Witten's Problems, Set Six — N^o. 2 (solution written by D. Freed)

Problem:

(2) In this exercise we will compute the one-loop renormalization of $U(1)$ gauge theory with fermions in four dimensions. It is of considerable theoretical importance, and also is part of the computation of the "Lamb shift" which was one of the historic tests of QED around 1950.

The Lagrangian is

$$L = \frac{1}{4e^2}\int d^4x F_{ij}F^{ij} + \int d^4x(\psi, (D+im)\psi). \tag{1}$$

Here the gauge field is $A = A_i dx^i$ – a connection on a line bundle $\mathcal{L}$ – the curvature is $F = \frac{1}{2}F_{ij}dx^i \wedge dx^j$, and ψ is a charged spinor field (section of $(S_+ \oplus S_-) \otimes \mathcal{L}$).

As explained in Faddeév's lectures, to quantize the theory one must "pick a gauge." One can pick a gauge in which the kinetic energy for $B = A/e$ is just

$$L' = \frac{1}{2}\int d^4x \sum_{i,j}(\partial_i B_j)^2. \tag{2}$$

This means that the free propagator of B is

$$\langle B_i(x)B_j(0)\rangle = \delta_{ij}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}. \tag{3}$$

The inverse of the free propagator is therefore, in momentum space, $D^0_{ij} = \delta_{ij}k^2$. The inverse of the exact propagator of B is $D_{ij} = D^0_{ij} + \Sigma_{ij}$, where Σ_{ij}, the "self-energy," is to be computed from loops.

A very fundamental property of Σ_{ij} is that in momentum space $k^i\Sigma_{ij}(k) = 0$. (An explanation of why it is so will be given later, maybe in the homework session.) This implies that

$$\Sigma_{ij}(k) = (k^2\delta_{ij} - k_ik_j)F(k^2), \tag{4}$$

with a scalar function F.

The problem here is to compute F in the one-loop approximation. (More exactly, you will compute Σ_{ij}, find it has the form (4), and compute F, after some renormalization.) The importance is twofold. (1) The sign of F – which can be predicted without computation, but in this exercise we will just compute it – is the basic quantity that determines the different behavior of abelian and nonabelian quantum gauge theories in four dimensions (this is a slight simplification as the definition of the analog of F is a bit more elaborate in the nonabelian case). (2) The renormalized F is observable and, as I mentioned above, its calculation is part of the computation of the Lamb shift.

To compute F in one-loop approximation, just write down the one-fermion-loop diagram. The fermion propagator is in momentum space $1/(\Gamma \cdot k + im)$ where $\Gamma \cdot k$ is Clifford multiplication by k. In the one loop diagram for Σ_{ij}, there are two vertices on the fermion loop; at these vertices one multiplies by $e\Gamma_i$ or $e\Gamma_j$ respectively (the e comes from using $B = A/e$). The loop gives a supertrace over the spin representation of $SO(4)$; don't forget that it is four-dimensional as both S_+ and S_- are included here! (To avoid confusion, in this context supertrace just means -1 times the ordinary trace, since the objects propagating in the loop are all fermions. No distinction is made between S_+ and S_-. We explained in a previous homework session where the -1 comes from.) You will have to rationalize the fermion propagator, writing $1/(\Gamma \cdot k + im) = (\Gamma \cdot k - im)/(k^2 + m^2)$, introduce Feynman propagators to combine the denominators, perform the k integral, and at that point you will know that you are on the right track when you demonstrate that Σ is of the form claimed in (4). Before you demonstrate (4) it will appear that you are dealing with a possibly quadratically divergent integral, but once you've confirmed (4) it should be clear that there is only a logarithmic divergence which you'll renormalize in the by now familiar fashion (just subtracting the value of the integral at any given value of k, such as $k = 0$).

Solution:

We work in Euclidean space.

