Lectures on supersymmetry

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Lecture 1. Why sypersymmetry

0. The aim of these lectures is to describe what physicists have in mind when they talk about supersymmetry (SUSY).

I should say that I am a mathematician, not a physicist, and these lectures are for mathematicians.

First I thought that these will be more or less straightforward course of lectures on supermanifolds. But then I (and David Kazhdan) realized that the situation is more interesting.

Physicists have constructed many models with supersymmetry; for some of them they found nice models formulated in terms of supermanifolds (super space in physical terminology). But for many interesting models with supersymmetry their "supermanifold" realization is unknown — and this may be an interesting problem for mathematicians.

1. What are sypersymmetries.

First, what are symmetries?

Physicists have three types of pictures for their theories.

(1) Classical picture – fields on a space-time + Lagrangian;

(2) Quantum field theory (QFT)

(3) Scattering theory (theory of S-matrix).

Usually there are many symmetries of the classical picture, but many of them give trivial symmetries of the scattering theory (of the S-matrix). It turns out that if we want to describe the symmetries of the S-matrix then the answer is very restrictive. In my lectures I will mostly deal only with classical picture. But in the first lecture I will describe an important result from the scattering theory which explains why physicists are so interested in supersymmetry.

2. The group P and its representations.

2.1.

Let V be a 4-dimensional real vector space ($V \simeq \mathbb{R}^4$) with a quadratic form of signature $\begin{pmatrix} + & & & \\ & - & & \\ & & - & \\ & & & - \end{pmatrix}$.

Let G denote the universal cover of the group $\mathrm{SO}(V)$: we have

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{Z}_2 & \to & G & \to & \mathrm{SO}(V) & \to & 1 \\ & & & & \wr\wr & & & & \\ & & & & \mathrm{Spin}(V) & & & & \end{array}$$

As a real group, G is isomoprhic to $\mathrm{SL}(2,\mathbb{C})$. We will list now all finite-dimensional irreducible complex representations of G.

Let L denote a standard 2-dimensional complex representation of G (coming from the isomorphism $G \simeq \mathrm{SL}(2,\mathbb{C})$). The action of G on L preserves a volume form ω and we have $L \simeq L^*$, as G-representations (* denotes the linear dual).

Claim. Any irreducible representation of G has a form:

$M_i \otimes \overline{M}_j$, where M_k denotes the $k+1$-dimensional representation $\mathrm{Sym}^k(L)$ and where $\overline{}$ is the complex conjugation. ■

Example. The space V as a real representation of G identifies with the space of Hermitian forms on L.

We have

$$\tau\colon V_{\mathbb{C}} \simeq M_1 \otimes \overline{M}_1 = L \otimes \overline{L}$$

2.2. Let P denote the Poincaré group, i.e. the semidirect product

$$P = G \ltimes V.$$

We will be interested in unitary representations of P. Note that the action of the central element $z \in P$ $\left(z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}) \simeq G\right)$ defines a $\mathbb{Z}_2$-grading on the space of each such representation:

$$H = H_+ \oplus H_-$$

Let H be an irreducible unitary representation of P. Then $H\big|_V$ has a form $\int\limits_{p \in V^*} H_p d\mu$, where $d\mu$ is a G-invariant measure on V^* concentrated on one G-orbit $\mathcal{O} \subseteq V^*$ (in our case each orbit admits a G-invariant measure). The structure of a P-representation on $\int\limits_{p \in V^*} H_p d\mu$ is determined by the action of $\mathrm{Stab}(p_0) \subseteq G$ (p_0 is an arbitrary point of $\mathcal{O}$) on H_{p_0}, which must be an irreducible unitary representaion of this group.

We say that an irreducible P-representation H is positive if the corresponding G-orbit $\mathcal{O} \subseteq V^*$ is one of the following three types:

a) $\mathcal{O}_m^+$, $m \in \mathbb{R}^{+*}$: $\{p \in V, < p, p >= m^2, \ p_0 > 0\}$;

b) $\mathcal{O}_0^+$: $\{p \in V, < p, p >= 0, \ p_0 > 0\}$;

c) $\{0\}$.

Here we have used the identification $V \simeq V^*$ and an orientation of the time-coordinate x_0 in $V \simeq \mathbb{R}^4$.

We say that a unitary representation H of P is "good" if it satisfies the following three properties:

1) It is a sum of finitely many irreducibles $H = \bigoplus\limits_i H$;

2) Each of the H_i's is positive in the above sense;

3) For each of the H_i's the representations of stabilizers H_{p_i} are finite dimensional.

Example. Let H be an irreducible P-representation satisfying 2) and 3).

a) If $\mathcal{O} = \mathcal{O}_m^+$, choose $p = (m, 0, 0, 0)$; $\mathrm{Stab}_p = \mathrm{SU}(2)$. Irreducible unitary representations of $\mathrm{SU}(2)$ are enumerated by positive half-integers and the corresponding P-representations will be denoted as $H_{m,s}$ $s \in \frac{1}{2}\mathbb{Z}^+$.

b) If $\mathcal{O} = \mathcal{O}_0^+$, choose $p = (1, 1, 0, 0)$; $\mathrm{Stab}_p = U(1) \ltimes N$, where N is a nilpotent group. Irreducible Stab_p-representations are enumerated by half-integers and the corresponding P-representations will be denoted as $H_{0,s}$ $s \in \frac{1}{2}\mathbb{Z}$.

c) If $\mathcal{O} = \{0\}$, $\mathrm{Stab}_0 = G$ and H_p must be trivial. Therefore, H is trivial too.

