Appendix: signs

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Some classical constructions give objects which are graded, and which can profitably be viewed as "super", i.e. subject to Koszul's sign rule. Examples: cohomology groups of all kind (H^*, Tor_*, $\mathrm{Ext}^* \ldots$), the de Rham complex and other standard complexes, … .

To handle their analogue in the super world, two points of view have been used. As we will see, they are basically equivalent. However, they lead to different sign conventions.

I. One considers them as graded $\bmod 2$ graded (i.e. super) objects. The grading which is the analogue of the grading in the classical case will be called the cohomological grading. The commutativity isomorphism for the tensor product of graded $\bmod 2$ graded vector spaces is defined as follows: for v of bidegree (p, n): parity p and cohomological degree n, and w of bidegree (q, m),

$$\sigma_{\mathbf{I}}\colon V \otimes W \to W \otimes V \quad \text{is} \quad v \otimes w \longmapsto (-1)^{pq+nm} w \otimes v.$$

II. One considers that the classical construction already lives in the super world, with parity = cohomological degree $\bmod 2$. In the super world, one continues to obtain super objects; they also have a cohomological degree, with no influence on signs: the commutativity isomorphism of graded $\bmod 2$ graded vector spaces is defined as

$$\sigma_{\mathbf{II}}\colon V \otimes W \to W \otimes V\colon v \otimes w \to (-1)^{pq} w \otimes v$$

for v of parity p and w of parity q. Of course, one could consider modules instead of vector spaces.

Here are advantages of the point of view **I**.

(A) For any tensor category $\mathcal{T}$, say additive with "associative and commutative" tensor product, one can define a new tensor category $\mathcal{T}^{\mathrm{gr}}$ of graded objects of $\mathcal{T}$, with the commutativity of $\otimes$ being given by:

$$\sigma^{\mathrm{gr}}\colon V \otimes W \to W \otimes V$$

induces on $V^n \otimes W^m$ the map $V^n \otimes W^m \to W^m \otimes V^n$ of $\mathcal{T}$, multiplied by $(-1)^{nm}$. For $\mathcal{T}$ the tensor category of super vector spaces, $\mathcal{T}^{\mathrm{gr}}$ is the category **I** of graded mod 2 graded vector spaces. The point of view **I** hence allows for the following way of reasoning: to prove a statement in superworld, prove it first in any tensor category (keeping in mind the case of vector spaces). At the end, specialize to the category of super vector spaces. In the general categorical story, no sign of super origin are there to confuse us: they are hidden in the compatibilities obeyed by the associativity and commutativity isomorphisms for $\otimes$.

Example (of this kind of reasoning)**.** Suppose we want to see the following for finite dimensional super vector spaces. Known: if W is the dual of V, then V is the dual of W. As V is the dual of W, repeating "known" for W and V, we get that: W is the dual of V. Wished: the duality map is the same as the one we started with.

The dual of V is well defined up to unique isomorphism as an object W provided with a morphism ev: $W \otimes V \to 1$ with suitable properties, for 1 the unit object. To give a map $W \otimes V \to 1$ is the same thing as giving a map $V \otimes W \to 1$, by the isomorphism σ of $W \otimes V$ with $V \otimes W$.

For super vector spaces, the structural map ev: $W \otimes V \to k$ identifies W with the space of linear forms on V, the evaluation map ev becoming $w \otimes v \mapsto w(v)$. If one prefers to say that it is the map $V \otimes W \to k$ which makes of W the dual of V, one is rather led to identify the dual W of V with the space of linear forms on V by $w \mapsto$ linear form ($v \mapsto$ image of $v \otimes w$). This gives two ways to identify the dual (in the categorical sense) with the space of linear forms on V, those two ways differ by a sign on the odd part, and one easily gets confused. In the general categorical setting, no sign occurs.

(B) One has not to decide early on whether an object should be seen as having a cohomological degree.

Example. Let A be a commutative super k-algebra. The point of view **I** suggest to define the module Ω^1_A of Kähler differential as being an A-module (hence bimodule) Ω, provided with a morphism of super k vector spaces $d\colon A \to \Omega$ such that

$$d(ab) = a.db + da.b, \tag{B.1}$$

which is universal. Later, when considering the de Rham complex, one may decide that Ω^1_A is of cohomological degree 1. In the point of view **II**, deciding that in the classical case (purely even A) Ω^1_A is odd, requires d to be an odd map, and that (B.1) be replaced by

$$d(ab) = (-1)^{p(a)}a.db + da.b \tag{B.2}$$

(C) The point of view **I** minimizes the use of the parity change functor Π. Using this functor leads to nightmares of signs, for the following reasons.

