ON INTEGRATION IN A SPACE WITH COMPLEX DIMENSION

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These are comments on dimensional regularization of Feynman integrals, which was discussed in the lectures of K.Gawedzki and D.Gross. This material can be found in physical literature, but in a slightly different form.

We will first define a certain remarkable operator (the D-dimensional integral), which acts on functions of nonnegative symmetric bilinear forms, and then show how it applies to dimensional regularization.

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1. The D-dimensional integral.

Let E be a real vector space of dimension N with a volume element dv_E. Denote by S^2E^* the space of symmetric bilinear forms on E, and by $S^2_+E^* \subset S^2E^*$ the set of nonnegative symmetric bilinear forms. We think of elements of S^2E^* as of selfadjoint operators $E \to E^*$. For any such operator A, we can canonically define its determinant $\det(A) \in \mathbb{R}$.

Let dA be a volume element on S^2E^* defined by dv_E. That is: identify E^* with E by any positive operator $X : E \to E^*$ with deteminant 1, regard $A \in S^2E^*$ as a selfadjoint operator on E using this identification, and define dA to be the volume element on S^2E^* corresponding to the norm $||A||^2 = \mathrm{Tr}(A^2)$. (It is clear that this definition does not depend on X).

Fix a complex number D with $\mathrm{Re}D > N - 1$.

Lemma 1. *For any $B \in S^2E$ such that $B > 0$*

$$\int_{S^2_+E^*} (\det A)^{(D-N-1)/2} e^{-\mathrm{tr}(AB)} dA = \Gamma_N(D/2)(\det B)^{-D/2},$$

$$\Gamma_N(D/2) = (2\pi)^{N(N-1)/4} \prod_{l=0}^{N-1} \Gamma\left(\frac{D+l}{2}\right). \tag{1}$$

Proof. Denote this integral by $f(B)$. It is easy to see that for any $S \in GL(E)$, $f(S^*BS) = f(B)|\det S|^{-D}$. From this it follows that $f(B) = C(D)(\det B)^{-D/2}$.

It remains to compute $C(D)$. Let us pick B such that $\det B = 1$, and identify E^* with E using the isomorphism $B : E^* \to E$. Then E becomes a Euclidean space with the inner product B^{-1}. Thus the constant $C(D)$ has the form

$$\int_{A:E\to E, A^*=A, A\geq 0} (\det A)^{(D-N-1)/2} e^{-\mathrm{tr}A} dA$$

Since this integral is invariant under conjugation by $SO(E)$, we can reduce it to the quotient space $S^2_+E^*/SO(E) = [0,\infty)^N/S_N$. It is known from the theory of symmetric spaces that the pushforward of dA from $S^2_+E^*$ to $S^2_+E^*/SO(E)$ has the form $b_N \prod_{1\le i<j\le N} |s_i - s_j| ds_1...ds_N$, where s_i are the eigenvalues of $A \in S^2_+E^*/SO(E)$, and

$$b_N = \frac{2^{N(N-1)/4}\pi^{N(N+1)/4}}{\prod_{l=1}^N \Gamma(l/2)},$$

Thus, $C(D)$ has the form

$$(2) \qquad b_N N!^{-1} \int_{[0,\infty)^N} \prod_i t_i^{(D-N-1)/2} \prod_i e^{-t_i} \prod_{1\le i<j\le N} |t_i - t_j| dt_1...dt_N.$$

To compute this integral, we use (a special case of) Selberg's integral formula:

$$\int_{0\le t_i\le 1} \prod_i t_i^a \prod_i (1-t_i)^b \prod_{1\le i<j\le N} |t_i - t_j| dt_1...dt_N =$$

$$(3) \qquad \prod_{l=1}^N \frac{\Gamma(a+\frac{l+1}{2})\Gamma(b+\frac{l+1}{2})\Gamma(\frac{l}{2}+1)}{\Gamma(a+b+\frac{2N-l+3}{2})\Gamma(\frac{3}{2})}.$$

Let us set $s_i = bt_i$ in Selberg's integral, and send b to $+\infty$. In the limit, using Stirling's formula, and setting $a = (D-N-1)/2$, we get

$$C(D) = b_N N!^{-1} \prod_{l=1}^N \frac{\Gamma(\frac{D+l-N}{2})\Gamma(\frac{l}{2}+1)}{\Gamma(\frac{3}{2})} =$$

$$(4) \qquad b_N \pi^{-N/2} \prod_{l=1}^N \Gamma\left(\frac{D+l-N}{2}\right)\Gamma\left(\frac{l}{2}\right) = \Gamma_N(D/2)..$$

The Lemma is proved.

