Witten's Problems, Set Seven — N°. 1 (solution written by D. Freed)

Problem:

(1) (a) Show that in ϕ^4 theory the anomalous dimension of ϕ vanishes in the one loop approximation and that the one-loop beta function is therefore given by a one-loop diagram that we have computed before. The one-loop beta function is in particular $\beta(\lambda) = A\lambda^2$ where λ is the ϕ^4 coupling, and A is the coefficient of the one-loop divergence.

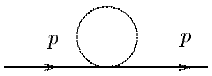
(b) Convince yourself that the sign of the logarithmic divergence in this one-loop diagram (whose significance we by now appreciate) can be determined without any computation; it is determined by the fact that in *free* scalar field theory the two point function $\langle\phi^2(x)\phi^2(0)\rangle$ is positive. (This is actually a point that has been made previously.)

Solution:

(a) The anomalous dimension of ϕ gets contributions from 2-point function diagrams. All such (connected) diagrams have naive quadratic divergences. Let p be the momentum flowing in (and by conservation flowing out) and m the mass. Then with a cutoff Λ the diagram diverges as

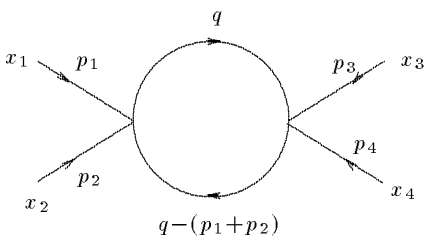
$$f\left(\frac{p}{m}\right)\Lambda^2 + g\left(\frac{p}{m}\right)\log\left(\frac{\Lambda}{m}\right)p^2 + h\left(\frac{p}{m}\right)\log\left(\frac{\Lambda}{m}\right)m^2$$

for some functions f, g, h. This follows by dimensional analysis. (In fact, since the second derivative of a diagram with respect to p is finite, we see that f, g, h are simple polynomials.) Now the only 1-loop diagram is shown, and the loop integral is independent



of p. For this diagram, then, $g \equiv 0$ and there is no renormalization to the coefficient of $|d\phi|^2$ and so no contribution to the anomalous dimension of ϕ.

The diagram which contributes to the 1-loop β-function is the integral



$$\hat{D}(p_1,p_2,p_3,p_4) = \lambda^2 \int \frac{d^4q}{(2\pi)^4}\frac{1}{(q^2+m^2)((q-p_1-p_2)^2+m^2)} \cdot \delta(p_1+p_2+p_3+p_4)$$

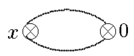
which has a leading divergence $A\lambda^2 \log \frac{\Lambda}{m}$ if an ultraviolet cutoff Λ is imposed. (Here $\hat{\Gamma}$ is a distribution supported on the subspace $(p_1 + p_2 + p_3 + p_4 = 0.)$

(b) We want to understand *a priori* that $A > 0$. For this it is best to Fourier transform to position space. This is straightforward and leads to

$$D(x_1, x_2, x_3, x_4) = G(x-y)^2 \delta(x_1 - x)\delta(x_2 - x)\delta(x_3 - y)\delta(x_4 - y),$$

where G is the propagator. (So D is supported on the subspace $x_1 = x_2$, $x_3 = x_4$ and its value there is the square of the Green's function.) In general, the position space prescription for computing a *one particle irreducible* Feynman diagram as part of the effective action is the same as usual — we attach propagators to the legs and integrate over the positions in the internal vertices — except that we replace the propagators on the *external* legs by δ-functions. For convenience we take $y = 0$ and write $D(x) = G(x)^2$.

Now introduce the composite operator $\mathcal{O}(x) = \phi^2(x)$. Then it is immediate that the two point function $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle_{\text{free}}$ in the *free* theory is computed by the diagram shown.