Write $A = eB$ and add to the lagrangian density a gauge fixing term

$$\tfrac{\alpha}{2} \left| \sum_i \partial_i B_i \right|^2 d^4 x,$$

where α is a real number to be determined. Then the free part of the lagrangian for the field B is

$$\begin{aligned}
&\tfrac{1}{4} \sum_{i \neq j} \|\partial_i B_j - \partial_j B_i\|^2 + \tfrac{\alpha}{2} \|\sum_i \partial_i B_i\|^2 \\
&\quad = \tfrac{1}{2} \sum_{i \neq j} \|\partial_i B_j\|^2 - \tfrac{1}{2} \sum_{i \neq j} \langle \partial_i B_j, \partial_j B_i \rangle + \tfrac{\alpha}{2} \sum_i \|\partial_i B_i\|^2 \\
&\qquad + \tfrac{\alpha}{2} \sum_{i \neq j} \langle \partial_i B_i, \partial_j B_j \rangle .
\end{aligned} \tag{1}$$

Here we write $\langle \,\cdot\, , \,\cdot\, \rangle$ and $\| \,\cdot\, \|$ for the integrated inner products. Take $\alpha = 1$. Then the second and fourth terms in (1) cancel using integration by parts, and the free lagrangian for B is

$$\tfrac{1}{2} \sum_{i,j} \|\partial_i B_j\|^2.$$

So the free propagator for B in momentum space is

$$(D^0_{ij})^{-1}(k) = \tfrac{\delta_{ij}}{k^2}. \tag{2}$$

The total lagrangian density (after cancelling the terms in (1)) is now

$$\mathcal{L}' = \Big\{ \tfrac{1}{2} \sum_{i,j} |\partial_i B_j|^2 + (i(\Gamma \cdot \partial + m)\psi, \psi) + e(\Gamma \cdot B\psi, \psi) \Big\} d^4x. \tag{3}$$

Here Γ^μ is Clifford multiplication in the μ^{th} coordinate direction, and $i\Gamma \cdot \partial = i\Gamma^\mu \partial_\mu$ is the flat space Dirac operator. Our convention is

$$\Gamma^\mu \Gamma^\nu + \Gamma^\mu \Gamma^\nu = 2\delta^{\mu\nu}. \tag{4}$$

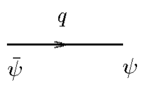
The Γ^μ are *hermitian* operators, so that $i\Gamma \cdot \partial$ is hermitian, while im is skew-Hermitian. Note that ψ is an odd variable so that $(i\Gamma \cdot \partial\psi, \psi)$ is purely imaginary, while $(im\psi, \psi)$ is real.* The fermionic propagator in momentum space is

$$\tfrac{1}{\Gamma \cdot k + im} = \tfrac{\Gamma \cdot k - im}{k^2 + m^2}. \tag{5}$$

We remark that our inner product is linear in the first variable and conjugate linear in the second variable. Think of A as a connection in a complex line bundle L, so that the spinor fields are sections of $S \otimes L$, where S is the (complexified) spin space. We view

$$Q(\psi, \bar{\psi}) = (i(\Gamma \cdot \partial + m)\psi, \bar{\psi}) \tag{6}$$

as a bilinear form, where ψ is a section of $S \otimes L$ and $\bar{\psi}$ a section of $S \otimes \bar{L}$. The hermitian inner product on the right hand side of (6) is written as linear in both variables. (The spin space S is quaternionic in the Euclidean signature, so there is an identification $S \cong \bar{S}$. Note that in Minkowski space the spinors are real.) The ordering of the variable is important, since the spinor fields are odd (anticommuting). The same holds for the propagator Q^{-1}, and so fermion lines in Feynman diagrams have a definite orientation:



$$\tfrac{\Gamma \cdot q - im}{q^2 + m^2}$$

Any loop in a Feynman diagram requires a trace, and by general principles in a theory with fermions it should be a *supertrace*. The reasoning is as follows. The edges around the loop are propagators, which are bilinear forms. The vertices represent polynomials and we are instructed to contract the polynomials with the propagators. After some contractions we can view the loop as a composition of operators, where when we close the loop we must take a trace. In general, if V is a graded vector space we agree to contract V^* and V

*Continued to Minkowski space the lagrangian is real.

by letting V^* operate on the left. An operator on V is identified (without signs) with an element of $V \otimes V^*$: the operator T corresponds to $Te_i \otimes e^i$, where $\{e_i\}$ is a basis of V and $\{e^i\}$ the dual basis of V^*. The natural contraction of an operator is the composition

$$\operatorname{End} V \xrightarrow{\simeq} V \otimes V^* \longrightarrow V^* \otimes V \longrightarrow \mathbb{C}$$

which is the supertrace.