Now consider the function ρ_D^E on S^2E^* which is supported at $S^2_+E^*$ and is given by the formula

$$(5) \qquad \rho_D^E(A) = \pi^{ND/2}\Gamma_N(D/2)^{-1}(\det A)^{(D-N-1)/2}.$$

By Lemma 1, we have

$$(6) \qquad \int_{S^2_+E^*} \rho_D^E(A) e^{-\mathrm{tr}(AB)} dA = \pi^{ND/2}(\det B)^{-D/2},$$

$\mathrm{Re}D > N-1$.

If $\mathrm{Re}D > N-1$, the function $\rho_D^E(A)$ is a locally L^1-function, so it defines a distribution on S^2E^*. Let us compute the Fourier transform of $\rho_D^E(A)$, i.e.

$$\hat{\rho}_D^E(B) = \int_{S^2E^*} \rho_D^E(A) e^{-i\mathrm{tr}(AB)} dA,$$

$B \in S^2E$ (the integral is understood in the sense of distributions).

Lemma 2.

$$\hat{\rho}_D^E(B) = \lim_{J\to+0} \pi^{ND/2} \det(iB+J)^{-D/2}. \tag{7}$$

where $J \in S^2E^*, J > 0.$

Remark. Here we use the branch of the function $z^{-D/2}$ which is equal to $e^{-D\ln z/2}$ for real positive z.

Proof. By analytic continuation, formula (6) holds for arbitrary $B \in (S^2E^*)_{\mathbb{C}}$ with $\mathrm{Re}B > 0$. In particular, we can replace B with $iB + J$, where $B \in S^2E^*$. In the limit when $J \to +0$, we obtain formula (7).

Let $\mu : S^2E \to \mathrm{Hom}(E^*, E)$ be the tautological embedding. Let Δ be the polynomial of degree N on S^2E, defined by the equation $\Delta(B) = \det(\mu(B))$. We regard Δ as a differential operator of order N on S^2E^* with constant coefficients.

Corollary 3. *If $ReD > N + 1$, then*

$$\Delta\rho_D^E(A) = \pi^N \rho_{D-2}^E(A). \tag{8}$$

Proof. The proof is obvious after passing to Fourier transforms.

Let $\mathcal{S}(S^2E^*)$ be the space of complex-valued Schwarz functions on S^2E^*. For any $f \in \mathcal{S}(S^2E^*)$ and $\mathrm{Re}D > N - 1$, define

$$I_D^E(f) = \int_{S^2E^*} \rho_D^E(A) f(A) dA. \tag{9}$$

Corollary 4. *$I_D^E(f)$ extends to an entire function of D.*

Proof. For $ReD > N - 1 - 2K$, $K \in \mathbb{Z}_+$, we can define $I_D^E(f)$ by the formula

$$I_D^E(f) = (-\pi)^{-KN} \int_{S^2E^*} \rho_{D+2K}^E(A) \Delta^K f(A) dA. \tag{10}$$

According to Corollary 3, if $\mathrm{Re}D >> 0$, formula (10) defines the same function as (9). Thus, (10) defines an analytic continuation of I_D^E.

Proposition 5. *Let $D \geq 0$ be a nonnegative integer, and V be a vector space of dimension D, with a positive definite symmetric bilinear form β. Then for any $f \in \mathcal{S}(S^2E^*)$ one has*

$$I_D^E(f) = \int_{\mathrm{Hom}(E,V)} f(x^*(\beta)) dx, \tag{11}$$

where $x^*(\beta)$ denotes the inverse image of β under x (a nonnegative symmetric bilinear form on E).

Proof. It is enough to check that (11) holds for functions $e^{-\mathrm{tr}(AB)} 1_{S_+^2E^*}(A)$ where 1_Y is the characteristic function of the set Y, and $\mathrm{Re}(B) > 0$. This follows from the fact that the function $f(A)$ admits a Fourier representation.

