Lectures on supersymmetry

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Lecture 2

1. Super liner algebra.

1.1. A super vector space (over a field) is a $\mathbb{Z}_2$-graded vector space, standard notation: $V = V_{\bar{0}} \oplus V_{\bar{1}}$. We let p denote the function of parity defined on homogeneous elements. Super vector spaces form an abelian category endowed with a functor of change of parity that we denote as Π. There is also a tensor structure on the category of super vector spaces:

$$(*) \qquad (V_{\bar{0}} \oplus V_{\bar{1}}) \otimes (W_{\bar{0}} \oplus W_{\bar{1}}) := \underbrace{V_{\bar{0}} \otimes W_{\bar{0}} \oplus V_{\bar{1}} \otimes W_{\bar{1}}}_{\substack{\uparrow \\ \text{even component}}} \oplus \underbrace{V_{\bar{0}} \otimes W_{\bar{1}} \oplus V_{\bar{1}} \otimes W_{\bar{0}}}_{\substack{\uparrow \\ \text{odd component}}}$$

The braiding operation $S : V \otimes W \to W \otimes V$ is defined by the rule:

$$(**) \qquad S_{V \otimes W}(X \otimes Y) = (-1)^{p(X)p(Y)} Y \otimes X.$$

If $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a super vector space, we set $(m, n) := (\dim V_{\bar{0}}, \dim V_{\bar{1}})$ to be its dimension. All types of algebras in the super world are defined as algebras in the tensor category of super vector spaces. We will give below a more down-to-Earth definition of basic algebraic structures.

1.2.

Associative super algebra is a $\mathbb{Z}_2$-graded algebra (of course, the product operation must respect the $\mathbb{Z}_2$-grading). Left and right modules over an associative super algebra are defined in an obvious manner, and the corresponding categories will be denoted by $A-$ mod and mod $-A$, respectively. On each of these categories we have the parity change functor Π.

An associative algebra is said to be commutative if $\forall\, a, b\ a \cdot b = (-1)^{p(a)p(b)} b \cdot a$.

Lemma-Construction. *Let A be a commutative super algebra. Then every left A-module M acquires in a functorial way a right A-action. Moreover, the right and the left A-actions on M commute.*

Proof. For $a \in A$, $m \in M$ we let $m \cdot a := (-1)^{p(a)p(m)} am$. This is a right action and we have:

$$(b \cdot m)a = (-1)^{p(a)(p(b)+p(m))}(a \cdot b)(m) = (-1)^{p(a)p(m)} b \cdot (am) = b \cdot (m \cdot a) \qquad \blacksquare$$

Corollary. *If $M, N \in A$-mod, we can define new left modules $\mathcal{H}om_{\text{mod-}A}(M,N)$; $M \underset{A}{\otimes} N$.*

Therefore, super commutative algebras play the same role in the super world as usual commuttive algebras do in the usual mathematics.

Finally, a Lie super algebra is a vector space $L = L_{\bar{0}} \oplus L_{\bar{1}}$ with a bracket operation $[\ \ ,\ \ ]:\ L \otimes L \to L$ with:

$$[X, Y] = (-1)^{p(X)p(Y)}[Y, X]$$
$$[X,[Y, Z]] + (-1)^{p(Z)(p(X)+p(Y))}[Z, [X, Y]] + (-1)^{p(X)(p(Y)+p(Z))}[Y, [Z, X]] = 0$$

Note that, if $X \in L_{\bar{1}}$, $[X, X]$ does not necessarily vanish. We have a functor:

$$\text{Associative algebras} \longrightarrow \text{Lie algebras}.$$

This functor preserves the underlying vector space and

$$[X, Y] = XY + (-1)^{p(X)p(Y)} YX.$$

In particular, if $X \in A_{\bar{1}}$, $[X, X] = 2X^2$

1.3.

Let V be a super vector space, or more generally a module over a commutative super aglebra A.

We have a natural action of the symmetric group S^n on

$$\underbrace{V \underset{A}{\otimes} V \otimes \cdots \underset{A}{\otimes} V}_{n \text{ times}} \qquad (\text{Remember, } \sigma(X \otimes Y) := (-1)^{p(X)p(Y)} Y \otimes X)$$

$\mathrm{Sym}^n(V)$ is defined to be $(V^{\otimes n})^{S^n}$.

The space $\mathrm{Sym}(V) = \oplus \mathrm{Sym}^n(V)$ has a natural structure of graded commutative superalgebra.

If $V = V_{\bar{0}}$, $\mathrm{Sym}^n(V)$ in the super sense identifies with the usual $\mathrm{Sym}^n(V)$, whereas if $V = V_{\bar{1}}$, $\mathrm{Sym}^n(V)$ in the super sense coincides with $\Lambda^n(V)$.

We define the exterior algebra $\Lambda(V)$ as $\mathrm{Sym}(\Pi(V))$. This again is a commutative superalgebra.

1.4.

