Fall Term Exam, N°. 2 (solution by D. Freed)

Problem:

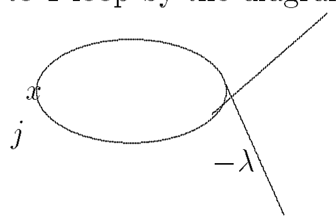
Compute in ϕ^4 theory in four dimensions the anomalous dimension of the operator ϕ^2, in one–loop approximation.

Solution:

Consider the Lagrangian density in Euclidean space

$$\mathcal{L} = \frac{1}{2}\,|d\phi|^2 + \frac{m^2}{2}\,\phi^2 + \frac{\lambda}{4!}\,\phi^4 - j\phi^2.$$

The coefficient j is renormalized to 1-loop by the diagram shown,



which contributes to the correlation function $\langle \phi^2\phi\phi\rangle$. Evaluating in momentum space, we obtain

$$\frac{-j\lambda}{2}\int \frac{d^4q}{(2\pi)^4}\quad \frac{1}{(q^2+m^2)\,((k-q)^2+m^2)},$$

where the factor of 2 comes from the symmetry of the diagram. This is logarithmically divergent, and cutting off at Λ we compute the divergence as $\Lambda \longrightarrow \infty$ to be

$$\frac{-j\lambda}{2}\cdot\frac{1}{(2\pi)^4}\cdot \operatorname{vol} S^3 \cdot \ell n\,\Lambda/\mu = \frac{-j\lambda}{16\pi^2}\,\ell n\,\Lambda/\mu,$$

where μ is a fixed constant.

To cancel this divergence, we renormalize j to

$$\hat{\jmath}_0 = j\left(1+\frac{\lambda}{16\pi^2}\,\ell n\,\Lambda/\mu\right).$$

The anomalous dimension of j is computed as $\gamma_j = \frac{\mu}{j}\,\frac{\partial j}{\partial\mu}$ with $\hat{\jmath}_0$ fixed and $\Lambda \longrightarrow \infty$. Thus

$$0 = \gamma_j\left(1+\frac{\lambda}{16\pi^2}\,\ell n\,\Lambda/\mu\right) - \frac{\lambda}{16\pi^2} + \frac{\beta}{16\pi^2}\,\ell n\Lambda/\mu,$$

where $\beta = \mu\,\frac{\partial\lambda}{\partial\mu}$ is the β-function. Since $\gamma_j = O(\lambda)$ and $\beta = O(\lambda^2)$ we find $\gamma_j = \frac{\lambda}{16\pi^2}$. Now the anomalous dimension of the Lagrangian vanishes, and so the anomalous dimension γ_{ϕ^2} of ϕ^2 equals $-\gamma_j$, or

$$\gamma_{\phi^2} = -\frac{\lambda}{16\pi^2}.$$