Since we will use dimensional regularization, we compute the scaling dimensions of the fields in d dimensional space:

$$[B] = \tfrac{d-2}{2}, \quad [\psi] = \tfrac{d-1}{2}, \quad [m] = 1, \quad [e] = \tfrac{4-d}{2}$$

so we replace e by a dimensionless constant λ and a mass parameter μ (i.e., $[\mu] = [m] = 1$):

$$e \to \lambda \mu^{\frac{4-d}{2}}.$$

It is convenient to define

$$\epsilon = \tfrac{4-d}{2},$$

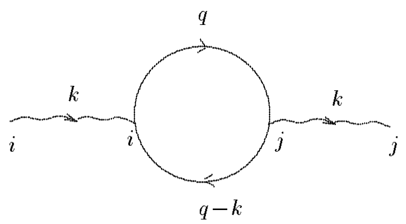
so that the interaction term in the lagrangian density is

$$\lambda \mu^{\epsilon} (\Gamma \cdot B\psi, \psi).$$

Now the 1-loop correction to the 2-point function of B comes by evaluating the graph shown. Denote its value as $-\Sigma_{ij}(k)$. Summing over all chains made out of this one particle irreducible graph we see that the corrected 2-point function is D_{ij}^{-1}, where

$$D_{ij} = D_{ij}^0 + \Sigma_{ij}. \tag{7}$$

(See Witten's lectures for an explanation.)



We remark that if V is an inner product space and Q: $V \to \operatorname{Sym}^2 V^*$ a quadratic form which is equivariant under the orthogonal group $O(V)$, then

$$Q_v(\xi_1, \xi_2) = E(|v|^2)(\xi_1, \xi_2) + F(|v|^2)(v, \xi_1)(v, \xi_2), \quad v, \xi_1, \xi_2 \in V,$$

for some functions E, F. Also, if $Q_v(v,\xi)=0$ for all ξ, then

$$Q_v(\xi_1,\xi_2)=F(|v|^2)\{(v,\xi_1)(v,\xi_2)-|v|^2(\xi_1,\xi_2)\}.$$

This is the form taken by $\sum$. (This is proved using the Ward identities, which we have not yet discussed in lectures.)

At last we evaluate the diagram in d dimensions as

$$-\Sigma_{ij}(k)=\lambda^2\mu^{2\varepsilon}\int\tfrac{d^dq}{(2\pi)^d}\left[-\mathrm{Tr}\left\{\Gamma^i\tfrac{\Gamma\cdot(q-k)-im}{(q-k)^2+m^2}\Gamma^j\tfrac{\Gamma\cdot q-im}{q^2+m^2}\right\}\right]. \tag{8}$$

We first expand the numerator of the expression in braces:

$$\begin{aligned}&\mathrm{Tr}\Big[\Gamma^i(\Gamma^\mu(q_\mu-k_\mu)-im)\Gamma^j(\Gamma^\nu q_\nu-im)\Big]\\&\qquad=\mathrm{Tr}\Big[(q_\mu-k_\mu)q_\nu\Gamma^i\Gamma^\mu\Gamma^j\Gamma^\nu-m^2\Gamma^i\Gamma^j\Big]\end{aligned} \tag{9}$$

Here we use the fact that the trace of an odd product of Γ matrices vanishes.* (Note that $\mathrm{Tr}(1)=4$ and $\mathrm{Tr}\Gamma^i=\mathrm{Tr}\Gamma^i\Gamma^j=\mathrm{Tr}\Gamma^i\Gamma^j\Gamma^k=\mathrm{Tr}\Gamma^1\Gamma^2\Gamma^3\Gamma^4=0$ if i, j,k are distinct.) For the moment we leave (9) unevaluated and introduce Feynman parameters into (8):

$$\begin{aligned}\tfrac{1}{(q-k)^2+m^2}\,\tfrac{1}{q^2+m^2}&=\int_0^1 d\alpha\,\tfrac{1}{[\alpha(q-k)^2+(1-\alpha)q^2+m^2]^2}\\&=\int_0^1 d\alpha\,\tfrac{1}{[(q-\alpha k)^2+m^2+\alpha(1-\alpha)k^2]^2}\end{aligned}$$

Let

$$M^2=m^2+\alpha(1-\alpha)k^2 \tag{10}$$

and make the shift of integration $q\to q+\alpha k$ in (8). Then using (9) we obtain

$$\begin{aligned}-\Sigma_{ij}(k)=-\lambda^2\mu^{2\varepsilon}\int_0^1 dx\int\tfrac{d^dq}{(2\pi)^d}\\ \tfrac{\mathrm{Tr}\left[(q_\mu+(\alpha-1)k_\mu)(q_\nu+\alpha k_\nu)\Gamma^i\Gamma^\mu\Gamma^j\Gamma^\nu-m^2\Gamma^i\Gamma^j\right]}{(q^2+M^2)^2}\end{aligned} \tag{11}$$