Let A be a commutative super algebra and let M and N be two free A-modules, i.e.

$$M = \underbrace{A \oplus A \oplus \cdots \oplus A \oplus A}_{p} \oplus \underbrace{\Pi A \oplus \Pi A \oplus \cdots \oplus \Pi A}_{q}$$
$$N = \underbrace{A \oplus A \oplus \cdots \oplus A \oplus A}_{p'} \oplus \underbrace{\Pi A \oplus \Pi A \oplus \cdots \oplus \Pi A}_{q'}$$

and let $(e_1, \ldots, e_p, e_{p+1}, \ldots, e_{p+q})$, $(f_1, \ldots, f_{p'}, f_{p'+1}, \ldots, f_{p'+q'})$ denote their bases.

Then every element in $\mathcal{H}om(M, N)$ (cf. Corollary 1.2) can be presented in a matrix form

$$T = \begin{array}{c c} & \begin{array}{cc} \overbrace{\quad}^{p} & \overbrace{\quad}^{q} \end{array} \\ \begin{array}{c} p'\{ \\ q'\{ \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array}$$

T is even if all the entries of A, D are even and those of B and C are odd.

T is odd if the entries of A, D are odd and those of B and C are even.

When $M = N$, $\mathcal{E}nd(M)$ will be identified with the matrix algebra which we denote as $Mat_{p,q}(A)$

Lemma. *Consider an even morphism str* : $\mathrm{Mat}_{p,q}(A) \to \mathrm{str}A$ *given by the formula* $\mathrm{str}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \mathrm{tr}\, A - (-1)^{p(D)}\mathrm{tr}\, D$. *Then* $\mathrm{str}([X,Y]) = 0$.
∎

The function (str) can also be defined using the tensor structure mentioned in 1.1.

2. Super calculus.

2.1. Super domains.

A super domain of dimension (p,q) is an open subset $U \subseteq \mathbb{R}^p$ with a super algebra of functions $C^\infty(U) \otimes \Lambda^\cdot(\xi_1,\ldots,\xi_q)$. For each $x \in U$ we have the evaluation map $ev_x\colon C^\infty(U) \otimes \Lambda^\cdot(\xi_1,\ldots,\xi_q) \to \mathbb{R}$ defined as

$$\begin{cases} ev_x(f \otimes 1) = f(X) \\ ev_x(1 \otimes \xi) = 0 \end{cases}.$$

The triple $(U, C^\infty(U)\otimes\Lambda^\cdot(\xi_1,\ldots,\xi_q), ev_x)$ will be denoted by calligraphic $\mathcal{U}$ and the algebra $C^\infty(U) \otimes \Lambda^\cdot(\xi_1,\ldots,\xi_q)$ will be denoted by $C^\infty(\mathcal{U})$.

If $\mathcal{U}$ and $\mathcal{V}$ are two super domains, a morphism $\varphi\colon \mathcal{U} \to \mathcal{V}$ is a pair (φ_0, φ^*), where $\varphi_0\colon U \to V$ is a map of the underlying topological spaces and $\varphi^*\colon C^\infty(\mathcal{V}) \to C^\infty(\mathcal{U})$ is a super algebra map, s.t. $\forall\, x \in U$, $\forall\, f \in C^\infty(\mathcal{V})$, $ev_x(\varphi^*(f)) = ev_{\varphi_0(x)}(f)$.

Remarks. Note that neither a function $\in C^\infty(\mathcal{U})$ nor a morphism $\mathcal{U} \to \mathcal{V}$ are determined by their values at $x \in U$.

Example.

Let $\mathcal{U} = (\mathbb{R}^1, C^\infty(\mathbb{R}^1) \otimes \Lambda(\xi_1,\xi_2))$. Consider a morphism $\varphi\colon \mathcal{U} \to \mathcal{U}$ given by $\varphi_0 = \mathrm{Id}$; $\varphi^*(\xi_1) = \xi_1$, $\varphi^*(\xi_2) = \xi_2$; $\varphi^*(x) = x + \xi_1\xi_2$. This is an example of an automorphism which is trivial on the underlying topological space.

Lemma.

(a) *A map of super domains* $\varphi\colon \mathcal{U} \to \mathcal{V}$ *is completely determined by the image of the coordinate functions*

$$y_1,\ldots,y_{p'},\xi_1,\ldots,\xi_{q'} \in C^\infty(\mathcal{V}).$$

(b) *For any* $\widetilde{y}_1,\ldots,\widetilde{y}_{p'},\widetilde{\xi}_1,\ldots,\widetilde{\xi}_{q'} \in C^\infty(\mathcal{U})$ $\exists!$ *map* $\varphi\colon \mathcal{U} \to \mathcal{V}$ *with* $\varphi^*(y_i) = \widetilde{y}_i$*:* $\varphi^*(\xi_j) = \widetilde{\xi}_j$ *provided that* $\forall\, x \in U$, $(\widetilde{y}_1(x), \widetilde{y}_2(x), \ldots, \widetilde{y}_{p'}(x)) \in V$.

2.2. Supermanifolds.

A supermanifold $\mathcal{M}$ is a topological space M endowed with a sheaf of super algebras $C^\infty(\mathcal{M})$ which is locally isomorphic to a super domain.

Morhisms between two supermanifolds $\mathcal{M}$ and $\mathcal{N}$ are defined as in the case of super domains.

Analogous definitions can be also given in the context of real analytic, complex, or algebraic manifolds.

