"Problem Set Six"

(1) In free scalar field theory in four dimensions, with mass m, calculate the operator product expansion $\phi^2(x)\phi^2(x')|_{x\to x'}$ up to and including all operators of dimension four or less. (This means, relative to what we did in class, working out contributions of some operators with derivatives. This should not be a long exercise.)

(2) In this exercise we will compute the one-loop renormalization of $U(1)$ gauge theory with fermions in four dimensions. It is of considerable theoretical importance, and also is part of the computation of the "Lamb shift" which was one of the historic tests of QED around 1950.

The Lagrangian is

$$L = \frac{1}{4e^2}\int d^4x F_{ij}F^{ij} + \int d^4x(\psi, (D + im)\psi). \tag{1}$$

Here the gauge field is $A = A_i dx^i$ – a connection on a line bundle $\mathcal{L}$ – the curvature is $F = \frac{1}{2}F_{ij}dx^i \wedge dx^j$, and ψ is a charged spinor field (section of $(S_+ \oplus S_-) \otimes \mathcal{L}$).

As explained in Faddeév's lectures, to quantize the theory one must "pick a gauge." One can pick a gauge in which the kinetic energy for $B = A/e$ is just

$$L' = \frac{1}{2}\int d^4x \sum_{i.j}(\partial_i B_j)^2. \tag{2}$$

This means that the free propagator of B is

$$\langle B_i(x)B_j(0)\rangle = \delta_{ij}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}. \tag{3}$$

The inverse of the free propagator is therefore, in momentum space, $D^0_{ij} = \delta_{ij}k^2$. The inverse of the exact propagator of B is $D_{ij} = D^0_{ij} + \Sigma_{ij}$, where Σ_{ij}, the "self-energy," is to be computed from loops.

A very fundamental property of Σ_{ij} is that in momentum space $k^i\Sigma_{ij}(k) = 0$. (An explanation of why it is so will be given later, maybe in the homework session.) This implies that

$$\Sigma_{ij}(k) = (k^2\delta_{ij} - k_ik_j)F(k^2), \tag{4}$$

with a scalar function F.

The problem here is to compute F in the one-loop approximation. (More exactly, you will compute Σ_{ij}, find it has the form (4), and compute F, after some renormalization.)

The importance is twofold. (1) The sign of F – which can be predicted without computation, but in this exercise we will just compute it – is the basic quantity that determines the different behavior of abelian and nonabelian quantum gauge theories in four dimensions (this is a slight simplification as the definition of the analog of F is a bit more elaborate in the nonabelian case). (2) The renormalized F is observable and, as I mentioned above, its calculation is part of the computation of the Lamb shift.

To compute F in one-loop approximation, just write down the one-fermion-loop diagram. The fermion propagator is in momentum space $1/(\Gamma \cdot k + im)$ where $\Gamma \cdot k$ is Clifford multiplication by k. In the one loop diagram for Σ_{ij}, there are two vertices on the fermion loop; at these vertices one multiplies by $e\Gamma_i$ or $e\Gamma_j$ respectively (the e comes from using $B = A/e$). The loop gives a supertrace over the spin representation of $SO(4)$; don't forget that it is four-dimensional as both S_+ and S_- are included here! (To avoid confusion, in this context supertrace just means -1 times the ordinary trace, since the objects propagating in the loop are all fermions. No distinction is made between S_+ and S_-. We explained in a previous homework session where the -1 comes from.) You will have to rationalize the fermion propagator, writing $1/(\Gamma \cdot k + im) = (\Gamma \cdot k - im)/(k^2 + m^2)$, introduce Feynman propagators to combine the denominators, perform the k integral, and at that point you will know that you are on the right track when you demonstrate that Σ is of the form claimed in (4). Before you demonstrate (4) it will appear that you are dealing with a possibly quadratically divergent integral, but once you've confirmed (4) it should be clear that there is only a logarithmic divergence which you'll renormalize in the by now familiar fashion (just subtracting the value of the integral at any given value of k, such as $k = 0$).