(i) Let 1^- be k, viewed as an odd k-vector space. The functor Π is best viewed as being the tensor product with 1^-. One has to decide whether it is $V \mapsto 1^- \otimes V$ or $V \mapsto V \otimes 1^-$. The two are canonically isomorphic, but lead to different sign conventions.

(ii) One has natural isomorphisms $(\Pi V) \otimes W \xrightarrow{\sim} (V \otimes W)$ and $(V \otimes \Pi W) \xrightarrow{\sim} \Pi(V \otimes W)$, exchanged by the commutativity of $\otimes$. The diagram

$$\begin{array}{ccc} \Pi V \otimes \Pi W & \longrightarrow & \Pi(V \otimes \Pi W) \\ \downarrow & & \downarrow \\ \Pi(\Pi V \otimes W) & \longrightarrow & \Pi\Pi(V \otimes W) \end{array}$$

is anticommutative, rather than commutative.

Here are advantages of the point of view **II**.

(A) One has only one parity to consider, for applying the sign rule.

(B) Some constructions are more natural. For example, if D^- is the standard odd line (coordinate ring $k[\theta]$, θ odd, $\theta^2 = 0$), the de Rham complex of a super manifold M is (up to a completion) the space of functions on the super manifold

$$\underline{\mathrm{Hom}}(D^-, M).$$

In the point of view **I**, one has to apply to the de Rham complex the functor "associated simply graded object" explained below.

I prefer the point of view **I**. Bernstein prefers the point of view **II**.

The points of view **I** and **II** are equivalent, in the sense that the tensor categories $\mathcal{T}_{\mathbf{I}}$ and $\mathcal{T}_{\mathbf{II}}$ of graded mod 2 graded vector spaces introduced in **I** and **II** are equivalent: we have an equivalence of categories

$$\widetilde{\mathbf{s}}\colon \mathcal{T}_{\mathbf{I}} \longrightarrow \mathcal{T}_{\mathbf{II}}$$

and an isomorphism of functors

$$\alpha\colon \widetilde{\mathbf{s}}(V \otimes_{\mathbf{I}} W) \xrightarrow{\sim} \widetilde{\mathbf{s}}(V) \otimes_{\mathbf{II}} \widetilde{\mathbf{s}}(W)$$

compatible with the associativity and commutativity isomorphisms for $\otimes_{\mathbf{I}}$ and $\otimes_{\mathbf{II}}$.

The functor $\widetilde{\mathbf{s}}$ is the regrading functor: $\widetilde{\mathbf{s}}(V)$ has the same underlying vector space as V, with v of parity p and cohomological degree n acquiring the parity $p+n$, and keeping its cohomological degree.

On the underlying vector space, $\otimes_{\mathbf{I}}$ and $\otimes_{\mathbf{II}}$ are both the usual tensor product. For v in $V^{p,n}$ and w in $W^{q,m}$, α is defined as

$$\alpha\colon v \otimes w \longmapsto (-1)^{nq} v \otimes w.$$

The compatibility with the commutativity isomorphisms is the commutativity of the diagram

$$\begin{tikzcd} {} \arrow[r, "(-1)^{nq}"] \arrow[d, "(-1)^{pq+nm}"'] & {} \arrow[d, "(-1)^{(p+n)(q+m)}"] \\ {} \arrow[r, "(-1)^{mp}"] & {} \end{tikzcd} .$$

The compatibility with the associativity isomorphisms comes from the identity

$$n_1(p_2 + p_3) + n_2 p_3 = n_1 p_2 + (n_1 + n_2) p_3 \ .$$

The functor "*associated simply* (mod 2) *graded*" $\mathbf{s}$ is the composite of $\widetilde{\mathbf{s}}$ with the functor "forgetting the cohomological degree" to super vector spaces.

Example. Let A be a graded $\quad$ mod 2 graded algebra (point of view **I**). It is given by a multiplication

$$. : A \otimes A \to A.$$

Applying $\mathbf{s}$, we obtain a super algebra, with a of parity p and cohomological degree n becoming of parity $p+n$. The new product

$$*\colon \mathbf{s}(A) \otimes \mathbf{s}(A) \xleftarrow{\sim} \mathbf{s}(A \otimes A) \xrightarrow{\mathbf{s}(\cdot)} \mathbf{s}(A)$$

is

$$x * y = (-1)^{nq} x.y$$

for x in $A^{p,n}$ and y in $A^{q,m}$.

The same applies to graded $\quad$ mod 2 graded modules.

Example: the already mentioned case of the Kähler differential of A.

