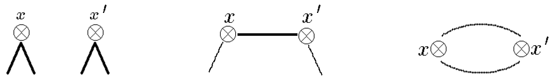
Witten's Problems, Set Six — N°. 1 (solution written by D. Freed)

Problem:

(1) In free scalar field theory in four dimensions, with mass m, calculate the operator product expansion $\phi^2(x)\phi^2(x')|_{x\to x'}$ up to and including all operators of dimension four or less. (This means, relative to what we did in class, working out contributions of some operators with derivatives. This should not be a long exercise.)

Solution:

We represent $\phi^2(x)$ by  and then there are 3 diagrams in $\phi^2(x)\phi^2(x')$:



I II III

The first diagram is the ordinary Taylor series, which to operators of dimension four is simply

$$I = \phi(x)^4 + \cdots$$

The second diagram occurs 4 times and is

$$\begin{aligned}
\mathrm{II} &= 4G(x-x')\phi(x)\phi(x') \\
&= 4G(x-x')\Big[\phi(x)^2 + (x'-x)^\mu \phi(x)\partial_\mu\phi(x) + \tfrac{1}{2}(x'-x)^\mu(x'-x)^\nu \phi(x)\partial_\mu\partial_\nu\phi(x) \\
&\qquad + \cdots\Big].
\end{aligned}$$

The third diagram occurs twice and is simply

$$\mathrm{III} = 2G(x-x')^2.$$

Since $G(x-x') \sim \frac{1}{|x-x'|^2}$ as $x' \to x$, we have the operator product expansion up to operators of dimension four:

$$\begin{aligned}
\phi^2(x)\phi^2(x') \underset{x'\to x}{\sim} \frac{2}{|x-x'|^4} &+ \frac{4\phi(x)^2}{|x-x'|^2} + 4\frac{(x-x')^\mu}{|x-x'|^2}\phi(x)\partial_\mu\phi(x) \\
&+ \left\{\phi(x)^4 + 4\frac{(x-x')^\mu(x-x')^\nu}{|x-x'|^2}\phi(x)\partial_\mu\partial_\nu\phi(x)\right\} \\
&+ \cdots
\end{aligned}$$