3. Scattering theory and Coleman-Mandula theorem.

3.1. The scattering picture consists of the following data:

1) H a "good" representation of P;
2) A unitary operator (called S-matrix) $S: \mathcal{H} \to \mathcal{H}$, where $\mathcal{H} := \mathrm{Sym}(H)$ such that S commutes with the action of P.

Here $\mathrm{Sym}(H)$ is understood in the super-sense, i.e.

$$\mathrm{Sym}(H) \simeq \mathrm{Sym}(H_+) \otimes \Lambda(H_-).$$

Definition. An even symmetry of the S-matrix is an (unbounded) operator $B: \mathcal{H} \to \mathcal{H}$ such that

a) B is even and preserves the $\mathbb{Z}$-grading on $\mathcal{H}$;
b) B satisfies the Leibnitz rule;
c) $[B, S] = 0$.

Example. Consider $\mathfrak{p} = \mathrm{Lie}(P)$. Then $\mathfrak{p}$ acts n $\mathcal{H}$ by symmetry operators.

A theorem of Coleman and Mandula classifies all possible symmetries of the S-matrix in a "non-degenerate" situation.

3.2. Assume that the pair (H, S) satisfies the following additional conditions:

a) H has irreducible summands with strictly positive masses only, i.e. only the orbits of the type $\mathcal{O}_m^+$ are allowed.
b) S is non-degenerate.

The last condition is rather obscure; let us, nevertheless, give a definition.

Let $\Sigma = \bigcup_i \mathcal{O}_m^+$ be the support of H in V^* and consider the operator $S^2: \mathrm{Sym}^2 H \to \mathrm{Sym}^2 H$

$$S^2: \mathrm{Sym}^2 H \overset{i}{\hookrightarrow} \mathcal{H} \xrightarrow{S} \mathcal{H} \xrightarrow{pr} \mathrm{Sym}^2 H.$$

$\mathrm{Sym}^2 H$ identifies with the space of L_2-sections of a vector bundle with fibers $H_p \otimes H_q$ over the manifold $\mathrm{Sym}^2(\Sigma) \setminus \Delta$ (Δ is the diagonal).

The operator S^2 is given by a kernel

$$T^{p,q}_{p',q'}\ (p \neq q;\ p' \neq q'):\ H_p \otimes H_q \to H_{p'} \otimes H_{q'}.$$

From the $V \subseteq P$-invariance of S it is easy to deduce that $T^{p,q}_{p',q'}$ is concentrated on the submanifold of $(\mathrm{Sym}^2(\Sigma) \setminus \Delta) \times (\mathrm{Sym}^2(\Sigma) \setminus \Delta)$ given by $p + q - p' - q' = 0$, i.e.

$$T^{p,q}_{p',q'} = \delta(p + q - p' - q')\widetilde{T}^{p,q}_{p',q'}.$$

The non-degeneracy condition says that $\widetilde{T}^{p,q}_{p',q'}$ is a boundary value of an analytic function and that the matrix $\widetilde{T}^{p,q}_{p',q'}$ is non-degenerate if p is close to p' and q is close to q'.

Theorem (Coleman-Mandula). *Let the pair (H, S) satisfy (a) and (b) as above. Then*

a) *Any symmetry B of S has a form $B = B_1 + B_2$, where $B_1 \in \mathfrak{p}$, $[B_2, \mathfrak{p}] = 0$.*

b) *The Lie-algebra of symmetries of S has a form $\mathcal{S} = \mathfrak{p} \oplus I$, where I is a finite-dimensional, reductive Lie algebra.* ■

3.3.

Let us now describe the extension of the Coleman-Mandula result to the case of non-necessarily even symmetries.

Definition. An odd symmetry of S is an operator B: $\mathcal{H} \to \mathcal{H}$ such that

a) B is odd and preserves the $\mathbb{Z}$-grading on $\mathcal{H}$.

b) B satisfies the super-Leibnitz rule:
$B(X \otimes Y) = BX \otimes Y + (-1)^{p(X)} X \otimes BY$.

c) $[B, S] = 0$.

Consider the Lie super-algebra $\mathcal{S} = \mathcal{S}_{\bar{0}} \oplus \mathcal{S}_{\bar{1}}$ of symmetries of S. We have the following result:

Theorem (Haag, Lopuszansky, Sohnius). *Let (H, S) satisfy a) and b) of* 3.2. *Then $\mathcal{S}$ has the following structure:*

1) *$\mathcal{S}_{\bar{0}} = \mathfrak{p} \oplus I$ (as in Coleman-Mandula theorem).*

2) *$\mathcal{S}_{\bar{1}}$ is finite dimensional, $[\mathcal{S}_{\bar{1}}, V] = 0$ and as a representation of G it has a form $\bigoplus_i (L \oplus \overline{L})$.*

3) *If* $Q \in L_i$, $Q' \in L_j$, *we have*

$$[Q, \overline{Q}'] = \delta_{ij} \cdot \tau(Q, \overline{Q}) \in V_{\mathbb{C}} \qquad (\tau \text{ of } 2.1)$$

Here $Q \to \overline{Q}$ *denotes the operation of passing to the adjoint operator acting on* H.

4) $[Q, Q'] = \delta_{ij} \cdot \iota$, *where* ι *belongs to* I *and is central.* ∎