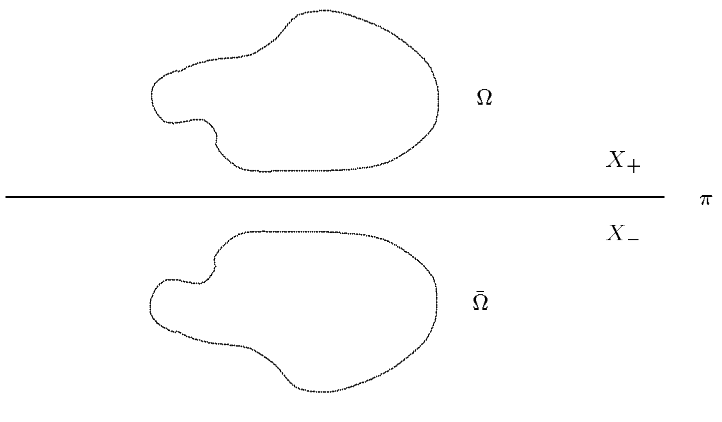


We evaluate this diagram as $G(x)^2$, so it is precisely equal to $D(x)$. It is manifest that $D(x) \geq 0$, of course, since it is written as the square of a real function. But we would like to invoke instead *reflection positivity* since the argument is useful in more general situations.

Reflection positivity asserts that if $\mathcal{O}_\Omega$ is a (complex) operator in a Euclidean field theory, supported in some set Ω lying on one side of a hyperplane, and if $\bar{\Omega}$ denotes the reflection of Ω in the hyperplane, then

$$\left\langle \mathcal{O}_\Omega \overline{\mathcal{O}_{\bar{\Omega}}} \right\rangle \geq 0.$$

If $\mathcal{O}_\Omega$ takes values in a vector space V, then the assertion is that $\left\langle \mathcal{O}_\Omega \overline{\mathcal{O}_{\bar{\Omega}}} \right\rangle$ is a nonnegative hermitian form on V^*.



We can understand this heuristically by writing a path integral over the half-space. Namely, suppose for simplicity that we are dealing with a complex scalar field ϕ with action $S(\phi)$. Then for ϕ_0 a function on π we set

$$Z_{X_+}(\phi_0) = \int\limits_{\substack{\phi\colon X_+\to\mathbb{C} \\ \phi|_\pi=\phi_0}} \mathcal{O}_\Omega(\phi)e^{-S_{X_+}(\phi)}D\phi.$$

This should be viewed as a vector in some Hilbert space of functions of ϕ_0. Now $X_- \cong \bar{X}_+$, where $\bar{X}_+$ is X_+ with the reversed orientation. The Euclidean action has the property that $S_{\bar{Y}} = \overline{S_Y}$ for any spacetime Y: orientation reversal conjugates the action. It follows that

$$\begin{aligned}\overline{Z_{X_+}(\phi_0)} &= \int\limits_{\substack{\phi\colon X_+\to\mathbb{C} \\ \phi|_\pi=\phi_0}} \overline{\mathcal{O}_\Omega(\phi)}e^{-\overline{S_{X_+}(\phi)}}D\phi \\ &= \int\limits_{\substack{\phi\colon X_-\to\mathbb{C} \\ \phi|_\pi=\phi_0}} \overline{\mathcal{O}_\Omega(\phi)}e^{-S_{X_-}(\phi)}D\phi\end{aligned}$$

The norm square of $Z_{X_+}(\phi)$ in $\mathcal{H}$ is computed by integrating over ϕ_0:

$$\begin{aligned}\|Z_{X_+}(\phi_0)\|^2_{\mathcal{H}} &= \int\limits_{\phi_0\colon \pi\to\mathbb{C}} Z_{X_+}(\phi_0)\overline{Z_{X_+}(\phi_0)}D\phi_0 \\ &= \int\limits_{\phi\colon X\to\mathbb{C}} \mathcal{O}_\Omega(\phi)\overline{\mathcal{O}_{\bar{\Omega}}(\phi)}e^{-S_X(\phi)}D\phi \\ &= \left\langle \mathcal{O}_\Omega\overline{\mathcal{O}_{\bar{\Omega}}}\right\rangle,\end{aligned}$$

where the measure is normalized so that the partition function is one. The left hand side is obviously positive, hence so is the right.

In our example $\mathcal{O}_{(x)} = \phi^2(x)$ is real, and to show that $D(x) = \langle \mathcal{O}(x)\mathcal{O}(0)\rangle_{\text{free}} \geq 0$ we use reflection positivity for the perpendicular bisector of the line segment connecting 0 and x. Now $D(x)$ diverges as $x \to 0$, and since $D(x) \geq 0$ the coefficient of the divergence is positive. This implies that the divergence in D with a momentum cutoff Λ has a positive coefficient.