Again, we emphasize that neither of the geometric constructions on supermanifolds that will be described is determined by its action on the points of M. Nevertheless, the following construction is often convenient:

Lemma-Construction. *For every supermanifold $\mathcal{M}$ of dimension (p, q) there exists a canonical sub-supermanifold $\overline{\mathcal{M}}$ of dimension $(p, 0)$ (i.e. a usual manifold) with a property that every map $\mathcal{N} \xrightarrow{\varphi} \mathcal{M}$ with $\mathcal{N}$ being completely even must factor as*

$$\begin{tikzcd} & \mathcal{M} \\ \mathcal{N} \arrow[ur, "\varphi"] \arrow[dr, "\overline{\varphi}"'] & \\ & \overline{\mathcal{M}} \arrow[uu, hook] \end{tikzcd}$$

Proof. We set $\overline{\mathcal{M}}$ to have the same underlying topological space as $\mathcal{M}$ does. $C^\infty(\overline{\mathcal{M}})$ is the quotient of $C^\infty(\mathcal{M})$ by the ideal generated by all odd functions. ■

$\overline{\mathcal{M}}$ is called the underlying manifold of the supermanifold $\mathcal{M}$.

We let $S\mathcal{M}an$ denote the category of supermanifolds. As in the case of usual manifolds, $S\mathcal{M}an$ admits the direct product construction: $\mathcal{M}, \mathcal{N} \to \mathcal{N} \times \mathcal{M}$.

As in the usual geometry, many important constructions for supermanifolds are carried out in families, where by a family $\mathcal{M}_S$ of supermanifolds over a base S which is a supermanifold too, we mean a submersive morphism

$$\begin{array}{c} \mathcal{M}_S \\ \big\downarrow \\ S \end{array} \qquad \text{(cf. 2.3).}$$

We will often consider sheaves of $C^\infty(\mathcal{M})$-modules on a supermanifold $\mathcal{M}$. A sheaf is called locally free, if locally on M, we have $\mathcal{F} \simeq C^\infty(\mathcal{M}) \underset{\mathbb{R}}{\otimes} V$, where V is a super vector space.

If x is a point of a supermanifold $\mathcal{M}$ and if $\mathcal{F}$ is a sheaf on $\mathcal{M}$, the fiber $\mathcal{F}_x$ of $\mathcal{F}$ at x is defined as a super vector space $\mathcal{F}/m_x\mathcal{F}$, where m_x is the maximal ideal in $C^\infty(\mathcal{M})$ corresponding to x.

2.3. Tangent sheaf.

Consider the sheaf of all (even and odd) derivations of $C^\infty(\mathcal{M})$. This is a sheaf of locally free $C^\infty(\mathcal{M})$-modules, called the tangent sheaf and we denote it by $\mathcal{T}(\mathcal{M})$.

The dual sheaf to $\mathcal{T}(M)$ denoted by $\mathcal{T}(\mathcal{M})^*$ is called the cotangent sheaf. We have a canonical even morphism $D : \mathcal{O}(\mathcal{M}) \to \mathcal{T}(\mathcal{M})$ defined by $D(f)(X) = (-1)^{p(f)p(X)}X(f)$.

The sheaf $\Omega^1(\mathcal{M}) = \Pi\mathcal{T}(\mathcal{M})^*$ is called the sheaf of 1-forms on $\mathcal{M}$. We have canonical odd morphism $d : \mathcal{O}(\mathcal{M}) \to \Omega^1(\mathcal{M})$.

Lemma-Definition. *The sheaf* $\underset{i\geq 0}{\oplus} \Lambda^i(\Omega^1(\mathcal{M}) \overset{\text{def}}{\simeq} \underset{i\geq 0}{\oplus} \Omega^i(\mathcal{M})$ *has a structure of a commutative super algebra such that:*

1) *The differential* d: $\Omega^i(\mathcal{M}) \to \Omega^{i+1}(\mathcal{M})$ *is odd and satisfies the Leibnitz rule.*
2) *For* $f \in C^\infty(\mathcal{M}) \overset{\text{def}}{\simeq} \Omega^0(\mathcal{M})$ *and* $X \in \Gamma(\mathcal{M}, \mathcal{T}(\mathcal{M}))$ *we have*

$$df(X) = (-1)^{p(x)}X(f). \qquad \blacksquare$$

Note that, unlike the usual situation, the De Rham complex $\Omega^0(\mathcal{M}) \to \Omega^1(\mathcal{M}) \to \dots$ is infinite unless $\mathcal{M}$ is even.

Let φ: $\mathcal{M} \to \mathcal{N}$ be a morphism between two supermanifolds. If $x \in M$ and $y = \varphi_0(x) \in N$ we have a map of superspaces called the differential of φ at x:

$$T_x\varphi\colon \mathcal{T}_x(\mathcal{M}) \to \mathcal{T}_y(\mathcal{N}).$$

As in the classical case, we have the "implicit function theorem":

Theorem.

