Problem Set Seven

(1) (a) Show that in ϕ^4 theory the anomalous dimension of ϕ vanishes in the one loop approximation and that the one-loop beta function is therefore given by a one-loop diagram that we have computed before. The one-loop beta function is in particular $\beta(\lambda) = A\lambda^2$ where λ is the ϕ^4 coupling, and A is the coefficient of the one-loop divergence.

(b) Convince yourself that the sign of the logarithmic divergence in this one-loop diagram (whose significance we by now appreciate) can be determined without any computation; it is determined by the fact that in *free* scalar field theory the two point function $\langle\phi^2(x)\phi^2(0)\rangle$ is positive. (This is actually a point that has been made previously.)

(2) Consider a four-dimensional scalar field theory with n scalar fields ϕ_i and a Lagrangian

$$\begin{aligned} L = \int d^4x \Bigg(&\frac{1}{2}\sum_{i,\alpha}\left(\frac{\partial\phi_i}{\partial x^\alpha}\right)^2 \\ &+ \sum_{ijkl}\lambda_{ijkl}\phi^i\phi^j\phi^k\phi^l + \dots \Bigg), \end{aligned} \tag{1}$$

where the $\dots$ are possible quadratic and cubic terms in ϕ_j that are irrelevant for the present purposes.

The renormalization group operator will read

$$D = \mu\frac{\partial}{\partial\mu} + \sum_{ijkl}\beta_{ijkl}\frac{\partial}{\partial\lambda_{ijkl}} + \dots \tag{2}$$

(the $\dots$ are anomalous dimension terms) with some functions β_{ijkl}.

(a) Write down the one-loop approximation to β_{ijkl}. You are not expected to do any real calculation here; write down the β_{ijkl} in terms of the λ_{ijkl} and the coefficient called A is problem 1 above.

(b) Show that in this approximation the renormalization group flow

$$\frac{d\lambda_{ijkl}}{dt} = \beta_{ijkl} \tag{3}$$

is a gradient flow (with respect to the obvious metric on the space of couplings).

As Gross mentioned, proving that this is true in general (or at least that there are no closed orbits) is a longstanding open problem.

The special case of this problem for two-dimensional sigma models is actually quite significant in string theory.

(More precisely, what one wants there is that if g is the metric on a sigma model target space, then the renormalization group flow for g should be, after possibly redefining g in a local fashion by a substitution such as $g_{ij} \to g_{ij} + R_{ij} + \ldots$, with R_{ij} the Ricci tensor and $\ldots$ analogous terms of higher degree, of the form

$$\frac{dg}{dt} = \frac{\delta}{\delta g}\Gamma \tag{4}$$

with Γ the integral over the target of a polynomial in the Riemann tensor and its covariant derivatives.)

At the end of one of Gawedzki's lectures, when I asserted that locally there is no obstruction to perturbing a Ricci-flat metric to give a zero of β, I omitted to mention that this is so if and only if it is true in sigma models that the renormalization group flow is a gradient flow in the above sense. Otherwise, there is an obstruction coming from the Bianchi identity.

(3) Consider in four dimensions a Yukawa theory with a spin one-half field ψ and a scalar ϕ. The part of the Lagrangian involving ψ is

$$L = \int d^4x \left(\bar{\psi}(i\Gamma \cdot \partial - \phi)\psi\right). \tag{5}$$

Compute the anomalous dimension of the operator $\bar{\psi}\,\psi$ in the one-loop approximation.

(4) The goal here is to explain the claim that the magnetic coupling is responsible for the negative beta function of nonabelian gauge theories.

We consider scalar, spinor, and vector fields in some representation R of a compact gauge group G, on a four-manifold M.

The Laplacian for a scalar field, a spinor field, or a section of the tangent bundle will be called ∇_0, $\nabla_{1/2}$. and ∇_1, respectively. The operator governing small fluctuations of the scalar, spinor, or vector field will be called X_0, $X_{1/2}$, or X_1. In each case $X_j = \nabla_j + G_j$ where G is a term of order zero (absent for $j = 0$) which Gross called the magnetic moment coupling. In the case of spin one, the vector fields are really gauge fields, and the operator X_1 is accompanied by some operators X_0 for ghost fields that are obtained upon quantization. (This was explained by Fadde'ev.)

The one-loop beta function comes from the logarithmic divergence in the determinant of X_j. As we are interested in terms that depend on the gauge field and not the metric of M, we may as well take M to be flat. In this case, $\nabla_{1/2}$ is just four copies of ∇_0, and ∇_1 is likewise just four copies of ∇_0 (as the spinor and tangent bundles of M are trivial rank four bundles with flat Levi-Civita connection - – at least, let's agree to consider the full rank four spin bundle).

Here is now to determine the X_j for $j = 1/2, 1$.

(a) For $j = 1/2$, X_j is the square of the Dirac operator D. (The log of the determinant of D is conveniently studied as one-half the log of the determinant of $X_{1/2} = D^2$.) A standard calculation shows that

$$X_{1/2} = D^2 = \nabla_{1/2} + \Gamma \cdot F \tag{6}$$

where F is the curvature of the gauge connection and $\Gamma \cdot F$ is Clifford multiplication.

If the second "magnetic moment" term were absent, the log determinant of $X_{1/2}$ would be four times that of X_0 but because one want a minus sign for the fermion loop, the contribution of the fermions to the beta function would be negative. The $\Gamma \cdot F$ term makes a contribution of the opposite sign, which is larger in magnitude, so fermions actually make a positive contribution.

(b) Now consider the gauge fields.

Maxwell's equations read

$$d * dA = 0. \tag{7}$$

You might therefore believe that if we have a non-abelian gauge connection B and we write $A = B + a$, with a a small fluctuation, and we write d_B for the covariant derivative with the connection B, the linearization of the Yang-Mills equations would be

$$d_B * d_B a = 0. \tag{8}$$

Not only is that not so, but the equations (8) are inconsistent, in the sense that they imply the first order constraint $d_B d_B(*d_B a) = 0$ which is trivial when $B = 0$ but generically non-trivial; thus the space of solutions is not "flat" (if I may risk the expression) in a deformation away from $B = 0$. That certainly corresponds to a pathology. By linearizing the Yang-Mills equations you get not (8) but

$$(d_B * d_B)a + (*F_B) \wedge a = 0. \tag{9}$$

(Verify this!) Equations (9) have a "flat" behavior (the constraint got by applying d_B to the equations is trivial) if B obeys the Yang-Mills equations. (Verify this!) More fundamentally, (9) has the gauge invariance $a \to a + d_B\epsilon$ (for any $\epsilon \in \Omega^0(M, \mathrm{adG})$) if B obeys the Yang-Mills equations. (Verify this too!)

In gauge fixing the theory, one would have introduced a gauge fixing term as Fadde'ev explained. This can conveniently be taken to be $(d_B^* a)^2$, and if so the gauge-fixed version of (9) is

$$\nabla_1 a + (*F_B) \wedge a = 0. \tag{10}$$

In (10) we see finally the gauge-fixed operator X_1 of the vector field. The "magnetic coupling" $a \to (*F_B) \wedge a$ is the term responsible for asymptotic freedom, in the sense that without this term, X_1 would be four copies of X_0 and would make a positive contribution to the beta function.