Appendix:
Signs: de Rham and integral forms complexes

P. Deligne

1. Some standard conventions for ordinary manifolds don't sit well with the principle of treating p-forms as objects of parity p. Here are troublesome conventions:

(a) for a p-form $\alpha_1 \wedge \ldots \wedge \alpha_p$, and vector fields $X_1, \ldots, X_p$,

$$\alpha_1 \wedge \ldots \wedge \alpha_p(X_1, \ldots, X_p) = \det(\alpha_i(X_j)),$$

with typical term $\alpha_i(X_1) \ldots \alpha_p(X_p)$, despite the fact that X_i passed over α_j for $i < j$.

(b) for f a function and X a vector field,

$$Xf = df(X) = i_X df = X \perp df.$$

In the middle equality, df and X are permuted without sign consequence.

(c) In $\mathbb{R}^n$, the density corresponding to the n-form $dx_1 \wedge \ldots \wedge dx_n$, when $\mathbb{R}^n$ is oriented by $(e_1, \ldots, e_n)$, is positive.

The bigraded points of view, by keeping separate the "cohomological" grading, used above, and the parity, allows to keep those formuli while treating parity consistently.

2. Let M be a supermanifold. The vector fields on M are the derivations of the structural sheaf $\mathcal{O}$:

$$D(fg) = Df.g + (-1)^{p(D)p(f)} fDg$$

The sheaf of 1-forms is the dual, the duality being defined by

$$T \otimes \Omega^1 \to \mathcal{O}\colon D \otimes \alpha \longmapsto i_D \alpha.$$

One defines

$$\alpha(D) = (-1)^{p(D)p(\alpha)} i_D \alpha.$$

To the extent possible, it is best to consider only vector fields and forms of even parity, considering instead of an odd D (resp. α) the product εD (resp. $\varepsilon\alpha$) for ε an auxiliary odd constant. If we work over a basis B, as we should, odd constant means odd function on B.

The bilinearity of $i_D\alpha$ and of $\alpha(D)$ is as dictated by the sign rule: with $(uD)(f) = u(Df)$, we have

$$i_{fD}\alpha = f i_D\alpha \ , \qquad i_D f\alpha = (-1)^{p(D)p(f)} f i_D \alpha$$
$$(f\alpha)(D) = f(\alpha(D)) \ , \qquad \alpha(fD) = (-1)^{p(\alpha)p(f)} f\alpha(D).$$

The de Rham complex Ω^* is $\wedge\Omega^1$, with $\Omega^p = \overset{p}{\wedge}\,\Omega^1$ in cohomological degree p, and with a differential d characterized by the following identities

(1) d is an even derivation of cohomological degree one;

(2) from $\mathcal{O}$ to Ω^1, d is defined by

$$i_D df = Df;$$

(3) $d^2 = 0$.

One may prefer to say that Ω^* is the symmetric algebra in the bigraded sense on Ω^1, put in cohomological degree 1. The defining formula (2) can be rewritten $df(D) = (-1)^{p(f)p(D)} Df$.

3. An even derivation D generate (at least locally) a 1-parameter group of automorphisms $\exp(tD)$, with

$$Df = \partial_t(\exp(tD)^* f) \quad \text{at} \quad t = 0. \tag{3.1}$$

By transport of structures, this group acts on all kinds of tensor fields, and the Lie derivative $\mathcal{L}_D$ is defined by the right side of (3.1). If D is odd, one defines $\mathcal{L}_D$ by

$$\varepsilon\mathcal{L}_D = \mathcal{L}_{\varepsilon D} \tag{3.2}$$

for ε an odd constant.

One should beware that $f \mapsto \exp(tD)^* f$ is the inverse of the action of $\exp(tD)$ on functions by transport of structures. Because of this,if derivations are viewed as forming the Lie algebra of a group of diffeomorphisms, with $D \mapsto$ infinitesimal diffeomorphism $\exp(\varepsilon D)$, i.e. D generating the one-parameter subgroup $\exp(tD)$, the bracket in this Lie algebra is the opposite of the bracket of vector fields.

Instead of considering $\exp(tD)$ for $t \in \mathbb{R}$, one can work $\mod t^2$: $\exp(tD)$ is then the family of automorphisms parametrized by the scheme with coordinate ring $\mathbb{R}[t]/(t^2)$ defined by

$$\exp(tD)^* f := f + tDf \qquad (\text{for } t^2 = 0) \tag{3.3}$$

and the Lie derivative of any tensor g is given by

$$\exp(tD)^* g = g + t\mathfrak{L}_D g \qquad (\text{for } t^2 = 0) \tag{3.4}$$

For D an odd derivation, the same formuli hold when t is taken to be odd.

A vector field D defines a derivation i_D of Ω^*, of cohomological degree -1 and parity that of D, characterized as extending the already defined i_D on Ω^1. Cartan's formula for the Lie derivative acting on Ω^* is

$$\mathfrak{L}_D = i_D d + d i_D. \tag{3.5}$$

As d is of parity 0 and i_D and d are of odd cohomological degree, the second member is the bracket $[i_D, d]$ in the bigraded sense.

