Appendix: the 2-form on a space of extremals

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In his lectures, Bernstein considered lagrangian densities $\mathcal{L}(\phi)$ on V, for ϕ in the space $\mathcal{M}$ of maps $V \to M$, or more generally for ϕ a section of a fiber bundle M_V over V. The case of maps to M correspond to $M_V = M \times V$. He considered the case where the density $\mathcal{L}(\phi)$ at $v \in V$ depends only on ϕ and $d\phi$ at v, and explained how to construct a closed 2-form ω on the space $\mathcal{E}$ of extremals. The construction depended on a cohomology class of hypersurfaces Γ with an oriented normal bundle: Γ has a "past" side and a "future" side. In practice, Γ is a "space-like" hypersurface. The construction uses integration on Γ, hence requires some control at the infinity of Γ, and a similar control on homologies among Γ's. We will ignore this by pretending that Γ is compact and that cohomology is taken with compact support.

In this appendix, our aim is to explain a small modification of the construction which works as well when $\mathcal{L}(\phi)$ depends on higher derivatives of ϕ. One can also take ϕ to be for instance a bundle with connection, rather than a map to some M. If V is a supermanifold, the new construction makes clear that the 2-form obtained is the same whether one works in term of superfields ϕ: $V \to M$ and of superdensities $\mathcal{L}(\phi)$ on V, or in term of "components" of ϕ and of the Lagrangian density $\overline{\mathcal{L}}(\phi)$ on V_{red} deduced from $\mathcal{L}(\phi)$ by "integrating out" the odd variables, for some projection $V \to V_{\mathrm{red}}$.

What we don't get are 2-forms ω_Γ, depending on Γ, on the space $\mathcal{M}$ of all ϕ, and inducing ω on the space $\mathcal{E}$ of extremals.

The basic construction is local around a space-like hypersurface Γ and, to explain it, we will assume that $V = \Gamma \times \mathbb{R}$, the projection to $\mathbb{R}$ being called "time".

For a 2-parameter family $\phi[a_1, a_2]$ of extremals, we now define the value at $(a_1, a_2) = (0, 0)$ of the pull back of ω by $(a_1, a_2) \mapsto \phi[a_1, a_2]$. We choose a family $\phi_1[a_, a_2]$ of maps to

M, deforming $\phi[a_1, 0]$, agreeing with $\phi[a_1, a_2]$ in the future, and agreeing with $\phi[a_1, 0]$ in the past:

$$
\begin{aligned}
\phi_1[a_1, 0] &= \phi[a_1, 0], & \\
\phi_1[a_1, a_2] &= \phi[a_1, a_2] & \text{in the (far) future,} \\
\phi_1[a_1, a_2] &= \phi[a_1, 0] & \text{in the (far) past.}
\end{aligned}
$$

We similarly choose ϕ_2, having the same properties with the roles a_1 and a_2 permuted.
Definition:

$$\omega(\partial_1, \partial_2) = \int_V (\partial_1 \partial_2 \mathcal{L}(\phi_1) - \partial_2 \partial_1 \mathcal{L}(\phi_2)) \quad \text{at} \quad (a_1, a_2) = (0, 0).$$

The integrand vanishes in the future, where $\phi_1 = \phi_2$, as well as in the past, where both terms vanish. This makes the integral defined.

Using Γ, with equation $t = 0$, we now express ω as an exterior derivative $d\alpha_\Gamma$ and show independence of the choices of ϕ_1 and ϕ_2. Let Y be the function on $\mathbb{R}$ which is 1 for $x < 0$ and 0 for $x > 0$: $\frac{d}{dx} Y = -\delta(x)$. We will have to integrate on V up to $t = 0$ some smooth density μ vanishing in the past. In the supervariety case, this means integrating $Y(t)\mu$ (integration on a supervariety with boundary Γ). This depends on Γ, not just on Γ_{red}.

The 1-form α_Γ is defined only at extremals, but makes sense on tangent vectors to the space $\mathcal{M}$ of all ϕ. Fix an extremal $\phi[0]$ and consider a family $\phi[a]$ deforming it. We now define α_Γ at the corresponding tangent vector. Let ϕ_1 be a deformation of $\phi[0]$ agreeing with $\phi[a]$ for $t > -\varepsilon$, and with $\phi[0]$ in the past.
Definition:

$$\alpha_\Gamma(\partial_a) = \int Y(t) \partial_a \mathcal{L}(\phi_1[a]) \quad \text{at} \quad a = 0$$

The integrand vanishes in the future, because $Y(t)$ does, and in the past, where $\phi_1[a]$ is constant. This makes the integral defined. Because $\phi[0]$ is an extremal, it does not depend on the choice of ϕ_1: the density $\partial_a \mathcal{L}(\phi_1[a])$ is the variation of the density $\mathcal{L}$ for a variation $\delta_1 \phi$ of $\phi[0]$, and if we consider another deformation ϕ_2, $\delta_1 \phi - \delta_2 \phi$ has support in a time interval $[-A, -\varepsilon]$. It is the tangent vector for a deformation ϕ_3 with a support in $t < 0$ and

$$\int Y(t)[\partial_a \mathcal{L}(\phi_1[a]) - \partial_a \mathcal{L}(\phi_2[a])] \quad (\text{at } a = 0)$$
$$= \int Y(t)\partial_a \mathcal{L}(\phi_3[a]) \quad (\text{at } a = 0) = \int \partial_a \mathcal{L}(\phi_3[a]) \quad (\text{at } a = 0) = 0$$

To see that $\omega = d\alpha_\Gamma$, one uses the formula that for commuting vector fields X and Y, one has

$$(d\alpha)(X, Y) = X.\alpha(Y) - Y.\alpha(X) \ .$$

For the pull back of α_Γ by $(a_1, a_2) \mapsto \phi[a_1, a_2]$, this gives, provided that $\phi_1[a_1, a_2] = \phi[a_1 a_2] = \phi[a_1 a_2]$ for $t > -\varepsilon$:

$$d\alpha_\Gamma(\partial_1, \partial_2) = \partial_1 \int Y(t)\partial_2 \mathcal{L}(\phi_1) - \partial_2 \int Y(t)\partial_1 \mathcal{L}(\phi_2)$$
$$= \int Y(t)(\partial_1 \partial_2 \mathcal{L}(\phi_q) - \partial_2 \partial_1 \mathcal{L}(\phi_2))$$
$$= \int \partial_1 \partial_2 \mathcal{L}(\phi_1) - \partial_2 \partial_1 \mathcal{L}(\phi_2)$$

at $(a_1, a_2) = (0, 0)$: the definition of ω.

In the case treated by Bernstein, integration by parts shows that the form α_Γ is the restriction to $\mathcal{E}$ of the one he considers, on $\mathcal{M}$.

The modification we presented does not require the tangent space of $\mathcal{M}$ at ϕ to be the space of sections of a vector bundle on V. Basically, it requires only that it be the space of sections of a soft sheave of $\mathbb{R}$-vector spaces on V. One has to be able to deform ϕ in a prescribed way in different regions of V. For V a supermanifold, this does not preclude imposing suitable constraints on ϕ.