the divergence – that is, introduce a cutoff Λ, and describe just what sort of Λ-dependence you get for large Λ.

Renormalize the result for $n = 6$. That is, carry out the necessary renormalization and write down the renormalized, finite correction to the three point function.

(b) For $n < 6$ make the result as explicit as you can. Use Feynman's identity

$$\frac{1}{ABC} = \int_0^1 d\alpha\, d\beta\, d\gamma\, \delta(\alpha + \beta + \gamma - 1) \frac{1}{(\alpha A + \beta B + \gamma C)^3}. \tag{7}$$

After completing the square, you should be able to reduce the integral over the loop momentum q to

$$\int d\alpha\, d\beta\, d\gamma\, \delta(\alpha + \beta + \gamma - 1) \int d^n q \, \frac{1}{(q^2 + b^2)^3} \tag{8}$$

for some b (which is a polynomial in the three external momenta and in α and β). Set $n = 4$ and carry out the q integral explicitly.

For extra credit, get as far as you reasonably can with the α and β integrals.

(3) Now consider a theory in n-dimensional Euclidean space of a scalar ϕ and a spinor ψ with Lagrangian

$$L = \int d^n x \left(\frac{1}{2}|d\phi|^2 + \frac{1}{2}M^2\phi^2 + \frac{i}{2}(\psi, i(D + m)\psi) + \frac{g}{2}\phi(\psi, \psi) \right). \tag{9}$$

(ψ takes values in a representation of $Spin(n)$ that is associated with a real representation of $Spin(1, n-1)$. The precise details depend a bit on n, but this should not affect the calculation you will have to do. $(\ ,\ )$ is a quadratic form on the spin representation. D is the Dirac operator.)

The free propagator of the ψ field is in momentum space $1/(\Gamma \cdot k + im)$, where $\Gamma \cdot k$ is physicist's notation for Clifford multiplication by k.

The free propagator of ϕ is $1/(k^2 + M^2)$.

L is written so that there is only one interaction vertex, a cubic vertex with two ψ's and one ϕ. The factor to be included at such a vertex in a Feynman diagram is just $-g$ (times the identity matrix or natural bilinear form $(\ ,\ )$ on the fermion ψ, depending on how you look at it).

Compute the correction of order g^2 to the fermion two point function $\langle\psi(x)\psi(0)\rangle$. (Again, it is best to take the Fourier transform. Also, it is best to write the two point function as $1/(\Gamma \cdot k + im + i\Sigma)$ and think of computing a correction to Σ, which will be given by an “amputated one loop diagram.” Σ will be of the form $w(k^2)\Gamma \cdot k + iv(k^2)$ and the problem is to study both functions w and v.)

(a) Show that the result is finite for $n < 4$.

(b) Show that (for $n < 4$) the result is analytic in k^2 for $k^2 > -(m+M)^2$. (We are writing the formulas in Euclidean signature here; in Lorentz signature with metric $+-----$ this would read $k^2 < (m+M)^2$.) How does this compare to what you would have expected? In particular, your result is holomorphic near $k^2 = -m^2$, so that the correction changes the value of the fermion mass (by an amount that is small if g is small) but not the existence of a discrete state.

(c) For $n = 4$, what renormalizations are necessary to remove the infinity that you have found? Write a formula for the renormalized, finite, correction to the two point function, that is, to Σ. For $n = 4$, does the renormalized two point function have the same analytic behavior found in (b) for $n < 4$?