4. Let V be a vector space of dimension $(0,1)$, with generator D. The free Lie aglebra generated by V, i.e. by D, has $D, D^2 = \frac{1}{2}[D,D]$ as basis. The corresponding Lie group has as points over a base B the $\exp(tD^2 + \theta D)$, t and θ being, respectively, even and odd functions on B, with the group law given by the Campbell-Hausdorff formula:

$$(t_1, \theta_1) * (t_2, \theta_2) = (t_1 + t_2 - \theta_1\theta_2, \theta_1 + \theta_2). \tag{4.1}$$

The minus sign comes from $\frac{1}{2}[\theta_1 D, \theta_2 D] = -\frac{1}{2}\theta_1\theta_2[DD] = -\theta_1\theta_2 D^2$.

An odd vector field on a supermanifold M generates an action of that group. More precisely, if D^2 is the vector field defined by $DDf = D^2 f$, i.e. $D^2 = \frac{1}{2}[D, D]$, the automorphisms $\exp(tD^2 + \theta D)$ (at least locally defined) form a group, with

$$\exp(t_1 D^2 + \theta_1 D)\exp(t_2 D^2 + \theta_2 D) = \exp((t_1 + t_2 + \theta_1\theta_2)D^2 + (\theta_1 + \theta_2)D). \tag{4.2}$$

The reason for the change of sign from (4.1) is explained in 3.

5. A coordinate system $x^1, \ldots, x^n$ (first p even, last $n-p$ odd) defines derivations ∂_i, with $\partial_i x^j = \delta_i^j$. Our conventions so far force the unusual looking

$$df = \sum dx^i \partial_i f. \tag{5.1}$$

6. The bundle of densities is defined by the property that its sections are in duality with the space of functions with compact support. The duality pairing is written $\int \mu f$. Accordingly, the $\mathcal{O}$-module structure is given by

$$\int a\mu.f = (-1)^{p(a)p(\mu)} \int \mu.af \tag{6.1}$$

According to the sign rule,

$$\int f\mu := (-1)^{p(f)p(\mu)} \int \mu f, \tag{6.2}$$

and one also has $\int f(a\mu) = \int (fa)\mu$.

By “density”, we mean here C^∞-density: if, in a local coordinate system $(x_1, \ldots, x_p, \theta_1, \ldots, \theta_q)$, f is expanded as $\sum \theta^I f_I$, we consider only the linear forms $f \mapsto \int \mu f$ which are linear combinations of the $\int f_I(n_1, \ldots, n_p)\varphi(x_1, \ldots, x_p)dx_1 \ldots dx_p$ for C^∞ functions φ.

A local coordinate system $x_1, \ldots, x_p, \theta_1, \ldots, \theta_q$ defines a density $[x_1, \ldots, x_p, \theta_1, \ldots, \theta_q]$ on its domain of definition: for $f_{[1,q]}$ the (post) coefficient of $\theta_1 \ldots \theta_q$, as above, the integral

$$\int f_{[1,q]}(x_1, \ldots, x_p)dx_1 \ldots dx_p. \tag{6.3}$$

It is the integral, for the density $dx_1 \ldots dx_p$, of the restriction to $\theta_1 = \ldots = \theta_q = 0$ of $\partial_{\theta_q} \ldots \partial_{\theta_1} f$. It is of parity $(-1)^q$ (and cohomological degree 0). It is a basis of densities, which is hence of dimension $(1,0)$ for q even and $(0,1)$ for q odd.

Warning. Densities μ_1 on M_1 and μ_2 on M_2 give a density $\mu_1 \boxtimes \mu_2$ on $M_1 \times M_2$ by

$$\int_{M_1\times M_2} \mu_1 \boxtimes \mu_2 f_1 \boxtimes f_2 = (-1)^{p(f_1)p(\mu_2)} \int \mu_1 f_1 \int \mu_2 f_2.$$

One has, in local coordinates,

$$[x_1,\dots,x_{p(1)},\theta_1,\dots,\theta_{q(1)}] \boxtimes [y_1,\dots,y_{p(2)},\eta_1,\dots,\eta_{q(2)}] =$$
$$(-1)^{q(2)q(1)}[x_1,\dots,x_{p(1)},y(1),\dots,y_{p(2)},\theta_1,\dots,\theta_{q(1)},\eta_1,\dots,\eta_{q(2)}].$$

The densities given by two coordinate systems are related by the Berezinian of a jacobian matrix, and a sign: $+$ if the orientations given by $x_1,\dots,x_p$ and $y_1,\dots,y_p$ on the reduced space agree, $-$ otherwise. This is best expressed by telling that the bundle of densities is canonically isomorphic to the tensor product Ber$\otimes$or of the Berezinian of Ω^1 (cohomological degree p, parity q) by the orientation line bundle (cohomological degree $-p$).

