Appendix: Even rules

P. Deligne

Let A be a commutative super ring and M, N be A-modules. A morphism f: $M \to N$ is a parity respecting map f such that $f(am) = af(m)$.

The "even rules" point of view is that

(a) one should think to M in terms of the functor which to any extension of scalars $A \to A'$ attach the even part $(M')^+$ of $M' := A' \otimes_A M$;

(b) it amounts to the same to give a morphism of modules f: $M \to N$ or to give for each commutative super A-algebra A' a $(A')^+$-linear morphism $F[A']$ from $(M')^+$ to $(N')^+$, functorial in A'.

More precisely, the map $f \mapsto F$:

$$F[A'] = f'|M'^+ ,$$

with f' deduced from f by extension of scalars, is bijective.

Proof. To a functorial $F[A']$: $(M')^+ \to (N')^+$ one attaches a morphism f: $M \to N$ as follows. On M^+, $f := F[A]$. Take now $A' = A[\theta]$ with θ odd. An element n of N'^+ can be written uniquely in the form $n_0 + \theta n_1$ with $n_0 \in N^+$ and $n_1 \in N^-$. Define φ_0 and φ_1 on M^- by

$$F[A'](\theta m) = \varphi_0(m) + \theta_1(m),$$

and define f on M^- to be φ_1.

Apply functoriality to $A' \to A$: $\theta \mapsto 0$. The corresponding map from $(M')^+$ to M^+ maps θm $(m \in M^-)$ to zero, and it follows that $\varphi_0 = 0$: for $A' = A[\theta]$, we have

$$F(A')(m_0 + \theta m_1) = f(m_0) + \theta f(m_1)$$

for m_0 in M^+ and m_1 in M^-.

We now check that f is A-linear. The A^+-linearity is clear. If a is inA^- and m in M^+,

$$\theta f(am) = F(A')(\theta am) = \theta a F(A')(m) = \theta a f(m) \ .$$

For m in M^-, applying functoriality to $A' \to A$: $\theta \to a$, we get $f(am) = af(m)$ as well.

If F is obtained from a morphism f, one has for $A' = A[\theta]$

$$f'(m_0 + \theta m_1) = f(m_0) + \theta f(m_1)$$

and it follows that this construction gives back f.

Conversely, the f deduced from F gives back F by extensions of scalars: for $a' \in A'$ and $m \in M$ even,

$$F(a'm) = a'F(m) = a'f(m)$$

by $A \to A'$ functoriality. For a' and m odd, functoriality for $A[\theta] \to A'$: $\theta \mapsto a'$ gives

$$F(a'm) = a'f(m)$$

as well.

A similar, more complicated argument, gives the

Theorem. *Let M_i be a finite family of A-modules. The data of a morphism*

$$f \colon \otimes M_i \to N$$

amounts to the data of a A'^+-multilinear map, functorial in A', of the $(M_i')^+$ to $(N')^+$.

The proof is left to the reader.

Corollary. *Let B be an A-module. To give on B a structure of A algebra amounts to give on $(B')^+$ a structure of $(A')^+$ algebra, functorially in A'. The algebra B will be associative (resp. commutative, resp. Lie, ...) if and only if the algebras $(B')^+$ are.*

Note that the $(B')^+$ are purely even objects: at the cost of having to think functorially, which one wants to do anyway, this removes signs from the definitions.

Example. Let X be a supermanifold and L be an (even) function on X. The locus where L is stationary is the subspace S of X (a sub-supermanifold in good cases) defined by the vanishing of the section dL of the cotangent bundle of X.

When thinking functorially, one thinks to X as determined by the functor which to any supermanifold B attaches the set B-points of X, where a B-point is a morphism $x\colon B \to X$. Intuitively: a family of points of X parametrized by B. In this language, describing S means describing which B-points $x\colon B \to X$ map to S, i.e. factors through $S \hookrightarrow X$. Such B-points will be called stationary.

By definition, a B-point x is stationary if for any even or odd section s on B of x^*T_X (the "fiber of T_X at x"), $\langle dL, s\rangle = 0$ on B. If s is odd, let $B' = B \times \mathbb{R}^{0,1}$: we add an odd parameter θ. Then, θs is even and $\langle dL, s\rangle = 0$ on B if and only if $\langle dL, \theta s\rangle = 0$ on B'. We see that a B-point x is stationary if and only if after any change of basis $B' \to B$, the B'-point $B' \to B \to X$, still noted x, is such that for any *even* section s of x^*T_M, $\langle dL, s\rangle = 0$ on B'.

This can be rephrased as follows: after any change of basis $B' \to B$, one wants that for any family of points $x(t)$ $(-\varepsilon < t < \varepsilon)$: $B' \times]-\varepsilon, \varepsilon[\to X$, one has

$$\tfrac{d}{dt} L(x) = 0 \quad \text{at} \quad t = 0$$

or, more cumbersomely written: the pull back of L to $B' \times]-\varepsilon, \varepsilon[$ is such that $\frac{\partial L}{\partial t} = 0$ at $t = 0$.

This formulation is convenient when X is "infinite dimensional", and when what makes good sense is not the space X and the function L, but the notion of map $x\colon B \to X$ and of the pull back by x of L.

Minority report: Odd rules.

The notion of super Lie algebra can be expressed nicely as follows:
Data: after any extension of scalars $A \to A'$, a quadratic map x^2 from L^- to L^+, functorial in A'.

Applying the theorem to the associated bilinear form, from $(\Pi L)^+ \otimes (\Pi L)^+$ to L^+, we see that $x \mapsto x^2$ defines a bracket $[x, y]\colon L \otimes L \to L$ such that for x and y in L^-, $[x, y] = (x + y)^2 - x^2 - y^2$. In particular, for x in L^-, $[x, x] = 2x^2$.

Axiom: after any extension of scalars $A \to A'$, for any x in L^-, one has

$$[x, x^2] = 0 \ .$$

Provided 6 is invertible in A, this definition is equivalent to the one in term of the Jacobi identity (which is a polarized version of the axiom).