Witten's Problems, Set Five — N^o. 2

(solution written by D. Freed)

(a) We should compute $\mathrm{Tr}\, e^{-TH}$ for

$$\frac{1}{2m}\left(-\frac{\partial^2}{\partial x^2} + m^2\omega^2 x^2\right) \ .$$

First, we can compute directly in the Hamiltonian picture. Introduce

$$\begin{aligned} A &= \frac{1}{\sqrt{2m}}\left(\frac{\partial}{\partial x} + m\omega x\right) \\ A^* &= \frac{1}{\sqrt{2m}}\left(-\frac{\partial}{\partial x} + m\omega x\right) \end{aligned}$$

Then $H = A^*A + \frac{\omega}{2}$ and $[A, A^*] = \omega$. So the spectrum of H is

$$\mathrm{Spec}\, H = \left\{\frac{\omega}{2} + n\omega\right\}_{n\in\mathbb{Z}^{\geq 0}}$$

and

$$\begin{aligned} \mathrm{Tr}\, e^{-TH} &= e^{-\omega T/2}\sum_{n=0}^{\infty} e^{-n\omega T} \\ &= e^{-\omega T/2}\frac{1}{1-e^{-\omega T}} \\ &= \frac{1}{2\sinh^{\omega T/2}} \end{aligned} \tag{1}$$

To compute the heat kernel trace as a path integral we use the Feynman-Kac formula

$$\mathrm{Tr}\, e^{-TH} = \int e^{-L(X)} dX, \tag{2}$$

where $X\colon\ S^1(T) \to \mathbb{R}$ is a function on the circle $S^1(T) = \mathbb{R}/T\mathbb{Z}$, and

$$L(X) = \frac{m}{2}\int_0^T \left\{\dot{X}(t)^2 + \omega^2 X(t)^2\right\} dt$$

Since X is periodic we can integrate by parts to obtain

$$L(X) = (QX, X)/2 \tag{3}$$

$$Q = m\left(-\frac{\partial^2}{\partial t^2} + \omega^2\right) \tag{4}$$

where the inner product in (3) is the L^2 inner product on $S^1(T)$. So the path integral (2) is formally

$$\operatorname{Tr} e^{-TH} = \frac{1}{\sqrt{\det Q}} \tag{5}$$

Now Q has eigenfunctions $\{e^{2\pi int/T}\}_{n\in\mathbb{Z}}$ with corresponding eigenvalues $\left\{m\left(\frac{4\pi^2n^2}{T^2} + \omega^2\right)\right\}_{n\in\mathbb{Z}}$. Therefore, the determinant is the divergent product

$$\begin{aligned}
\det Q &= \prod_{n\in\mathbb{Z}} m\left(\frac{4\pi^2n^2}{T^2} + \omega^2\right) \\
&= \prod_{n\in\mathbb{Z}^+}\left(\frac{4\pi^2m}{T^2}\right)^2 \prod_{n=1}^{\infty}\left(1 + \left(\frac{\omega T}{2\pi n}\right)^2\right)\cdot m\omega^2 \\
&= \prod_{n\in\mathbb{Z}^+}\left(\frac{4\pi^2m}{T^2}n^2\right)^2 \cdot \frac{\sinh^2\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)^2}\cdot m\omega^2 \ .
\end{aligned}$$

We evaluate the infinite constants using the Riemann ζ-function:

$$\begin{aligned}
\prod_{n\in\mathbb{Z}^+}\left(\frac{4\pi^2m}{T^2}n^2\right)^2 &= \left(\frac{4\pi^2m}{T^2}\right)^{2\zeta(0)} e^{-4\zeta'(0)} \\
&= \frac{T^2}{4\pi^2m}\,4\pi^2 \ ,
\end{aligned}$$

since $\zeta(0) = -1/2$ and $\zeta'(0) = -\frac{1}{2}\ln 2\pi$. Altogether we have

$$\det Q = \left(2\sinh\frac{\omega T}{2}\right)^2 \ ,$$

and plugging in (5) we obtain agreement with (1).

(b) We first evaluate the path integral. We must diagonalize (4) with antiperiodic boundary conditions. So the eigenfunctions are $\{e^{2\pi int/T}\}_{n\in\mathbb{Z}+1/2}$ with corresponding eigenvalues

$\left\{m\left(\frac{4\pi^2n^2}{T^2}+\omega^2\right)\right\}_{n\in\mathbb{Z}+1/2}$. So

$$\begin{aligned}
\det\, Q &= \prod_{n\in\mathbb{Z}+1/2} m\left(\frac{4\pi^2n^2}{T^2}+\omega^2\right)\\
&= \prod_{n\in\mathbb{Z}+1/2}\left(\frac{4\pi^2m}{T^2}n^2\right)\prod_{n\in\mathbb{Z}+1/2}\left(1+\left(\frac{\omega T}{2\pi n}\right)^2\right)^2 \qquad (6)\\
&= 4\cosh^2\frac{\omega T}{2}\ ,
\end{aligned}$$

from which the path integral is

$$\frac{1}{\sqrt{\det\, Q}}=\frac{1}{2\cosh\,\frac{\omega T}{2}}. \qquad (7)$$

In (6) we use the fact that the ζ-normalized number of integers is zero to compute

$$\prod_{n\in\mathbb{Z}+1/2}\left(\frac{4\pi^2m}{T^2}n^2\right)=\prod_{n\in\mathbb{Z}+1/2}n^2=\prod_{n\text{ odd}}n^2=\frac{\prod\limits_{n\neq 0}n^2}{\prod\limits_{\substack{n\text{ even}\\ n\neq 0}}n^2}=\frac{\prod\limits_{n\neq 0}n^2}{\prod\limits_{n\neq 0}4n^2}=4\ ,$$

since the ζ-normalized number of nonzero integers is -1.

On the Hilbert space the operator J is the action on functions induced by $x\to -x$. Notice that J anticommutes with A^*, so that its action on the n$^{\text{th}}$ eigenfunction $\frac{(A^*)^n}{n!}\,\Omega$ is multiplication by $(-1)^n$. Hence

$$\begin{aligned}
\operatorname{Tr} Je^{-TH} &= e^{-\omega T/2}\sum_{n=0}^{\infty}(-1)^n e^{-n\omega T}\\
&= e^{-\omega T/2}\frac{1}{1+e^{-\omega T}}\\
&= \frac{1}{2\cosh\omega T/2}
\end{aligned}$$

This agrees with (7).