Fall Term Exam, N^0. 1 (solution by L. Jeffrey and S. Wu)

Problem:

Prove via path integrals the character valued index theorem for the Dirac operator.

That is, let M be a spin manifold and let g be an isometry of M, with a lift to the spin bundles. Calculate

$$Tr(-1)^F g$$

by path integrals.

That is, write down the path integral for sections of a suitable M bundle over a circle. Taking the circumference of the circle to be small, compute the integral explicitly and obtain the Atiyah-Bott formula for (1).

Solution:

We use a supersymmetric quantum field theory where the fields are maps $X : \mathbf{R}^{1,1} \to M$ satisfying a periodicity condition. We write the maps as $X(t, \theta)$ where t is the even variable and θ the odd variable: decomposing into bosonic and fermionic fields we write

$$X(t, \theta) = x(t) + \theta\psi(t).$$

The Lagrangian is

$$L(X) = \int dt d\theta g_{IJ} \frac{dX^I}{dt} DX^J$$

where we have introduced a metric g_{IJ} on M and the operator

$$D = \frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t}.$$

In components the action becomes

$$L(X) = \int dt\{g_{IJ}\frac{dx^I}{dt}\frac{dx^J}{dt} + \frac{i}{2}g_{IJ}\psi^I\frac{D\psi^J}{Dt}\},$$

where $D\psi^J/Dt$ denotes the covariant derivative (with respect to the Levi-Civita connection) in the direction of the tangent vector $x_*(d/dt)$ in M. The supersymmetry operator is

$$Q = \frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial t}.$$

Upon quantization the space of states is identified with the sections $\mathcal{H}_\pm$ of the spin bundle $\mathcal{S}_\pm$ on M and the operator Q is identified with the Dirac operator $\mathcal{D}$ on M. We introduce the operator $(-1)^F$ on $\mathcal{H}_\pm$ which is defined to have the values $+1$ on $\mathcal{H}_+$ (states with an even number of fermions) and (-1) on $\mathcal{H}_-$ (states with an odd number of fermions). The index of the Dirac operator $\mathcal{D}$ is then

$$\mathrm{Index}(\mathcal{D}) = \mathrm{Trace}(-1)^F = \mathrm{Trace}(-1)^F e^{-\beta H}, \tag{1}$$

where we have introduced the Hamiltonian H and a positive parameter β. Notice that since the Dirac operator $\mathcal{D}$ gives an isomorphism between the eigenspaces $\mathcal{H}_+^\lambda$ of H on $\mathcal{H}_+$ with a positive eigenvalue λ and the corresponding eigenspaces $\mathcal{H}_-^\lambda$ of the action of H on $\mathcal{H}_-$, the trace of $(-1)^F e^{-\beta H}$ is in fact independent of β. This can be computed by the path integral

$$\mathrm{Trace}(-1)^F e^{-\beta H} = \int \mathcal{D}X e^{-\int_0^\beta L(X)dt}, \tag{2}$$

where we integrate over fields $X(t)$ satisfying periodic boundary conditions $X(t+1) = X(t)$.

The corresponding character valued index is

$$\mathrm{Index}_g(\mathcal{D}) = \mathrm{Trace}(-1)^F g = \mathrm{Trace}(-1)^F g e^{-\beta H}. \tag{3}$$

Since the isometry g commutes with the Dirac operator, the trace $\mathrm{Trace}(-1)^F g e^{-\beta H}$ is again independent of β. It is given by the path integral

$$\mathrm{Trace}(-1)^F g e^{-\beta H} = \int \mathcal{D}X e^{-\int_0^\beta L(X)dt}, \tag{4}$$

which is the same as (2) but with the boundary conditions on the fields changed to

$$x(t+1) = gx(t), \quad \psi(t+1) = g\psi(t). \tag{5}$$

We now rescale the time variable t by replacing it by $t' = \beta t$ where $t \in [0,1]$: this is equivalent to replacing the action $\int_0^\beta L(X)dt$ by

$$\frac{1}{\beta}\int_0^1 \left\{ g_{IJ}\frac{dx^I}{dt}\frac{dx^J}{dt} + \frac{i}{2} g_{IJ}\psi^I \frac{D\psi^J}{Dt} \right\}, \tag{6}$$

where we have also rescaled the fermionic variables ψ^I by multiplying them by a factor $1/\sqrt{\beta}$.

