Fall Term Exam, N^0. 5 (solution by D. Freed)

Problem:

QCD is an $SU(3)$ gauge theory with fermions (quarks) which describes the nuclear forces. For this problem you only need to know a few facts. For instance, the theory contains a "current" operator, which is simply a Lorentz vector J^i (that is a local operator of spin one) that is conserved, $d_j J^i = 0$ (or $d * J = 0$), has zero anomalous dimension (we'll learn why later), and has the property that its two point function needs an additive c-number renormalization.

The renormalization two point function of J is therefore the limit as a cutoff Λ goes to infinity of

$$\int d^4x\ e^{ik\cdot x}\langle J^i(x)J^j(0)\rangle - A(\Lambda/\mu, g)(k^i k^j - \delta^{ij} k\cdot k). \tag{1}$$

Here g is the (dimensionless) coupling constant, Λ the cutoff, μ a mass that must be introduced in renormalization, and A is a funciton of the indicated arguments which must be "subtracted" to make the two point function finite. Also, the theory is asymptotically free and for present purposes you can ignore quark bare masses and assume that there are no parameters in the theory except g, μ, and Λ (and of course Λ is eventually eliminated by taking it to infinity). There is no multiplicative renormalization of J in (8) because the anomalous dimension vanishes. Because J is conserved, the expression in (8) is of the form

$$(k^i k^j - \delta^{ij} k^2) B(k^2, g, \mu) \tag{2}$$

where Λ is omitted as we want to consider the limit $\Lambda \to \infty$.

I suppose it also helps to know that J is built in terms of quarks as $J = \overline{q}\Gamma^i q$, so that the one loop approximation to $\langle JJ\rangle$ is given by a one loop diagram with a fermion loop that we computed in homework. (For those who are getting a paper and not internet copy of this, the loop is drawn below at the end of problem 7.) But all you really need to know is that in the free field theory, that is if $g = 0$, $A = a\ \ln(\Lambda/\mu)$, where a is a constant, independent of g.

(a) Write down the renormalization group equation obeyed by B.

(b) Use this equation, together with the fact that QCD is asymptotically free, to determine the large k^2 behavior of B.

This large k^2 behavior, after including some additional theoretical wrinkles to put it in a form that can be compared with experiment, is one of the classic tests of QCD.

Solution:

First, a reminder about the renormalization group equation. A perturbative theory depends on some coupling constants g and masses m. (For simplicity we assume there is just one of each.) Renormalization introduces a scale μ. The renormalization group operator is

$$\mathcal{D} = \mu\,\frac{\partial}{\partial\mu} + \beta\,\frac{\partial}{\partial g} + \gamma_m\, m\,\frac{\partial}{\partial m},$$

where β and γ_m (the *anomalous* dimension of mass) are in general functions of g and $\frac{m}{\mu}$. They describe how the parameters of the theory flow when μ is varied. Their precise form depends on the renormalization procedure; for dimensional regularization/minimal subtraction, they are independent of $\frac{m}{\mu}$. Now any operator $\mathcal{O}$ in the theory has an anomalous dimension $\gamma_{\mathcal{O}}$ (which is also a function of g and $\frac{m}{\mu}$). Since the operators are renormalized they also depend on μ, and the meaning of the anomalous dimension is that they satisfy the renormalization gorup equation

$$\mathcal{D}\mathcal{O} = \gamma_{\mathcal{O}}\mathcal{O}.$$

In general the operators mix under renormalization: the renormalization of a given operator $\mathcal{O}_0$ involve operators of equal and lower dimension. Thus $\gamma_{\mathcal{O}}$ is a triangular matrix acting on operators of dimension $\leq k$ for some fixed k. Of course, the mixing only occurs among operators with the same symmetries. In the renormalization group equation $\mathcal{D}\mathcal{O} = \gamma_{\mathcal{O}}\mathcal{O}$, then, $\mathcal{O}$ is a vector of operators and γ_{0} a matrix (of functions of g and $\frac{m}{\mu}$). Similar remarks apply to equation (1) below, though we write it as if there is no mixing. Notice that this holds for the elementary operators (i.e., the fields which appear in the Lagrangian) as well as for composite operators. What defines β, γ_m, and $\gamma_{\mathcal{O}}$ is that correlation functions are preserved under the renormalization group flow. That is, for operators $\mathcal{O}_1,\dots,\mathcal{O}_n$ and *distinct* points $x_1,\dots,x_n$, we have

$$\mathcal{D}\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle = (\gamma_{\mathcal{O}_1}+\cdots+\gamma_{\mathcal{O}_n})\;\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\rangle. \tag{1}$$

The correlation function blows-up as some points x, come together, and if we consider coinciding points this necessitates further renormalization. Such renormalizations will introduce additional terms in (1).

(a) The function B is defined by

$$\lim_{\Lambda\to\infty}\left\{\int\limits_{|x|\geq\frac{1}{\Lambda}} d^4x\; e^{-ik\cdot x}\,\langle J^i(x)J^j(0)\rangle - A\left({}^{\Lambda}/_{\mu}, g_{{}^{n}/\mu}\right)\;(k^ik^j-\delta^{ij}k^2)\right\}$$
$$= B(k^2,g,\mu)\,(k^ik^j-\delta^{ij}k^2), \tag{2}$$

where $g_{\Lambda/\mu}$ is the effective coupling constant with initial value g_1. Since there are no masses the renormalization group operator is

$$\mathcal{D} = \mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}.$$

We are given $\gamma_J = 0$, so differentiating both sides of (2) using (1), we obtain

$$\mathcal{D}B = -\lim_{\Lambda\to\infty}\mathcal{D}A \tag{3}$$

as the renormalization group equation of B. There is no operator mixing since J is the lowest dimensional vector operator.

Now asymptotic freedom means $\lim_{\Lambda\to\infty} g_\Lambda = 0$ and so to leading order

$$\lim_{\Lambda\to\infty} \mathcal{D}A = \lim_{\Lambda\to\infty} \mathcal{D}A({}^{\Lambda}/_{\mu}, 0) = \lim_{\Lambda\to\infty} \mu\frac{\partial}{\partial\mu}\,(a\,\ell n^{\Lambda}/_{\mu}) = -a.$$

Hence (3) becomes

$$\mathcal{D}B = a.$$

(b) It follows from (2) that the naive engineering dimension of B is zero. Thus simple scaling combined with (4) gives

$$\begin{aligned} B(e^{2t}k_0^2,\, g_1, \mu) &= B(k_0^2,\, g_1,\, e^{-t}\mu) \\ &= B(k_0^2,\, g_{e^t},\, \mu) - at. \end{aligned} \tag{5}$$

Now take $k_0^2 = 1$ and set $k^2 = e^{2t}$. Then taking $t \to \infty$ we learn

$$B(k^2, g_1, \mu) \sim -\frac{a}{2}\,\ell n\, {}^{k^2}/_{\mu^2} \qquad \text{as} \qquad k \longrightarrow \infty.$$

Observe that this is the answer for the free theory. In other words, in the absence of anomalous dimensions the high momentum behavior of a quantity in an asyptotically free theory is the same as the high momentum behavior in the free theory.