Fall Term Exam, N^0. 3 (solution by D. Freed)

Problem:

Determine the asymptotic behavior of perturbation theory in soft or cutoff scalar field theories, as follows. (Fermions and gauge fields introduce different behavior, not to mention other types of theories such as sigma models.)

(a) Consider the integral

$$Z(g) = \int_{\infty}^{\infty} d\phi \, \exp\left(-\frac{1}{2}\,\phi^2 - g\phi^4/4!\right). \tag{1}$$

Expand in an asymptotic series in g

$$Z(g) = \sum_k a_k g^k \tag{2}$$

and determine the large k behavior of a_k. Hint: write an integral representation for a_k and find a method of computing the integral for large k, for instance, Where is the maximum of the integrand?

(b) Interpret the answer to (a) as a statement about the number of graphs of a particular kind, weighted according to their automorphism groups.

(c) Now consider the path integral

$$Z(g) = \int D\phi \, e^{-L} \tag{3}$$

in a scalar field theory in n dimensions, with

$$L = \int d^n x \, \left(\frac{1}{2}|d\phi|^2 + \frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4\right). \tag{4}$$

Write down, again, an integral representation for the coefficients a_k in the asymptotic expansion of $Z(g)$ and determine the large k behavior, at least at a physical level of rigor, by finding the maximum of the integrand.

You should get a non-linear differential equation, and you should be able to prove that it has a spherically symmetric solution; it is believed that this solution does determine the large k behavior of the a_k in a cutoff ϕ^4 theory or in dimensions below four. In four dimensions, the much more complicated renormalization, which you've presumably ignored in the computation above, changes things (L must be redefined by incorporation of counterterms and the expansion in powers of g is not made as simply as you've presumably done above).

Solution:

(a) We write

$$Z(g) = \int_{-\infty}^{\infty} d\phi \; e^{-\phi^2/2 - g\phi^4/4!} \sim \sum_{k=0}^{\infty} a_k g^k$$

so that by Taylor's formula

$$a_k = \frac{(-1)^k}{k!(4!)^k} \quad \int_{-\infty}^{\infty} d\phi \; e^{-\phi^2/2} \; \phi^{4k}. \tag{1}$$

We can evaluate the integral in (1) exactly: set

$$\begin{aligned} F(J) &= \int_{-\infty}^{\infty} d\phi \; e^{-\phi^2/2 + J\phi} \\ &= \int_{-\infty}^{\infty} d\phi \; e^{-(\phi+J)^2/2} \; e^{J^2/2} \\ &= \sqrt{2\pi} \; e^{J^2/2}. \end{aligned}$$

The integral in (1) is the derivative

$$F^{(4k)}(J=0) = \sqrt{2\pi} \; \frac{(4k)!}{(2k)! 2^{2k}}, \tag{2}$$

so from (1) we have

$$a_k = \frac{(-1)^k (4k)! \; \sqrt{2\pi}}{k!(24)^k (2k)! \, 2^{2k}}. \tag{3}$$

Now we apply Sterling's formula

$$n! \sim \sqrt{2\pi} \; \left(\frac{n}{e}\right)^n \; \sqrt{n}$$

to deduce

$$a_k \sim (-1)^k \; \left(\frac{2k}{3e}\right)^k \; \sqrt{\frac{2}{k}}. \tag{4}$$

Alternatively, write the integral in (1) as

$$\int_{-\infty}^{\infty} d\phi \; \phi^{4k} \; e^{-\phi^2/2} = \int_{-\infty}^{\infty} d\phi \; e^{-\phi^2/2 + 2k \, \ell n \phi^2} \tag{5}$$

and find the maximum of the exponent at $\phi = \pm\sqrt{2k}$. At these points the second diverative of the exponents is -2, so the integral to a Gaussian approximation at either of these maxima is

$$\int_{-\infty}^{\infty} d\phi \; e^{-\phi^2/2+2k\,\ell n\phi^2} \sim (4k)^{2k} \; e^{-2k} \int_{-\infty}^{\infty} d\tilde{\phi} \; e^{-2\tilde{\phi}^2/2}$$
$$= (4k)^{2k} \; e^{-2k} \; \sqrt{\pi}.$$

Since there are two such maxima we find from (1) that

$$a_k \sim \frac{(-1)^k \, 2(4k)^{2k} \; e^{-2k} \; \sqrt{\pi}}{k!(24)^k}.$$

Using Sterling's formula we obtain agreement with (4).

(b) According to the Feynman calculus $|a_k|/_{\sqrt{2\pi}}$ counts the number of equivalence classes of graphs with the k 4-valent vertices, each graph Γ counted with a factor of ${}^1/_{\sharp\mathrm{Aut}\Gamma}$. We can check this directly by counting. Start with k 4-valent vertices, with the vertices numbered $1, 2, ..., k$ and the edges of each vertex labeled 1,2,3,4. Then each edge has a number ab attached, where a is the vertex number and b the label of the edge at that vertex. Now there is an obvious ordering of all edges (11, 12, 13, 14, 21, 22, 23, 24, etc.). We must attach the edges. For the first edge there are $(4k-1)$ choices, for the next available edge $(4k-3)$ choices, etc. So the total number of labeled graphs is

$$1 \cdot 3 \cdot 5 \cdot \cdots \cdot (4k-1) = \frac{(4k)!}{(2k)!\, 2^{2k}},$$

and since there are $k!$ automorphisms of the vertex numberings and $(4!)^k$ automorphisms of the edge labelings, our counting is consistent with (1).