We must first identify the solutions of the Euler-Lagrange equation. This equation is

$$dX^I/dt = 0;$$

given the twisted periodicity condition (5) the only solutions are constant paths $x_0(t) = x_0$ which send $\mathbf{R}$ to a point x_0 in the fixed point set $Z = M^g$ of g, and constant paths $\psi_0(t) = \psi_0$ which send $\mathbf{R}$ to a point ψ_0 in the tangent space $\Pi T_{x_0} M$.

If $X_0 = (x_0, \psi_0)$ is a solution of the Euler-Lagrange equation, we write an infinitesimal variation δX at X_0 as

$$\delta X = (\alpha, \eta)$$

(where α is the bosonic piece, a path in $T_{x_0}M$, and η is the fermionic piece, a path in $\Pi T_{x_0}M$). To avoid overcounting infinitesimal variations δX which are simply deformations within the class of solutions X_0 of the Euler-Lagrange equation, we impose the conditions

$$\int_0^1 \alpha(t)dt = 0, \quad \int_0^1 \eta(t)dt = 0. \tag{7}$$

We decompose the infinitesimal variations (α, η) further into

$$\alpha = \alpha^{||} + \alpha^{\perp}, \quad \eta = \eta^{||} + \eta^{\perp}$$

where $\alpha^{||}$ is a path in the tangent space $T_{x_0}Z$ at x_0 to the fixed point set Z, while $\alpha^{\perp}$ is a path in the fibre $\mathcal{N}_{x_0}$ of the normal bundle $\mathcal{N}$ to Z in M. The fermionic piece η is similarly decomposed into a path $\eta^{||}$ in the tangent space $T_{x_0}Z$ and a path $\eta^{\perp}$ in the fibre $\mathcal{N}_{x_0}$ of the normal bundle. These components separately satisfy the condition (7). The endpoint conditions become

$$\alpha^{||}(1) = \alpha^{||}(0), \quad \eta^{||}(1) = \eta^{||}(0) \tag{8}$$

$$\alpha^{\perp}(1) = g\alpha^{\perp}(0), \quad \eta^{\perp}(1) = g\eta^{\perp}(0) \tag{9}$$

where in (9) g acts as an automorphism of the fibres of the normal bundle to Z in M.

Expanding the Lagrangian to second order in (α, η) we find $L = L^{||} + L^{\perp}$, where

$$L^{||} = \frac{1}{\beta}\int_0^1 dt \left(\frac{1}{2}|\frac{d\alpha^{||}}{dt}|^2 - \frac{i}{4}\psi_0^I \psi_0^J R_{IJKL}(\alpha^{||})^K \frac{d(\alpha^{||})^L}{dt} + \frac{i}{2}(\eta^{||}, \frac{d\eta^{||}}{dt}) \right) \tag{10}$$

while

$$L^{\perp} = \frac{1}{\beta}\int_0^1 dt \left(\frac{1}{2}|\frac{d\alpha^{\perp}}{dt}|^2 - \frac{i}{4}\psi_0^I \psi_0^J R_{IJKL}(\alpha^{\perp})^K \frac{d(\alpha^{\perp})^L}{dt} + \frac{i}{2}(\eta^{\perp}, \frac{d\eta^{\perp}}{dt}) \right). \tag{11}$$

(Because the isometry g preserves the Levi-Civita connection, the fixed point set Z is totally geodesic. The solutions ψ_0 to the Euler-Lagrange equation are tangent to the fixed point set Z. Thus, in (10) the indices I, J, K, L run over a basis for the tangent bundle to the fixed point set Z, while in (11) the indices I, J run over a basis for the tangent space to Z while the indices K, L run over a basis for the normal bundle $\mathcal{N}$ to the fixed point set.)

The bosonic parts of these Lagrangians are given by a differential operator $(1/\beta)D_B$ of order 2 acting on $\alpha^{||}$ (resp. $\alpha^{\perp}$), where D_B is defined by

$$(D_B)_{KL} = -\frac{d^2}{dt^2}\delta_{KL} - F_{KL}\frac{d}{dt}, \tag{12}$$

where $F = (F_{KL})$ is the curvature $(i/2)\psi_0^I\psi_0^J R_{IJKL}$ of the Levi-Civita connection on M. Similarly the fermionic Lagrangians are given by a differential operator $(1/\beta)\ D_F$ (where $D_F = i\frac{D}{Dt}$ has order 1) acting on $\eta^{||}$ (resp. $\eta^{\perp}$). (At this point we may use normal coordinates to identify the covariant derivative D/Dt with the ordinary derivative d/dt.)