Example: the Berezinian**.** For V a super vector space, considering V as a space leads to consider the super algebra $\mathrm{Sym}(V^*)$ of functions on "V, viewed as a space". More generally, let V be a free super module of finite rank (p,q) on a commutative super ring A. One has an augmentation $\mathrm{Sym}^*_A(V) \to A$, and

$$\mathrm{Ber}(V) := \underline{\mathrm{Ext}}^*_{\mathrm{Sym}_A(V^*)}(A, \mathrm{Sym}^*_A(V^*)).$$

Those Ext are best considered as given by the derived functor of the functor $\underline{\mathrm{Hom}}(A, \quad)$, identified with the functor from $\mathrm{Sym}^*_A(V)$-modules to A-modules: $M \mapsto$ the sub-A-module of elements of M annihilated by the augmentation ideal.

The only non zero $\underline{\mathrm{Ext}}$ is $\underline{\mathrm{Ext}}^p$. It is a free A-module of dimension $(1,0)$ for q even, $(0,1)$ for q odd. Cohomological degree is p.

Basic examples: For $\mathrm{Sym}_A(V^*) = A[T]$, $\underline{\mathrm{Ext}}$ can be computed using the following resolution of $A[T]$, pictured as a quasi-isomorphism of complexes

$$\begin{array}{ccccc} A[T] & & A[T] & & \\ \big\downarrow \equiv & : & \big\downarrow & & \\ K^* & & A[T,T^{-1}] & \to & A[T,T^{-1}]/A[T] \ . \end{array}$$

This resolution is not injective, but it is good enough: $\underline{\mathrm{Ext}}^i(A, K^j) = 0$ for $i > 0$, as can be seen by using the projective resolutions $A[T] \xrightarrow{T} A[T]$ of the $A[T]$-module A, and this ensures that for any injective resolution K' and any morphism of resolutions $\alpha\colon K \to K'$, α induces an isomorphism, which is independent of α,

$$H^*\underline{\mathrm{Hom}}_{A[T]}(A, K) \longrightarrow H^*\underline{\mathrm{Hom}}_{A[T]}(A, K')\,.$$

A basis of $\underline{\mathrm{Ext}}^1$ is the image of $T^{-1} \in A[T, T^{-1}]/A[T]$. For $\mathrm{Sym}_A(V^*) = A[\theta]$, no resolution is necessary. A basis of $\underline{\mathrm{Ext}}^0$ is θ.

For $(V_i)_{i \in I}$ a finite family of free modules, by the associativity and commutativity isomorphisms for $\oplus$, one can define the direct sum of the V_i without having first to choose an ordering of I. Similarly, by the associativity and commutativity isomorphisms for $\otimes$, for the tensor product of the $\mathrm{Ber}(V_i)$, viewed as graded super modules (point of view **I**). A Künneth formula gives a canonical isomorphism

$$\mathrm{Ber}(\oplus V_i) \simeq \otimes \mathrm{Ber}(V_i)\,.$$

If V is filtered, a splitting of the filtration gives a direct sum decomposition $V \xrightarrow{\sim} \oplus \mathrm{Gr}^i_F(V)$ and the resulting isomorphism

$$\mathrm{Ber}(V) \xrightarrow{\sim} \otimes\, \mathrm{Ber}(\mathrm{Gr}^i_V(V))$$

does not depend on the choice of the splitting.

The compatibilities to which those constructions obeys involve no sign. Trouble begins when one wants to name an element of $\otimes \mathrm{Ber}(V_i)$: a family of $b_i \in \mathrm{Ber}(V_i)$ define an element $\otimes b_i$ of $\otimes \mathrm{Ber}(V_i)$ only once an ordering of I has been chosen, and for different orderings, $\otimes b_i$ maps to different elements, according ot the sign rule. In other words: trouble begins when one starts writing formuli, as formuli name elements. Written text is linear, and it is the usage that the linear order of a formula gives the linear order of I which is used.

Applying to Ber the functor "associated simply $\bmod 2$ graded" $\mathbf{s}$, one obtains $\mathrm{ber}(V) := \mathbf{s}\mathrm{Ber}(V)$. It is a free A-module of rank $(1, 0)$ or $(0, 1)$, depending on the parity of $p + q$;, and it obeys the same compatibility formalism.

Example: densities. On a supermanifold W of dimension (p, q), the line bundle of densities is the tensor product of the orientation local system with $\mathrm{Ber}(T^\vee)$, for T the tangent bundle. It is natural to give the orientation local system cohomological degree $-p$: the line bundle of densities has dimension $(0, 1)$ or $(0, 1)$, depending on the parity of q, and its cohomological degree zero.

With these conventions, the integration of densities with compact support is an even map.