Let φ: $\mathcal{M} \to \mathcal{N}$ *be a submersion (i.e.* $T_x\varphi$ *is onto* $\forall\, x$*). Then locally on* $\mathcal{M}$*, it can be identified with a projection* $\mathcal{M} = \mathcal{N} \times \mathcal{N}' \to \mathcal{N}$

2.4. Distributions.

A distribution on a manifold $\mathcal{M}$ is a subsheaf $\mathfrak{a} \hookrightarrow \mathcal{T}(\mathcal{M})$ which locally on M can be realized as a direct summand. Such $\mathfrak{a}$ leads to considering the Frobenius pairing:

$$\mathrm{Fr}\colon\ \mathfrak{a} \otimes \mathfrak{a} \to \mathcal{T}(\mathcal{M})/\mathfrak{a}.$$

It is easy to see that Fr is a map of $C^\infty(\mathcal{M})$-modules. We have an analog of the Frobenius theorem:

Theorem. $\mathrm{Fr} \equiv 0$ *if and only if the distribution* $\mathfrak{a}$ *is integrable, i.e. if locally on* M *there exists a submersive map* $\pi : \mathcal{M} \to \mathcal{N}$ *such that* $\mathfrak{a}$ *identifies with the kernel*

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathcal{T}(\mathcal{M}) \longrightarrow \pi^*\mathcal{T}(\mathcal{N}) \longrightarrow 0 \qquad \blacksquare$$

2.5.

A convenient way to think about supermanifolds is to view them as contravariant functors on the category $S\mathcal{M}an$. In other words, to a supermanifold $\mathcal{M}$ associate a functor

$$F_{\mathcal{M}}\colon\ \mathrm{SMan}^0 \to \mathrm{Sets}$$

given by $F_{\mathcal{M}}(S) = \mathrm{Hom}(S, \mathcal{M})$. By the general category theory, such a functor determines $\mathcal{M}$ completely. (Sometimes, the appropriate supermanifold does not exist and such a functor is a good substitute for it.) Let us consider several examples.

Example 1. $\mathbb{R}^{1,0}$, $\mathbb{R}^{0,1}$.

It is easy to see that

$$F_{\mathbb{R}^{1,0}}(S) \simeq C^\infty(S)_{\overline{0}}$$
$$F_{\mathbb{R}^{0,1}}(S) \simeq C^\infty(S)_{\overline{1}}.$$

Example 2.

Let V be a finite dimensional super vector space. Then one can canonically define a supermanifold which corresponds to V. This supermanifold corresponds to the functor

$$F_V(S) = (C^\infty(S) \otimes V)_{\overline{0}}.$$

More generally, let E be a locally free sheaf of $C^\infty(\mathcal{M})$-modules on a supermanifold $\mathcal{M}$. We define the total space of the corresponding super vector bundle $V(E)$ on $\mathcal{M}$ by means of the functor

$$F_{V(E)}(S) = \{\text{pairs } \alpha\colon S \to M, \sigma\},$$

where α is a morphism of supermanifolds and σ is an even section of $\alpha^*(E)$ on S. (Here α^* denotes the functor of pull-back.)

Example 3.

We will consider a particular case of the previous example for $E = \Pi(\mathcal{T}(\mathcal{M}))$. The supermanifold $V(E)$ will be also denoted as $\widetilde{\mathcal{M}}$.

Lemma. $F_{\widetilde{\mathcal{M}}}(S) \simeq \mathrm{Hom}(S \times \mathbb{R}^{0,1}, \mathcal{M})$ ■

In particular, the points of $\widetilde{\mathcal{M}}$ *bijectively correspond to morphisms* $\mathbb{R}^{0,1} \to \mathcal{M}$.

Supermanifolds obtained in this way will be sometimes denoted as $\mathrm{Mor}(\mathbb{R}^{0,1}, \mathcal{M})$.

3. Lie supergroups and Lie superalgebras.

3.1. Lie supergroups.

Let $\mathfrak{G}$ be a supermanifold, which we would like to endow with a structure of a Lie supergroup on $\mathfrak{G}$. It is very convenient to phrase the corresponding definition in terms of the corresponding functors.

Definition. 1) We say that $\mathfrak{G}$ is a Lie supergroup if the functor $F_{\mathfrak{G}}\colon S\mathcal{M}an^0 \to \mathrm{Sets}$ possesses a group structure (i.e. if for every $B \in S\mathcal{M}an$ the set $F_{\mathfrak{G}}(B)$ is a group structure and for any morphism $B_1 \to B_2$ of supermanifolds the corresponding morphism $F_{\mathfrak{G}}(B_2) \to F_{\mathfrak{g}}(B_1)$ is a homomorphism of groups.)

2) We say that a Lie supergroup $\mathfrak{G}$ acts on a supermanifold $\mathcal{M}$ if the functor $F_{\mathfrak{G}}$ acts on the functor $F_{\mathcal{M}}$ (i.e. if for every supermanifold B we are given an action of the group $F_{\mathfrak{G}}(B)$ on the set $F_{\mathcal{M}}(B)$ such that the obvious compatibility condition holds.)

Remark. One can give also the following (equivalent) definition of a Lie supergroup, or of its action on a supermanifold:

We can say that a structure of a Lie supergroup on a supermanifold $\mathfrak{G}$ is a pair of morphisms $m\colon \mathfrak{G}\times\mathfrak{G}\to\mathfrak{G}$ (multiplication) and $i\colon \mathfrak{G}\to\mathfrak{G}$ (taking the inverse) such that the obvious axioms hold. However, we will see below that in most examples the definition via functors is more convenient.