In the case

$$f(A) = e^{-\mathrm{tr}(AB)} 1_{S^2_+ E^*}(A), \tag{12}$$

evaluating the Gaussian integral on the right hand side of (11), we get $\pi^{ND/2}(\det B)^{-D/2}$. On the left hand side of (11), by Lemma 1, we get the same. The proposition is proved.

Remark. From Proposition 5 it is clear why formula (9) works only for $\mathrm{Re}D > N-1$. Indeed, for integer D between 0 and $N-1$ the distribution $\rho_D(A)dA$ is concentrated on forms of rank $\leq D$, so it is a purely singular distribution. Thus, the measure $\rho_D(A)dA$ is not absolutely continuous with respect to dA and hence cannot be represented as a function times dA. In fact, it turns out that this distribution is singular (not locally L^1) for all D with $\mathrm{Re}D \leq N-1$.

Proposition 5 motivates the following definition.

Definition. *$I_D^E(f)$ is called the D-dimenional integral of f.*

Another way of computing the D-dimensional integral is using Fourier transforms. By Plancherel formula, it follows from (9) that

$$I_D^E(f) = (2\pi)^{-N(N+1)/2} \int_{S^2 E^*} \hat{f}(B)\overline{\hat{\rho}_D^E(B)}dB, \tag{13}$$

where $\hat{f}$ is the Fourier transform of f. (It is clear that $\overline{\hat{\rho}_D^E(B)} = \lim_{J\to+0} \pi^{ND/2}\det(-iB+J)^{-D/2}$).

For $\mathrm{Re}D < 2$, this formula can be written as

$$I_D^E(f) = 2^{-N(N+1)/2}\pi^{N(D-N-1)/2} \int_{S^2 E^*} \hat{f}(B) e^{\frac{\pi i D}{4}\sigma(B)} |\det(B)|^{-D/2} dB, \tag{14}$$

where $\sigma(B)$ is the signature of B (the number of pluses minus the number of minuses). If $\mathrm{Re}D \geq 2$, the distribution $\overline{\hat{\rho}_D^E(B)}$ is singular, so in order to define $I_D^E(f)$ by a direct integration formula, we need to use the operator Δ. Namely, from Lemma 1 it follows that

$$\Delta\hat{\rho}_D^E(B) = (-1)^N p_N(D)\hat{\rho}_{D+2}^E(B), \quad p_N(D) = \pi^{-N}\prod_{l=1}^N \frac{D+l-N}{2} \tag{15}$$

So if $\mathrm{Re}D < 2+2K$, we can define I_D^E by the formula

$$\begin{aligned} I_D^E(f) &= 2^{-N(N+1)/2}\pi^{N(D-N-1)/2} \prod_{J=1}^K p_N(D-2J)^{-1} \times \\ &\int_{S^2 E^*} \Delta^K \hat{f}(B) e^{\frac{\pi i (D-2K)}{4}\sigma(B)} |\det(B)|^{(-D+2K)/2} dB, \end{aligned} \tag{16}$$

Example. If $N=1$, the D-dimensional integral of f is (up to normalization) just the Mellin transform of f:

$$I_D^E(f)=\pi^{D/2}\Gamma(D/2)^{-1}\int_0^\infty x^{\frac{D}{2}-1}f(x)dx.$$

Let $\mathcal{S}(S_+^2E^*)$ be the space of smooth functions f on $S_+^2E^*$, such that all the derivatives of f are rapidly decaying. It is easy to see that the restriction map $\mathcal{S}(S^2E^*)\to\mathcal{S}(S_+^2E^*)$ is surjective. So, I_D^E defines a linear functional $\mathcal{S}(S_+^2E^*)\to\mathbb{C}$.

2. D-dimensional integral with parameters.

Now we will extend our construction to a more general situation. Let $F\subset E$ be a subspace of dimension M with a volume element dv_F. Any symmetric bilinear form A on E defines a form A_F on F (the restriction of E).

Also, for any symmetric bilinear form B on E^* such that $B_{F^\perp}$ is nondegenerate we can define a form B^{F^*} on F^* as follows. Let $(F^\perp)^{\perp_B}$ be the B-orthogonal complement of $F^\perp$. As $B_{F^\perp}$ is nondegenerate, this space is naturally identified with $F^*=E^*/F^\perp$. By the definition, the form B^{F^*} is the restriction of B to $(F^\perp)^{\perp_B}=F^*$.