Now it is easy to compute that for any (co)vectors v and w,

$$v_\mu w_\nu\mathrm{Tr}(\Gamma^i\Gamma^\mu\Gamma^j\Gamma^\nu)=4(v_iw_j+v_jw_i-\delta_{ij}v\cdot w),$$

*We evaluate the Γ matrices in 4 dimensions before continuing the integral to d dimensions.

and

$$\mathrm{Tr}(\Gamma^i\Gamma^j) = 4\delta_{ij}.$$

Observe that $\int d^dq F(q) = 0$ for an odd function of q, simply by changing variables $q \to -q$. Thus the $\int d^dq$ in (11) reduces to

$$\begin{aligned} 4\int \tfrac{d^dq}{(2\pi)^d}\tfrac{(2q_iq_j-\delta_{ij}q^2)+\alpha(\alpha-1)(2k_ik_j-\delta_{ij}k^2)-\delta_{ij}m^2}{(q^2+M^2)^2} \\ = -8\alpha(1-\alpha)(k_ik_j-\delta_{ij}k^2)\int \tfrac{d^dq}{(2\pi)^d}\tfrac{1}{(q^2+M^2)^2} \\ +4\int \tfrac{d^dq}{(2\pi)^d}\left\{\tfrac{2q_iq_j}{(q^2+M^2)^2}-\tfrac{\delta_{ij}}{q^2+M^2}\right\} \end{aligned} \tag{12}$$

The last integral vanishes for $i \neq j$ (take $q_j \to -q_j$) and is independent of i for $i = j$, so reduces to

$$4\delta_{ij}\int \tfrac{d^dq}{(2\pi)^d}\left\{\tfrac{\frac{2}{d}q^2}{(q^2+M^2)^2}-\tfrac{1}{q^2+M^2}\right\}. \tag{13}$$

We claim that under dimensional regularization (13) vanishes. To see this consider

$$I(\lambda) = \int \tfrac{d^dq}{(2\pi)^d}\tfrac{1}{(\lambda q^2+M^2)}.$$

Then by scaling $I(\lambda) = \lambda^{-d/2}I(1)$, and differentiating in λ and $\lambda = 1$ we find $I'(1) = -d/2I(1)$. This implies (13) vanishes. Now we evaluate the remaining integral in (12) in d dimensions as

$$\int \tfrac{d^dq}{(2\pi)^d}\tfrac{1}{(q^2+M^2)^2} = \tfrac{\Gamma(\epsilon)}{(4\pi)^{d/2}}\tfrac{1}{(M^2)^\epsilon}.$$

(See problem set 4, problem 2 for the derivation of a similar formula.)

Finally, plugging back into (11) we have

$$-\Sigma_{ij}(k) = (k_ik_j-\delta_{ij}k^2)F(k^2), \tag{14}$$

where

$$F(k^2) = \tfrac{\lambda^2}{2\pi^2}\Gamma(\epsilon)(4\pi\mu^2)^\epsilon\int_0^1 d\alpha\,\alpha(1-\alpha)(m^2+\alpha(1-\alpha)k^2)^{-\epsilon}.$$

This is finite for $\epsilon > 0$ (which is $d < 4$) and has a simple pole at $\epsilon = 0$. Using the expansion

$$\Gamma(\epsilon) = \tfrac{1}{\epsilon} - \gamma + O(\epsilon), \qquad \gamma = \text{ Euler's constant},$$

we have

$$F(k^2) = \tfrac{\lambda^2}{2\pi^2}\left\{\tfrac{1}{6\epsilon} - \tfrac{\gamma}{6} - \int_0^1 d\alpha\,\alpha(1-\alpha)\ln\left(\tfrac{m^2+\alpha(1-\alpha)k^2}{4\pi\mu^2}\right) + O(\epsilon)\right\}. \tag{15}$$

The $\int_0^1 d\alpha$ can be evaluated in terms of elementary functions, but the result is complicated and unilluminating.