We compute the path integral (4) perturbatively. Since the result is independent of β, we may evaluate the integral in the limit $\beta \to 0$: however, since the propagator is multiplied by a factor β, this means that in the limit $\beta \to 0$ the exact answer is given by the leading order term in perturbation theory (i.e. the ratio of determinants arising from the Gaussian integral). In other words, the path integral (4) becomes

$$\frac{\mathrm{Pf}(D_F)}{\sqrt{\det D_B}}. \tag{13}$$

Since the operators D_B and D_F preserve the tangent bundle to Z and the normal bundle to Z (in other words D_B decomposes into operators

$$D_B = {D_B}^{||} \oplus {D_B}^{\perp},$$

and similarly for D_F: this will follow from the decomposition (10)-(11) of the Lagrangian given below) our path integral decomposes as

$$\int_Z \frac{\mathrm{Pf}({D_F}^{||})}{\sqrt{\det {D_B}^{||}}} \frac{\mathrm{Pf}({D_F}^{\perp})}{\sqrt{\det {D_B}^{\perp}}}.$$

The computation of

$$\frac{\mathrm{Pf}({D_F}^{||})}{\sqrt{\det {D_B}^{||}}}$$

is identical to the computation presented in the lecture and yields

$$\hat{A}(F^{||}) = \frac{1}{(2\pi)^{\dim Z/2}} \prod_{a=1}^{\dim Z/2} \frac{F_a^{||}/2}{\sinh F_a^{||}/2}. \tag{14}$$

$\hat{A}(Z) = \hat{A}(F^{||}/(2\pi))$ represents the A-hat genus of the fixed point set Z. Here, we have decomposed the curvature $F^{||}$ of the Levi-Civita connection of Z (which is a matrix valued two-form on Z with values in skew-symmetric square matrices of dimension $\dim Z$) into its Chern roots: in other words we have skew diagonalized it and written it in the form

$$F^{||} = \begin{pmatrix} 0 & F^1 & 0 & 0 & 0\dots & 0 \\ -F^1 & 0 & 0 & 0 & 0\dots & 0 \\ 0 & 0 & 0 & F^2 & 0\dots & 0 \\ 0 & 0 & -F^2 & 0 & 0\dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \dots \end{pmatrix} \tag{15}$$

The computation of the determinants for ${D_B}^\perp$ and ${D_F}^\perp$ differs from the computation presented in the lecture only because new periodicity conditions (5) for the fields have been introduced. We first observe that since g is orientation preserving, the normal bundle $\mathcal{N}$ to each component of the fixed point set Z of g is even dimensional. Moreover, the multiplicity of the eigenvalue -1 is even. Therefore the fiber $\mathcal{N}_{x_0}$ over $x_0 \in Z$ decomposes into subspaces V_b invariant under g, $b = 1, \ldots, (\dim_{\mathbf{R}} \mathcal{N})/2$. Each V_b has real dimension 2 and g acts on V_b as

$$g = \begin{pmatrix} \cos\vartheta_b & \sin\vartheta_b \\ -\sin\vartheta_b & \cos\vartheta_b \end{pmatrix} \tag{16}$$

Here a pair of eigenspaces of eigenvalue -1 corresponds to one V_b with $\vartheta_b = \pi$. (Alternatively, identifying each eigenspace V_b with $\mathbf{C}$ the isometry g acts as multiplication by $e^{i\vartheta_b}$. Of course, here the ϑ_b need not be distinct.) Since the isometry g preserves the Levi-Civita connection, the basis of the normal bundle which skew-diagonalizes the curvature (as in (15)) may be chosen to coincide with the basis of eigenspaces of g. Thus we may compute the action of ${D_B}^\perp$ and ${D_F}^\perp$ acting on paths $\alpha_b^\perp, \eta_b^\perp$ taking values in one eigenspace V_b.