Let us find an explicit expression of B^{F^*}. Choose a positive definite inner product on E and thus identify E^* with E. Then we have a decomposition $E=F\oplus F^\perp$, and B becomes a selfadjoint operator $E\to E$ with a decomposition

$$B=\begin{pmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{pmatrix},$$

where $B_{ij}^*=B_{ji}$. Now the operator corresponding to $B_{F^\perp}$ is B_{22}, and the operator corresponding to B^{F^*} is $B_{11}-B_{12}B_{22}^{-1}B_{21}$. From this formula it is obvious that for this operator to be defined, it is sufficient for B_{22} to be invertible.

The following Lemma is a generalization of Lemma 1.

Lemma 6. *For $Re D>N-1$ and any positive $B\in S^2E$, $C\in S^2F^*$*

$$\int_{S_+^2E^*}\delta(A_F-C)(\det A)^{(D-N-1)/2}e^{-\mathrm{tr}(AB)}dA=$$
$$\frac{\Gamma_N(D/2)}{\Gamma_M(D/2)}e^{-\mathrm{tr}(CB^{F^*})}(\det C)^{(D-M-1)/2}(\det B_{F^\perp})^{-D/2}, \tag{17}$$

Remark. The space S^2F^* inherits a volume element from F, so the delta-function $\delta(A_F-C)$ is well defined.

Proof. Denote the left hand side of (17) by $I(B,C)$. Let $S\in GL(E)$ be an automorphism which preserves F. Make a change of variable $A\to S^*AS$. Then $A_F\to S_F^*A_FS_F$, where S_F is the restriction of S to F. Thus we get

$$I(B,C)=|\det S|^{N-M}I(SBS^*,(S_F^*)^{-1}CS_F^{-1}). \tag{18}$$

Now choose B with $\det B_{F^\perp}=1$, and identify E^* with E using the map $B:E\to E^*$. Then E becomes a Euclidean space with the inner product B^{-1}. Now we can identify F^* with F using the map $B^{F^*}:F^*\to F$, thus F is also a Euclidean space.

After these identifications, integral (17) takes the form

$$\int_{A\in S^2_+E^*} \delta(A_F - C)(\det A)^{(D-N-1)/2} e^{-\mathrm{tr}A} dA. \tag{19}$$

Using the decomposition $E = F \oplus F^\perp$, we can write A in the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{20}$$

where $A^*_{ij} = A_{ji}$. This allows us to rewrite (19) as

$$\int_{A\in S^2_+E^*:A_{11}=C} (\det A)^{(D-N-1)/2} e^{-\mathrm{tr}A} dA. \tag{21}$$

To compute this integral, let us use the identity $\det A = \det A_{11} \det(A_{22} - A_{21}A_{11}^{-1}A_{12})$. Since we are integrating over such A that $A_{11} = C$, we get $\det A = \det C \det(A_{22} - A_{21}C^{-1}A_{12})$. After the change of variable $X = A_{22} - A_{21}C^{-1}A_{12}$, integral (21) takes the form

(22)

$$(\det C)^{(D-N-1)/2} e^{-\mathrm{tr}C} \int_{X\geq 0} (\det X)^{(D-N-1)/2} e^{-\mathrm{tr}X} dX \int_{\mathrm{Hom}(F^\perp,F)} e^{-\mathrm{tr}A^*_{12}C^{-1}A_{12}} dA_{12}$$

The second integral in (22) is Gaussian and equals

$$(2\pi)^{M(N-M)/2} (\det C)^{(M-N)/2}. \tag{23}$$

The first integral in (22) was computed in Lemma 1 and equals $\Gamma_{N-M}((D-M)/2)$. Using the identity

$$\frac{\Gamma_N(D/2)}{\Gamma_M(D/2)} = (2\pi)^{M(N-M)/2} \Gamma_{N-M}((D-M)/2). \tag{24}$$

we obtain the statement of the Lemma.