Example. Let us define the general linear Lie group $\mathfrak{GL}(p,q)$, which is a super analog of $\mathrm{GL}(n)$. It should be thought of as the group of linear automorphisms of $\mathbb{R}^{p,q}$. We define a functor $F_{\mathfrak{G}}\colon S\mathcal{M}an^0 \to \mathrm{Sets}$ by

$$F_{\mathfrak{G}}(B) = \mathrm{Aut}(C^\infty(B)\otimes\mathbb{R}^{p,q}) \qquad \text{(cf. 1.4)}$$

This functor is endowed with the obvious group structure and it is not difficult to see that this functor is represented by a supermanifold.

More generally, let V be any superspace of dimension (p,q). Then one can define a Lie supergroup $\mathfrak{GL}(V)$ by putting

$$F_{\mathfrak{G}}(B) = \mathrm{GL}_{C^\infty(B)}(V\otimes C^\infty(B))$$

(here $\mathrm{GL}_{C^\infty(B)}(V\otimes C^\infty(B))$ means the group of automorphisms of the module $V\otimes C^\infty(B)$ over the superalgebra $C^\infty(B)$).

3.2. Lie superalgebras.

If $\mathfrak{G}$ is a supergroup, it possesses a distinguished point e called the unit element. Therefore we can consider the tangent space $\mathcal{T}_e\mathfrak{G}$ to $\mathfrak{G}$ at the point e. It is easy to see that the superspace $\mathfrak{g}=\mathcal{T}_e\mathfrak{G}$ has a canonical structure of a Lie superalgebra (as in usual Lie theory, one can identify $\mathfrak{g}$ with the space of left invariant vector fields on $\mathfrak{G}$ and then consider their commutator).

Let G denote $\overline{\mathfrak{G}}$. Then the Lie algebra $\mathrm{Lie}\,G$ of G is canonically identified with $\mathfrak{G}_{\overline{0}}$ and we see that the pair $(G,\mathfrak{g})$ forms a Harish-Chandra pair. Let us recall this notion.

Definition. A Harish-Chandra pair is a pair $(G,\mathfrak{g})$ where G is a Lie group and $\mathfrak{g}$ is a Lie superalgebra together with an action of G on $\mathfrak{g}$ by automorphisms and a Lie-algebra map $i\colon \mathrm{Lie}\,G\to\mathfrak{g}_0$, such that the standard compatibility condition saying that the action of

$\operatorname{Lie} G$ on $\mathfrak{g}$ obtained by deriving the G-action coincides with the adjoint action of $\operatorname{Lie} G = \mathfrak{g}_0$ on $\mathfrak{g}$ holds.

It is easy to see that the assignment $\mathfrak{G} \to (\overline{\mathfrak{G}}, \operatorname{Lie} \mathfrak{G}) = (G, \mathfrak{g})$ is a correctly defined functor from the category of Lie supergroups to the category of Harish-Chandra pairs.

Proposition. *This functor is an equivalence between the category of Lie supergroups and the category of Harish-Chandra pairs for which the map*

$$i\colon \operatorname{Lie} G \to \mathfrak{g}_0$$

is an isomorphism.

This proposition allows to reduce any question about Lie supergroups to a question about Harish-Chandra pairs, which are intuitively easier to handle. For example, let us explain how to describe an action of a Lie supergroup on a supermanifold in terms of Harish-Chandra pairs.

Definition. 1) An action of a Lie superalgebra $\mathfrak{g}$ on a supermanifold $\mathcal{M}$ is a morphism of Lie superalgebras $\mathfrak{g} \to \operatorname{Vect} \mathcal{M}$ (here $\operatorname{Vect} \mathcal{M}$ denotes the algebra of global vector fields on $\mathcal{M}$).

2) An action of a Harish-Chandra pair $(G, \mathfrak{g})$ on $\mathcal{M}$ is an action of the Lie group G on $\mathcal{M}$ together with an action of the Lie superalgebra $\mathfrak{g}$ on $\mathcal{M}$, satisfying the obvious compatibility condition (i.e. that the two actions of $\operatorname{Lie} G$ on $\mathcal{M}$ obtained by deriving the G-action on $\mathcal{M}$ and by identifying $\operatorname{Lie} G$ with $\mathfrak{g}_0 \subset \mathfrak{g}$, coincide).

It is easy to see that to give an action of a Lie supergroup $\mathfrak{G}$ on $\mathcal{M}$ is the same as to give an action of the corresponding Harish-Chandra pair.

Example. Let $\mathfrak{G} = \mathbb{R}^{0,1}$. This is a Lie supergroup (with respect to addition) and as any Lie supergroup it acts on itself by translations. More precisely, for any supermanifold B the set $\operatorname{Hom}(B, \mathbb{R}^{0,1})$ is canonically identified with the vector space $C^\infty(B)_{\overline{1}}$,, which endows $F_{\mathfrak{G}}$ with a group structure.