(c) Now for the path integral we have, analogously to (1),

$$a_k = \frac{(-1)^k}{k!(4!)^k} \int D\phi \; e^{-\frac{1}{2}\int d^n x(|d\phi|^2+m^2\phi^2)} \left(\int d^n x \, \phi^4 \right)^k . \tag{6}$$

As in (5) we rewrite the integrand in (6) as

$$\exp\left\{ -\frac{1}{2} \int d^n x \, (|d\phi|^2 + m^2\phi^2) + k \; \ell n \int d^n x \; \phi^4 \right\}.$$

To find the maximum of the exponent we differentiate at ϕ in the direction $\dot{\phi}$ and set the result to zero:

$$0 = \int d^n x \, (-\Delta\phi - m^2\phi, \;\; \dot{\phi}) + \frac{4k \int d^n x \, \phi^3 \dot{\phi}}{\int d^n x \, \phi^4}$$

which implies

$$(\Delta + m^2)\phi = \frac{4k\,\phi^3}{\int d^n x\,\phi^4}. \tag{7}$$

(Our Laplacian Δ is nonnegative.)

We search for a spherically symmetric solution $\phi = \phi(r)$ for r the radial variable, so

$$\Delta\phi = -\frac{1}{r^{n-1}}\ \frac{d}{dr}\ \left(r^{n-1}\ \frac{d\phi}{dr}\right).$$

Thus (7) becomes

$$-\frac{d^2\phi}{dr^2} - \frac{n-1}{r}\ \frac{d\phi}{dr} + m^2\phi = \frac{k\phi^3}{\pi^2\ \int_0^\infty \phi^4(r)r^3\,dr}.$$

By rescaling ϕ it suffices to solve the equation

$$-\frac{d^2\phi}{dr^2} - \frac{n-1}{r}\ \frac{d\phi}{dr} + m^2\phi = k\phi^3; \tag{8}$$

namely, we replace the solution ϕ to (8) by

$$\left[\pi^2\ \int_0^\infty \phi^4(r)r^3\,dr\right]^{-2}\phi.$$

Since we want $\int_0^\infty \phi^4(r)\,r^3\,dr$ to be finite, we require

$$\lim_{r\to\infty} \phi(r) = 0. \tag{9}$$

Also, to make the first derivative term meaningful at $r = 0$, we require

$$\lim_{r\to 0} \phi'(r) = 0. \tag{10}$$

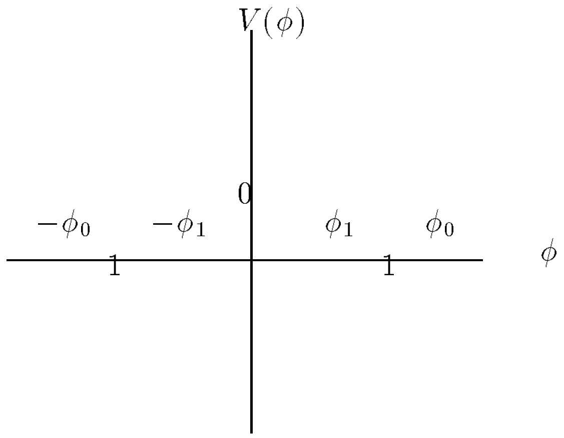
We now argue that a solution to (8) satisfying (9) and (10) exists.

For $n = 1$ equation (8) is Newton's equation

$$\frac{d^2\phi}{dr^2} = -V'(\phi) \tag{11}$$

for the potential

$$V(r) = \frac{k}{4}\,\phi^4 - \frac{m^2}{2}\,\phi^2.$$



There is a solution $\phi(r)$ to (11) with the desired boundary conditions where $\phi(0) = \phi_0$ is the positive zero of V. Picture a ball released in this potential at $\phi = \phi_0$. It rolls down and back up to $\phi = 0$, where it is at rest. For $n > 1$, we have

$$\frac{d^2\phi}{dr^2} = -V'(\phi) - \frac{n-1}{r}\ \frac{d\phi}{dr}.$$

In terms of our mechanical model, the ball feels an additional acceleration when moving to the left and an additional deceleration when moving to the right. The effect dies off with time (r) and is proportional to the speed. Also, the effect is bounded for small time if we have zero initial velocity (as prescribed by (9)). So now if we release the ball at $\phi = \phi_0$, it overshoots $\phi = 0$. Of course, if we release the ball at $\phi = \phi_1$, the minimum of the potential, it stays at rest. By continuity there is some value $\bar{\phi}$ between ϕ, and ϕ_0 where the ball released at $\phi = \bar{\phi}$ will roll exactly up to $\phi = 0$ as desired.