Now consider the distribution on S^2E^* (depending on C as a parameter)

$$\rho^{EF}_D(A, C) = \delta(A_F - C) \frac{\rho^E_D(A)}{\rho^F_D(C)} =$$

$$\pi^{(N-M)D/2} \frac{\Gamma_M(D/2)}{\Gamma_N(D/2)} \delta(A_F - C)(\det C)^{-(D-M-1)/2} (\det A)^{(D-N-1)/2}, \tag{25}$$

for $\mathrm{Re}D > N - 1$.

For $f \in \mathcal{S}(S^2_+E^*)$, define a function $I^{EF}_D(f)$ on $C > 0$, by setting

$$I^{EF}_D(f)(C) = \int_{S^2_+E^*} \rho^{EF}_D(A, C) f(A) dA. \tag{26}$$

Since $(\det C)^{-1} \leq (\det A)^{-1} (\mathrm{tr}A)^{N-M}$, the function $I^{EF}_D(f)$ extends continuously to $C \geq 0$, i.e. defines a continuous, rapidly decaying function on $S^2_+F^*$.

Remark. Although the power of $(\det C)^{-1}$ in (25) is more than that of $\det A$, for small $\det C$ the intersection of the region of integration in (26) with any fixed ball has area $\sim (\det C)^{(N-M)/2}$ (because of presence of the delta-function), which helps compensate the negative exponent of $\det C$.

Using Lemma 6, it is easy to compute the Fourier transform $\hat{\rho}^{EF}_D(B, C)$ of $\rho^{EF}_D(A, C)$ with respect to A.

Lemma 7.

$$\hat{\rho}_D^{EF}(B,C) = \lim_{J\to +0} \pi^{(N-M)D/2} e^{-i\mathrm{tr}(CB^{F^*})} \det(iB_{F^\perp}+J)^{-D/2}. \tag{27}$$

Proof. Analogous to the proof of Lemma 2.

Let Δ_2 be the operator Δ acting on $\mathcal{S}(S^2F^\perp)$.

Corollary 8. *If $ReD > N+1$, then*

$$\Delta_2 \rho_D^{EF}(A,C) = \pi^N \rho_{D-2}^{EF}(A,C). \tag{28}$$

(here Δ_2 acts on A).

Proof. The proof is immediate after passing to Fourier transforms.

Corollary 9. *For any $f \in \mathcal{S}(S_+^2E^*)$, $I_D^{EF}(f)$ is a Schwarz function of C, and for any $C \geq 0$ the function $I_D^{EF}(f)(C)$, regarded as a function of D, analytically continues to an entire function.*

Proof. Similar to the Proof of Corollary 4.

Proposition 10. *Let $D \geq 0$ be a nonnegative integer, and V be a vector space of dimension D with a positive definite symmetric bilinear form β. Then for any $f \in \mathcal{S}(S_+^2E^*)$ and $y : F \to V$ one has*

$$I_D^{EF}(f)(y^*(\beta)) = \int_{x\in \mathrm{Hom}(E,V):x|_F=y} f(x^*(\beta))dx. \tag{29}$$

Proof. As in Proposition 5, it is enough to consider $f(A) = e^{-\mathrm{tr}(AB)}$. In this case, on the r.h.s. of (19) is a Gaussian integral, whose value is

$$\pi^{(N-M)D/2} e^{-\mathrm{tr}(y^*(\beta)B^{F^*})} (\det B_{F^\perp})^{-D/2}.$$

Thus, the Proposition follows from Lemma 6.

Definition. *I_D^{EF} is called the D-dimensional integral with external parameters.*

As before, the D-dimensional integral can be computed using Fourier transform. Let $f \in \mathcal{S}(S^2E^*)$. By Plancherel formula,

$$I_D^{EF}(f)(C) = (2\pi)^{-N(N+1)/2} \int_{S^2E^*} \hat{f}(B)\overline{\hat{\rho}_{\bar{D}}^{EF}(B,C)}dB, \tag{30}$$

where $\hat{f}$ is the Fourier transform of f. Here

$$\overline{\hat{\rho}_{\bar{D}}^{EF}(B,C)} = \lim_{J\to +0} \pi^{(N-M)D/2} e^{i\mathrm{tr}(CB^{F^*})} \det(-iB_{F^\perp}+J)^{-D/2}.$$