For any supermanifold $\mathcal{M}$ consider $\widetilde{\mathcal{M}} = \operatorname{Mor}(\mathbb{R}^{0,1}, \mathcal{M})$. $\mathcal{M}$ is a supermanifold (cf. 2.5) which admits a natural action of $\mathbb{R}^{0,1}$ (since it acts on itself). The Lie algebra $\operatorname{Lie} \mathbb{R}^{0,1}$

is generated by the standard odd vector field $\frac{\partial}{\partial\xi}$. The action of its image in $\operatorname{Vect}\widetilde{\mathcal{M}}$ on $\Omega^*(\mathcal{M}) \subset C^\infty(\widetilde{\mathcal{M}})$ coincides with the De Rham differential.

3.3. Theory of connections.

We want to define the notion of connection in the superworld. Let us first explain what we mean by principal $\mathfrak{G}$-bundle over $\mathcal{M}$, where $\mathfrak{G}$ is a Lie supergroup. A principal $\mathfrak{G}$ bundle over $\mathcal{M}$ is a supermanifold P together with a free $\mathfrak{G}$-action and a natural isomorphism $\mathcal{M} \simeq P/\mathfrak{G}$ (the notions of quotient and free actions are again easily formulated in terms of the corresponding functors).

As in the usual differential geometry, one can define a connection on P to be a distribution $\mathfrak{a} \subset \mathcal{T}(P)$ of codimension equal to $\dim \mathcal{M}$ and transversal to the fibers of the projection $P \to \mathcal{M}$ which is, moreover, $\mathfrak{G}$-invariant.

A connection on a vector bundle $\mathcal{E}$ on $\mathcal{M}$ is by definition a first-order differential operator

$$\nabla\colon \mathcal{E} \otimes \Omega^*(\mathcal{M}) \to \mathcal{E} \otimes \Omega^*(\mathcal{M})$$

which is odd, satisfies the Leibnitz rule and has degree 1 with respect to the natural $\mathbb{Z}$-grading on $\Omega^*(\mathcal{M})$.

A connection on a principle $\mathfrak{G}$-bundle P on $\mathcal{M}$ can be defined as a compatible (w.r. to tensor products) family of connections on every vector bundle associated with a representation $\rho\colon \mathfrak{g} \to \mathfrak{GL}(V)$ of $\mathfrak{G}$.

4. Integration and determinants.

4.0. So far we have discussed only differential geometry in the supercase. Now we want to treat integration theory. It is easy to see that in supercase one cannot integrate differential forms over a supermanifold (since there is no natural notion of highest-degree forms). Instead, we can integrate sections of some vector bundle over $\mathcal{M}$. In order to define it we first have to discuss the notion of superdeterminant (or Berezinian). So, we are going back to linear algebra.

4.1. Let C be a commutative superalgebra and let $\mathcal{M}$ be a free C-module of rank (p, q). We want to define an analog of the notion of highest exterior power of $\mathcal{M}$. We have the following definition:

Let $S(\mathcal{M})$ denote the symmetric algebra of $\mathcal{M}$.

Lemma.

$$\operatorname{Ext}^i_{S(\mathcal{M})}(C, S(\mathcal{M})) = \begin{cases} 0 & \text{if } i \neq p \\ \text{free } C\text{-module of rank 1 and parity } q \text{ if } i = p \end{cases}$$

We define $\operatorname{Ber}_C(\mathcal{M}) := \operatorname{Ext}^p_{S(\mathcal{M})}(C, S(\mathcal{M}))^*$.

Properties of Ber

1) If $\mathcal{M}$ is of rank $(p, 0)$ then

$$\operatorname{Ber} \mathcal{M} = \Lambda^p_C \mathcal{M}$$

2) Suppose that $0 \to \mathcal{M}_1 \to \mathcal{M} \to \mathcal{M}_2 \to 0$ is a short exact sequence of C-modules. Then

$$\operatorname{Ber} \mathcal{M} = \operatorname{Ber} \mathcal{M}_1 \otimes \operatorname{Ber} \mathcal{M}_2.$$

3) Suppose that $\mathcal{M}$ is $(0, q)$-dimensional. Then

$$\operatorname{Ber} \mathcal{M} = S^q(\mathcal{M})^{-1}.$$

Let now $g \in \operatorname{Aut}_C(\mathcal{M})$ — an automorphism of $\mathcal{M}$. Then one can define $\operatorname{Ber}(g)$ to be the invertible element of $C_{\overline{0}}$ by which g acts on $\operatorname{Ber}(\mathcal{M})$. We obtain a homomorphism $\operatorname{Aut}_C(\mathcal{M}) \to C^*_{\overline{0}}$.

Berezin actually wrote a formula for $\operatorname{Ber}(g)$ (this was his original definition of Berezinian).

Namely, let us choose a basis of $\mathcal{M}$ over C. Suppose that in this basis g looks like $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. Then

$$\operatorname{Ber} g = \det(A - B\ D^{-1} C)(\det\ D)^{-1}$$

Looking at this formula, it is not easy (although possible) to prove that it is multiplicative.

Let us emphasize that, unlike the usual situation, Berezinian is defined only on even invertible matrices.