7. The generalized functions (distributions) on a supermanifold can be defined as the dual of densities with compact support, or as a suitable completion of the space of functions. It makes sense to take the pull back $f(g)$ of a generalized function f by a submersion g.

Example. Let Y be the function on $\mathbb{R}$ with value 1 for $x<0$ and 0 for $x>0$. On $\mathbb{R}^{1,2}$, with coordinates (t,θ_1,θ_2), we have

$$Y(t+\theta_1\theta_2) = Y(t) + \left(\tfrac{d}{dt}Y\right)(t).\theta_1\theta_2 = Y(t) - \delta(t).\theta_1\theta_2. \tag{7.1}$$

The integration up to 0 of a density μ with support bounded below is

$$\int_{-\infty}^{0} \mu := \int Y(t)\mu.$$

The formula (7.1) shows that it is not invariant by the automorphism α: $(t,\theta_1,\theta_2) \mapsto (t-\theta_1\theta_2,\theta_1,\theta_2)$ of $\mathbb{R}^{1,1}$, despite α being the identity on the reduced space: integration on an open set requires the choice of a codimension $(1,0)$ boundary submanifold.

8. Integration of a density μ with compact support is defined, hence invariant by diffeomorphisms. It follows that for any vector field v, one has

$$\int \mathfrak{L}_v(\mu) = 0 \tag{8.1}$$

Applying (8.1) to a product μf, we get

$$\int \mathcal{L}_v(\mu).f = -(-1)^{p(v)p(\mu)} \int \mu.vf \ , \tag{8.2}$$

from which results that for any g

$$\mathcal{L}_{vg}(\mu) = \mathcal{L}_v(g\mu) \tag{8.3}$$

(with $vg = (-1)^{p(v)p(g)} gv$). The map $v \otimes \mu \mapsto \mathcal{L}_v \mu$ hence induces a morphism

$$d\colon \text{ (tangent bundle)} \otimes \text{densities} \to \text{densities} \tag{8.4}$$

with

$$\int d\alpha = 0 \tag{8.5}$$

if α has compact support.

9. For an ordinary manifold of dimension n, one could construct Ω^* not from the bottom up, defining p-forms as wedge of 1-forms, but from the top down, defining a p-form as obtained by contracting a top form with $(n-p)$ tangent vectors. This corresponds to two descriptions of Ω^* as a Clifford module over the sum of the tangent and cotangent bundle: the usual description is being generated by $\mathcal{O}$, which is killed by the i_X, the other is being generated by Ω^n, which is killed by the $\alpha\wedge$. In the second description, d can be defined (as was done in 8.) by its relation (3.4) with the Lie derivative. Instead of Ω^n, one could use its twist by the orientation bundle, i.e. the bundle of densities. In the case of super varieties, one can similarly define an *integral forms* complex I^n $(n \leq 0)$ starting from the line bundle of densities. It differs (more than by a twist) from the de Rham complex, but one can arrange the sign conventions so that it obeys the same formalism. One disposes of contractions i_v, exterior products $\alpha\wedge$ for α a 1-form and exterior differential d, of cohomological degree $-1, 1$ and 1, as usual. Iterated contractions are linear and induce an isomorphism

$$\overset{p}{\wedge} T \otimes I^0 \to I^{-p}.$$

Exterior products are linear,

$$[i_v, \alpha\wedge] = i_v(\alpha)$$

and iterated products give a module structure over Ω^*. One has

$$[d, \alpha \wedge] = (d\alpha) \wedge \qquad \text{and}$$

$$[d, i_v] = \mathcal{L}_v$$

for v a vector field.

10. The basic property of integral $(-p)$-forms is that they can be integrated on $(p, 0)$-codimensional cooriented subvarieties. If the orientation of the normal bundle to W is defined by the basis dual to $dt_1, \ldots, dt_p$ for $t_1, \ldots, t_p$ a system of equations for W, our convention is that for any function with compact support on V, one has

$$\int_W \alpha | W.f = \int_V \delta(t_1) dt_1 \wedge \ldots \wedge \delta(t_n) dt_n \wedge \alpha.f \tag{10.1}$$

The rule (10.1) has the virtue that if t: $V \to \mathbb{R}$ is submersive above 0 and 1, one has for a codimension 1-form with compact support α

$$\int_{0 \,\leq\, t \leq 1} d\alpha = \int_{t=1} \alpha - \int_{t=0} \alpha \tag{10.2}$$

if χ is the characteristic function of the interval $[0, 1]$ of $\mathbb{R}$, the left side of (10.2) is by definition $\int \chi(t) d\alpha$ (cf. 7.) and an integration by part using (8.5), shows it equals $\int -d(\chi(t))\alpha = \int (\delta(t-1)dt)\alpha - \int \delta(t) dt \alpha$, as claimed by (10.2).