For Re$D < 2$, this formula can be written as

$$I_D^{EF}(f)(C) =$$

(31)
$$2^{-N(N+1)/2}\pi^{((N-M)D-N(N+1))/2}\int_{S^2E^*}\hat{f}(B)e^{\frac{\pi i D}{4}\sigma(B_{F^\perp})}e^{\mathrm{itr}(CB^{F^*})}|\det(B_{F^\perp})|^{-D/2}dB.$$

If Re$D \geq 2$, the distribution $\overline{\hat{\rho}_D^{EF}(B,C)}$ is singular, so in order to define $I_D^{EF}(f)$ by a direct integration formula, we need to use the operator Δ. Namely, from Lemma 6 it follows that

$$(\det C)^{-1}\Delta\hat{\rho}_D^{EF}(B,C) = (-1)^{N-M}p_{NM}(D)\rho_{D+2}^{EF}(B),$$

(32)
$$p_{NM}(D) = \pi^{M-N}\prod_{l=M+1}^{N}\frac{D+l-N}{2}$$

So if Re$D < 2+2K$, we can define I_D^{EF} by the formula

$$I_D^{EF}(f)(C) = 2^{-N(N+1)/2}\pi^{((N-M)D-N(N+1))/2}(\det C)^{-K}\prod_{J=1}^{K}p_N(D-2J)^{-1}\times$$
$$\int_{S^2E^*}\Delta^K\hat{f}(B)e^{\mathrm{itr}(CB^{F^*})}e^{\frac{\pi i(D-2K)}{4}\sigma(B)}|\det(B)|^{(-D+2K)/2}dB.$$

Thus we have defined an operator $I_D^{EF}:\mathcal{S}(S^2_+E^*)\to\mathcal{S}(S^2_+F^*)$. It is clear that $I_D^{FG}I_D^{EF} = I_D^{EG}$ if $E\supset F\supset G$, and $I_D^{E0} = I_D^E$.

Now we can define the D-dimensional integral of functions with polynomial growth. Let f be a smooth function on S^2E^* such that there exists $a\in\mathbb{R}$ for which f and all its derivatives satisfy the condition $\phi(A) = O(||A||^a)$, $||A||\to\infty$ (for any norm on S^2E^*). In this case, the integral $I_D^{EF}(f)$ is defined for Re$D << 0$. Indeed, the condition on f implies that the Fourier transform $\hat{f}$ is a distribution defined on functions with sufficiently many continuous derivatives having polynomial growth at infinity. Thus, we can use formula (30) to define $I_D^{EF}(f)$ for Re$D << 0$.

Remark. The operator I_D^{EF} has rather strange properties. For example, if P is any polynomial, then $I_D^E(P(A)) = 0$. Indeed, the Fourier transform of P is a distribution supported at 0, so for sufficiently negative D formula (14) gives 0. Thus, I_D^E is not continuous in any topology in which polynomials are dense.

Example. Let $N = 1$, $M = 0$, and f be a rational function, regular at $x\geq 0$. Write f as a sum of simple fractions: $f(x) = f_0(x) + \sum_i \alpha_i(x+a_i)^{-n_i}$, where f_0 is a polynomial, $a_i\in\mathbb{C}\setminus\mathbb{R}_-$, and n_i are positive integers. Then

$$I_D^E(f) = \pi^{D/2}\sum_i\alpha_i\frac{\Gamma(n_i-D/2)}{\Gamma(n_i)}a_i^{n_i-\frac{D}{2}}.$$

Thus we see that if f is satisfies the above condition of polynomial growth then $I_D^{EF}(f)(C)$ is holomorphic in D for Re$D << 0$. In general, we cannot expect that this function will analytically continue to the whole complex plane. However, for functions of a certain algebraic nature, it extends meromorphically to the whole complex plane.

3. D-dimensional integral of functions arising from Feynman diagrams.

Definition. *A function f on $S^2_+E^*$ is called a function of Feynman type if it has the form*

$$f(A)=\frac{P(A)}{\prod_{j=1}^{l}(tr(AB_j)+m_j^2)},$$

where P is a polynomial, $B_j\ge 0$, such that $(\sum t_iB_i)_{F^\perp}$ is nondegenerate for $t_i>0$, and $m_j\ne 0$ are real numbers.

Proposition 11. *If f is of Feynman type then $I_D^{EF}(f)$ extends meromorphically to the whole complex plane.*

Proof. We will prove the statement for $P=1$. For general P, the proof is similar to the case $P=1$.