There is yet another definiton of the Berezinian: by proposition 3.2 there exists a canonical morphism of Lie supergroups

$$\text{Ber: } \mathfrak{GL}(p, q) \to \mathfrak{GL}(1, 0)$$

given by the usual determinant $GL(n) \to GL(1)$ and by the supertrace

$$\mathrm{str} : \mathrm{Lie}(\mathfrak{GL}(p,q)) \simeq \mathrm{Mat}(p,q,\mathbb{R}) \to \mathbb{R} \simeq \mathrm{Lie}(\mathfrak{GL}(1,0))$$

and we can define Ber on even invertible B-valued matrices as the corresponding homomorphism

$$F_{\mathfrak{GL}(p,q)}(B) \to F_{\mathfrak{GL}(1,0)}(B).$$

Let us now list some further properties of $\mathrm{Ber}(\mathcal{M})$.

4) $\mathrm{Ber}(\Pi(\mathcal{M})) = \mathrm{Ber}(\mathcal{M})^{-1}$

5) $\mathrm{Ber}(\mathcal{M}^*) = \mathrm{Ber}(\mathcal{M})^{-1}$

6) (very important in integration theory).

Fix a basis $e_1 \dots e_p$, $e_{p+1} \dots e_{p+q}$ of $\mathcal{M}$. Then it defines a canonical element $e_1 \Lambda e_2 \dots \Lambda e_{p+q}) \in \mathrm{Ber}(\mathcal{M})$. Moreover, if $e_1^1 \dots e_{p+q}^1$ is another basis of $\mathcal{M}$ and g is the transition matrix from one basis to the other, then $\Delta(e_1^1 \dots e_{p+q}^1) = \mathrm{Ber}(g)\Delta(e_1 \dots e_{p+q})$. In fact, this property may serve as a definition of $\mathrm{Ber}(\mathcal{M})$.

Let now $\mathcal{F}$ be a locally free sheaf of $C^\infty(\mathcal{M})$-modules on a supermanifold $\mathcal{M}$. We can associate to it an invertible sheaf $\mathrm{Ber}(\mathcal{F})$ on $\mathcal{M}$.

4.2 Integration.

Let $\mathcal{M}$ be a supermanifold. We define an invertible sheaf $\mathrm{Vol}(\mathcal{M})$ on it as $\mathrm{Ber}(\Pi(\mathcal{T}(\mathcal{M})))$. When $\mathcal{M}$ is even, $\mathcal{T}(\mathcal{M})$ is even and $\mathrm{Vol}(\mathcal{M}) \simeq \Omega^{\mathrm{top}}(\mathcal{M})$ coincides with the sheaf of usual volume forms. Let us also fix an orientation on the underlying manifold.

Theorem. *We have a unique map $\Gamma_c(\mathcal{M}, \mathrm{Vol}(\mathcal{M})) \xrightarrow{\int} \mathbb{R}$ of parity p satisfying the following properties:*

1) *If $\mathcal{M}$ is even $\int$ is the usual integration of a volume form $\omega \to \int\limits_M \omega$*

2) *If $\mathcal{M} \simeq \mathcal{M}_1 \times \mathcal{M}_2$; $\omega_i \in \Gamma_c(\mathcal{M}_i, \mathrm{Vol}(\mathcal{M}_i))$, we have*

$$\int\limits_{\mathcal{M}} \omega = \int\limits_{\mathcal{M}_1} \omega_1 \times \int\limits_{\mathcal{M}_2} \omega_2$$

3) *If $\mathcal{M} \simeq \mathbb{R}^{0,1}$ and $\omega_0 \in \mathrm{Vol}(\mathbb{R}^{0,1})$ is given by $\xi \cdot d\xi$, $\int\limits_{\mathbb{R}^{0,1}} \omega_0 = 1$.*

Lemma. *For every* $X \in \Gamma(\mathcal{M}, \mathcal{T}(\mathcal{M}))$*;* $\omega \in \Gamma_c(\mathcal{M}, \mathrm{Vol}(\mathcal{M}))$ *we have* $\int\limits_{\mathcal{M}} (X\omega) = 0$. ■

As in the classical case, everything said above can be put also in the relative framework. Namely, consider a submersive morphism $\pi : \mathcal{M} \to \mathcal{N}$ and let ω be a volume form on $\mathcal{M}$ whose support is proper w.r. to π. We can then "integrate it along the fibers", i.e. $\exists$ a well defined volume form $\pi_! \omega \in \Gamma(\mathcal{N}, \mathrm{Vol}(\mathcal{N}))$ and if $\omega \in \Gamma_c(\mathcal{M}, \mathrm{Vol}(\mathcal{M}))$,

$$\int\limits_{\mathcal{M}} \omega = \int\limits_{\mathcal{N}} \pi_! \omega.$$

In particular, in order to compute $\int\limits_{\mathcal{M}} \omega$ for a supermanifold $\mathcal{M}$, one must choose a projection p: $\begin{array}{c}\mathcal{M}\\ \downarrow\\ \overline{\mathcal{M}}\end{array}$ which reduces $\int\limits_{\mathcal{M}} \omega$ to the usual integration $\int\limits_{\overline{\mathcal{M}}} p_!(\omega)$.

Note that the integration $\omega \to p_!(\omega)$ is "non-analytic".