To renormalize we must adjust the fields and parameters in the lagrangian (3), which we write now including the gauge fixing term:

$$\mathcal{L} = \left\{ \tfrac{1}{2}\Sigma_{i,j}|\partial_i B_j|^2 + \tfrac{\alpha-1}{2}\left|\Sigma_i \partial_i B_i\right|^2 + (i(\Gamma\cdot\partial + m)\psi, \psi) + e(\Gamma\cdot B\psi,\psi)\right\} dx^4. \tag{16}$$

The inverse propagator, written in terms of the Fourier variable k, is

$$D_{ij} = k^2\delta_{ij} + (\alpha-1)k_i k_j.$$

The renormalization scheme defines the bare field B_0 and bare coupling α_0 as

$$\begin{aligned} B_0 &= (1+P)^{1/2}B \\ \alpha_0 &= \tfrac{\alpha}{1+P} \end{aligned} \tag{17}$$

so that the bare inverse propagator is

$$\begin{aligned} D^0_{ij} &= (1+P)k^2\delta_{ij} + (\alpha - 1 - P)k_i k_j \\ &= D_{ij} - \Sigma_{ij} \end{aligned}$$

Comparing to (14) we have for $\alpha = 1$ to 1-loop accuracy, using (15),

$$P = -\tfrac{\lambda^2}{12\pi^2}\tfrac{1}{\epsilon} + \text{ finite}.$$

Finally, we renormalize the charge $e = \lambda\mu^\epsilon$ in (16) by setting

$$\begin{aligned} e_0 &= \lambda\mu^\epsilon\cdot(1+P)^{-1/2} \\ &\approx \lambda\mu^\epsilon\left(1 + \tfrac{\lambda^2}{24\pi^2}\tfrac{1}{\epsilon} + \text{ finite}\right). \end{aligned} \tag{18}$$

The β-function is

$$\beta = \mu\tfrac{\partial\lambda}{\partial\mu}\Big|_{\epsilon=0}$$

with e_0 fixed. Using minimal subtraction we set the finite part to zero and differentiate (18):

$$0 = \epsilon\left(\lambda + \tfrac{\lambda^3}{24\pi^2}\tfrac{1}{\epsilon}\right) + \beta\left(1 + \tfrac{\lambda^2}{8\pi^2}\tfrac{1}{\epsilon}\right) \tag{19}$$

So to leading order in ϵ we have $\beta = -\epsilon\lambda$, and then taking the regular part of (19) at $\epsilon = 0$ we find

$$\mu\tfrac{d\lambda}{d\mu} = \beta = \tfrac{\lambda^3}{12\pi^2}. \tag{20}$$

This ODE has solution

$$\frac{1}{\lambda^2(\mu)} - \frac{1}{\lambda^2(\mu_0)} = -\frac{1}{6\pi^2}\ln(\mu/\mu_0)$$

for some constant μ_0. Equivalently,

$$\lambda^2(\mu) = \frac{\lambda^2(\mu_0)}{1-\frac{\lambda^2(\mu_0)}{6\pi^2}\ln(\mu/\mu_0)}$$

which exhibits a (Landau) pole at

$$\mu = \mu_0 \, \exp\left(\frac{6\pi^2}{\lambda^2(\mu_0)}\right).$$

Finally, we given an *a priori* argument that the coefficient of the divergence in $F(k^2)$ is positive (see (15)). The same argument in a simpler situation appears in Problem 1 of Problem Set 7. Here we identify the one particle irreducible diagram we computed as minus the two point function of the *current* J_i in the free theory:

$$-\check{\Sigma}_{ij}(x) = -\lambda^2 \left\langle J_i(x)\overline{J_j(0)} \right\rangle_{\text{free}}.$$

Here $\check{\Sigma}_{ij}$ is the inverse Fourier transform of Σ_{ij}, and the current is

$$J_i(x) = \Gamma^i \psi_{(x)}\overline{\psi_{(x)}},$$

where the product of the spinors is taken to be a vector. The minus sign is due to the fact that conjugation reverses the sign of J_i, since we interchange ψ and $\bar{\psi}$. Now reflection positivity implies that $\left\langle J_i(x)\overline{J_j(0)} \right\rangle_{\text{free}}$ is a nonnegative quadratic form for $x \neq 0$. The behavior of the Fourier transform $-\Sigma_{ij}(k)$ as $k \to \infty$ is determined by the behavior of $-\check{\Sigma}_{ij}(x)$ as $x \to 0$, and it follows easily that $-\Sigma_{ij}(k)$ is nonpositive for large k. Using (14) this translates into $F(k^2) > 0$ for large k. This is consistent with (15).