In fact, it is convenient to replace $\alpha_b^\perp$ and $\eta_b^\perp$ by paths $\alpha_{b,\pm}^\perp, \eta_{b,\pm}^\perp$ in the complexification $V_b \otimes \mathbf{C}$: this decomposes naturally as $V_b \otimes \mathbf{C} = V_b^+ \oplus V_b^-$, where g acts on $V_b^\pm$ with the eigenvalue $e^{\pm i\vartheta_b}$. Thus we may convert the endpoint conditions (5) to periodic endpoint conditions by writing

$$\alpha_{b,\pm}^\perp(t) = \exp(\pm it\vartheta_b)\zeta_b^\pm(t), \tag{17}$$

$$\eta_{b,\pm}^\perp(t) = \exp(\pm it\vartheta_b)\xi_b^\pm(t), \tag{18}$$

where $\zeta_b^\pm : \mathbf{R} \to V_b^\pm$ and $\xi_b^\pm : \mathbf{R} \to \Pi V_b^\pm$ satisfy the usual periodic boundary conditions $\zeta_b^\pm(t+1) = \zeta_b^\pm(t)$, $\xi_b^\pm(t+1) = \xi_b^\pm(t)$.

If we substitute (17) into the definition of D_B we find that the operator D_B acting on the variables $\zeta_b^\pm$ becomes instead (using the decomposition of $F^\perp$ corresponding to (15), where the eigenspaces are the eigenspaces $V_b^\pm$: the subspaces which diagonalize the curvature coincide with the eigenspaces of g because the isometry g is assumed to preserve the Riemannian metric on M)

$$(D_B)_b^\pm = -(\frac{d}{dt} \pm i\vartheta_b)^2 \pm F_b(\frac{d}{dt} \pm i\vartheta_b). \tag{19}$$

Since the eigenvalues of $\frac{d}{dt}$ on the ζ_b are $2\pi in$ for $n \in \mathbf{Z}$, we find that the eigenvalues are given by

$$(2\pi in \pm i\vartheta_b)^2 \pm F_b(2\pi in \pm i\vartheta_b). \tag{20}$$

To compute the determinant of D_B we take the product over all the eigenvalues.

The computation of the character-valued index differs from the ordinary index in that the fermionic Pfaffian now depends on the values of the eigenvalues ϑ_b for the isometry. Substituting (18) into $D_F = id/dt$ we find that the eigenvalues of D_F acting on the periodic fermionic paths $\xi_b^\pm$ are

$$2\pi n \pm \vartheta_b.$$

We shall make use of the identity

$$\prod_{n\geq 1}(1-\frac{z^2}{4\pi^2 n^2}) = \frac{\sin z/2}{z/2} \tag{21}$$

Combining all factors we find

$$\det D_B = \prod_{\pm}\prod_{n\in\mathbf{Z}}\prod_{b}(2\pi i n + i\vartheta_b)^2 \pm F_b(2\pi i n \pm i\vartheta_b) \tag{22}$$

while the fermionic Pfaffian is given by

$$(\mathrm{Pf} D_F)^2 = \prod_{\pm}\prod_{b}\prod_{n}(2\pi n \pm \vartheta_b). \tag{23}$$

Combining (22) and (23) we see that (up to a multiplicative constant from regularization)

$$\frac{(\mathrm{Pf} D_F)^2}{\det D_B} = \prod_{n\in\mathbf{Z}}\prod_{b,\pm}\frac{1}{2\pi n \pm (\vartheta_b + iF_b)}, \tag{24}$$

$$= \prod_b \frac{1}{\vartheta_b + iF_b}\prod_{\pm}\prod_{n\geq 1}\frac{1}{4\pi^2 n^2 - (\vartheta_b + iF_b)^2}$$

$$= \prod_b \frac{1}{2\sin\Big((\theta_b + iF_b)/2\Big)}.$$

The final answer for the character-valued index of the Dirac operator is then

$$\begin{aligned}\mathrm{Index}_g(\mathcal{D}) &= \frac{1}{(2\pi)^{\dim Z/2}}\int_Z \hat{A}(F^{||})\prod_{b=1}^{\dim_{\mathbf{R}}\mathcal{N}/2}\frac{1}{2\sin\Big((\vartheta_b + iF_b)/2\Big)} \\ &= \int_Z \hat{A}(Z)\prod_{b=1}^{\dim_{\mathbf{R}}\mathcal{N}/2}\frac{1}{2\sin\Big((\vartheta_b + iF_b/(2\pi))/2\Big)}.\end{aligned} \tag{25}$$

References:

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