4.3. Integral forms.

The following formalism will be useful for us in the sequel. We introduce sheaves $\sum^{-i}$ on a supermanifold $\mathcal{M}$ as $\sum^{-i} = \mathcal{H}om(\Omega^i(\mathcal{M}), \mathrm{Vol}(\mathcal{M}))$. We have an action of vector fields on $\mathcal{M}$ on $\sum^{-i}$'s by Lie derivatives. The direct sum $\sum(\mathcal{M}) := \bigoplus\limits_i \sum^i$ is a $\mathbb{Z}$-graded module over the sheaf of superalgebras $\Omega(\mathcal{M}) := \bigoplus\limits_i \Omega^i(\mathcal{M})$. It is easy to check that $\forall X \in \Gamma(\mathcal{M}, \mathcal{F}(\mathcal{M}))$ extends uniquely to a derivation of degree -1 on $\bigoplus\limits_i \sum^i$ compatible with the module structure.

Lemma. $\exists!$ *odd differential of degree* 1 $d : \sum \to \sum$ *which makes it a DG-module over* $\Omega(\mathcal{M})$ *such that* $\forall X \in \Gamma(\mathcal{M}, \mathcal{T}(\mathcal{M}))$ *we have* $[i_X, d] = \mathrm{Lie}_X$. ■

5. SUSY Manifolds.

5.1.

Definition. A pre-SUSY manifold is a supermanifold $\mathcal{M}$ of dimension (p, q) equipped with a distribution $\mathfrak{a} \hookrightarrow \mathcal{T}(\mathcal{M})$ of dimension $(0, q)$. It is called a SUSY manifold if the Frobenius pairing

$$\mathrm{Fr}\colon\ \mathfrak{a} \otimes \mathfrak{a} \to \mathcal{T}(\mathcal{M})/\mathfrak{a}$$

is sufficiently non-degenerate.

Let us consider first the case of SUSY curves, i.e. $(p,q)=(1,1)$. In this case the non-degeneracy condition reads as: $\mathrm{Fr}\colon \mathfrak{a}\otimes\mathfrak{a}\to\mathcal{T}(\mathcal{M})/\mathfrak{a}$ is an isomorphism. In what follows, our assertions apply both to the case of usual supermanifolds and complex analytic ones.

Definition. A Spin curve (complex or real) is a curve C with a line bundle S and an isomorphism $S\otimes S=\mathcal{T}(C)$.

Proposition. *The category of SUSY curves is equivalent to the category of Spin curves.*

Proof. Let $(C,\mathfrak{a})$ be a SUSY curve. Consider the restriction $\mathfrak{a}|\overline{C}$. By definition,

$$\mathfrak{a}|\overline{C}\otimes\mathfrak{a}|\overline{C}\simeq\mathcal{T}(C)|\overline{C}/\mathfrak{a}|\overline{C}\simeq\tau(\overline{C}).$$

Therefore, $(\overline{C},\mathfrak{a}|\overline{C})$ is a Spin curve and we have constructed a functor SUSY curves $\to$ Spin curves.

Since locally on a curve all Spin structures are isomorphic, it is sufficient to prove the local faithfulness of this functor.

So, let $C=\mathbb{R}^{1,1}$ be with coordinates (x,ξ) and let $\mathfrak{a}$ be given by $C^\infty(C)\cdot(\partial_\xi+x\partial_x)$. It is easy to see then that there are no non-trivial automorphisms of this SUSY structure that induce the trivial automorphism of $(\overline{C},\mathfrak{a}|\overline{C})$. ∎

Let $S\mathcal{M}_g$ be the moduli space of Spin curves of genus g. The proposition proved above implies that it admits a natural super extension, namely the supermanifold of moduli of SUSY curves of genus g.

$$S\mathcal{M}_g\hookrightarrow \mathrm{SUSY}\,\mathcal{M}_g.$$

Moreover, $S\mathcal{M}_g$ identifies with $\overline{\mathrm{SUSY}\,\mathcal{M}_g}$.

5.2. In the sequel we will consider the example of a SUSY $(2,2)$-manifold.

Let $\mathcal{V}$ be a SUSY $(2,2)$-manifold and let us assume, moreover, that $\mathcal{V}$ is locally a product of two SUSY curves. This condition is equivalent to the existence of a decomposition:

$$\begin{aligned}
\mathfrak{a} &= \mathfrak{a}_1 \oplus \mathfrak{a}_2 \\
\mathcal{T}(\mathcal{V})/\mathfrak{a} &= (\mathcal{T}(\mathcal{V})/\mathfrak{a})_1 \oplus (\mathcal{T}(\mathcal{V})/\mathfrak{a})_2 \\
\mathrm{Fr}(\mathfrak{a}_1 \otimes \mathfrak{a}_1) &\simeq (\mathcal{T}(\mathcal{V})/\mathfrak{a})_1 \\
\mathrm{Fr}(\mathfrak{a}_2 \otimes \mathfrak{a}_2) &\simeq (\mathcal{T}(\mathcal{V})/\mathfrak{a})_2 \\
\mathrm{Fr}(\mathfrak{a}_1 \otimes \mathfrak{a}_2) &= \mathrm{Fr}(\mathfrak{a}_2 \otimes \mathfrak{a}_1) = 0.
\end{aligned}$$

Such a decomposition is canonical if it exists.

Lemma. *In the above situation,* $\mathrm{Vol}(\mathcal{V})$ *canonically identifies with* $\mathfrak{a}_1^* \otimes \mathfrak{a}_2^*$. ■