Using the formula $b^{-1}=\int_0^\infty e^{-tb}dt$, we can rewrite $I_D^{EF}(f)$ in the form

$$I_D^{EF}(f)=\int_{t_1,\dots,t_l\in[0,\infty)} e^{-\sum t_jm_j^2}I_D^{EF}(e^{-\mathrm{tr}(A(\sum t_jB_j))})dt. \tag{33}$$

Set $B(t)=\sum t_jB_j$. Using Lemma 6, we can rewrite this in the form

$$I_D^{EF}(f)(C)=\pi^{(N-M)/2}\int_{t_1,\dots,t_l\in[0,\infty)} e^{-\sum t_jm_j^2-\mathrm{tr}(CB^{F^*}(t))}(\det B_{F^\perp}(t))^{-D/2}dt. \tag{34}$$

Now we will use the following theorem of Bernstein, which is proved with the help of the theory of D-modules.

Bernstein's theorem. Let $Q(t)$ be any polynomial in l variables. Then there exists a polynomial differential operator $L(D)$ in l variables, with coefficients depending polynomially on D, and a polynomial q in D, such that $L(D)Q^{-D/2}=q(D)Q^{-1-D/2}$.

This theorem implies the following statement, which is a special case of a result of Bernstein.

Corollary. Assume that Q takes positive values for $t_j>0$, and let g be a rapidly decaying smooth function defined for $t_i\ge 0$, such that its derivatives are also rapidly decaying. Then the integral

$$I(D,g)=\int_{t_i\in[0,\infty)} g(t)Q(t)^{-D/2}dt, \tag{35}$$

convergent for Re$D<<0$, extends meromorphically to the whole complex plane.

The corollary is proved by induction in l (the number of t_j). The base of induction is trivial. Now, using Bernstein's theorem and integration by parts, we can write

$$\begin{gathered}I(D+2,g)=q(D)^{-1}\int g(t)L(D)(Q(t)^{-D/2})dt=\\ q(D)^{-1}\int (L(D)^*g(t))Q(t)^{-D/2}dt+C(D)=q(D)^{-1}I(D,L(D)^*g)+C(D),\end{gathered} \tag{36}$$

where $C(D)$ is the integral over the boundary, which is meromorphic by the assumption of induction.

Now we apply the corollary to our situation, and obtain the proof of the Proposition for $P = 1$.

Remark 1. Using Hironaka's theorem about the existence of a smooth resolution of singularities for a compact projective variety, it is possible to prove that I_D^{EF} is meromorphic in D in a much more general situation than in Proposition 11. For example, as was explained to me by M.Kontsevich, one can prove that if f is a rational function without singularities in $S_+^2 E^*$, then $I_D^{EF}(f)$ is meromorphic.

It is also possible to show that for rational f as above the poles of $I_D^{EF}(f)$ are all rational, and belong to a finite number of arithmetic progressions.

4. Dimensional regularization of Feynman integrals.

Proposition 11 allows to define the procedure of dimensional regularization of Feynman integrals in quantum field theory.

Suppose that we have a quantum field theory given by some Lagrangian, and we want to compute its correlation functions. According to Feynman rules, such a correlation function equals to the sum of amplitudes of all Feynman diagrams (graphs), which are given by some finite-dimensional integrals.

We will work in the Euclidean setting and in momentum space. For simplicity we will assume that our theory contains only scalar massive fields. Then the amplitude of any Feynman graph Γ is given by the formula

$$\int_{x \in \mathrm{Hom}(E,V): x|_F = y} g(x)dx, y \in \mathrm{Hom}(F, V), \tag{37}$$

where V is the momentum space (a Euclidean space of dimension d with inner product β), $E = H_c^1(\Gamma \setminus \partial\Gamma, \mathbb{R})$ (the cohomology with compact supports), F is the image of the map $H^0(\partial\Gamma, \mathbb{R}) \to H_c^1(\Gamma \setminus \partial\Gamma, \mathbb{R})$ in the long exact sequence of cohomology, and g is a function on $\mathrm{Hom}(E, V)$ of the form $g(z) = f(z^*(\beta))$, where f is a certain function of Feynman type attached to Γ.

Remark. It is clear from the long exact sequence that $E/F = H^1(\Gamma, \mathbb{R})$.

Since cohomology groups are defined over $\mathbb{Z}$, they have natural volume elements. This defines the volume element dx in (37).

The problem is that integral (37) is often divergent, since the function g does not decay rapidly enough.

Dimensional regularization is a method to give meaning to the integral (37). This is done as follows. Recall that since f is of Feynman type, $I_D^{EF}(f)$ is meromorphic in D.

Definition. *Dimensional regularization of integral (37) is the regular part of $I_D^{EF}(f)$ at $D = d$, i.e. $\frac{1}{2\pi i}\oint_{|D-d|=\varepsilon} I_D^{EF}(f) d\ln(D-d)$.*

Remark. In general, we have to consider functions f which (even before the regularization) depend meromorphically on D. This is caused by the fact that our graph Γ may contain divergent proper subgraphs, which have to have been regularized before we start working with Γ. Since regularization is performed by adding D-dependent counterterms, we will have to work with functions which depend meromorphically on D.

5. D-dimensional Stokes formula.

Let $F \subset E$ be vector spaces with volume elements, $\dim E = N$, $\dim F = M$, and D be any complex number. Define the space $\Omega^{\rm top}_{EF}$ of (Schwarz) top degree forms to be the space $\mathcal{S}(S^2_+E^*)$, and the space $\Omega^{\rm top-1}_{EF}$ of forms of degree top $-\ 1$ to be the space of Schwarz functions $\omega : S^2_+E^* \to \mathrm{Hom}(E/F, E)$. That is, $\Omega^{\rm top}_{EF} = \mathrm{Hom}(E/F, E) \otimes \mathcal{S}(S^2_+E^*)$.

If D is an integer, and V is a Euclidean space of dimension D with inner product β, then we have a linear map $\theta : \mathrm{Hom}(E, V) \to \mathrm{Hom}(F, V)$ given by restriction, and any $f \in \Omega^{\rm top}_{EF}$ indeed defines a top differential form $f(x^*(\beta))dx$ on each fiber of θ (dx is the natural volume form on the fiber).

Also, any $\omega \in \Omega^{\rm top-1}_{EF}$ defines a top $-$ 1-form on the fiber. Indeed, let $z : (E/F)^* \to V$ be a linear map. We regard z as a constant 1-form on the fibers of θ. Then the top $-$ 1-form corresponding to Ω is defined by the formula $(\omega \wedge z)(x) = \mathrm{tr}(\omega\beta(x, z))dx$, where $\beta(x, z) \in \mathrm{Hom}(E \otimes (E/F)^*, \mathbb{R})$ is regarded as an operator $E \to E/F$.

Recall that the group $GL(E)$ acts on $S^2_+E^*$ by $A \to S^*AS$, $S \in GL(E)$. Thus, any element X of the Lie algebra $\mathrm{End}(E)$ of $GL(E)$ defines a vector field on $S^2_+E^*$: $L_X f(A) = \frac{d}{dt}|_{t=0} f(e^{X^*t}Ae^{Xt})$.

For any $\omega = \sum X_i \otimes f_i \in \Omega^{\rm top-1}_{EF}$, $X_i \in \mathrm{Hom}(E/F, E)$, $f_i \in \mathcal{S}(S^2_+E^*)$, define the form $d\omega \in \Omega^{\rm top}_{EF}$ by the formula

$$d\omega = \sum_i (L_{X_i} f_i + D\mathrm{tr}(X_i) f_i), \tag{38}$$

where X_i are regarded as elements of $\mathrm{End}(E)$. Thus, we have defined a linear map $d : \Omega^{\rm top-1}_{EF} \to \Omega^{\rm top}_{EF}$. It is easy to see that if D is an integer, d coincides with the usual differential.

Proposition 12. *(Stokes formula.) $I^{EF}_D(d\omega) = 0$ for any $\omega \in \Omega^{\rm top-1}_{EF}$.*

Proof. It is enough to prove the formula for $\mathrm{Re}D >> 0$; for other values of D the formula follows by analytic continuation.

For $\mathrm{Re}D >> 0$, the formula follows by integration by parts from the definition (26) of I^{EF}_D, as $L_X((\det A)^{(D-N-1)/2}dA) = D\mathrm{tr}(X)(\det A)^{(D-N-1)/2}dA$.