Fall Term Exam, N^0. 6 (solution by D. Freed)

Problem:

Let $\mathcal{O}(x)$ be a local operator which annihilates the vacuum:

$$\mathcal{O}(x)|\Omega\rangle = 0. \tag{1}$$

Show at least at a physical level of rigor that $\mathcal{O}(x) = 0$.

Now extend this slightly. Suppose that for some distinct points x and y $\langle\Omega|\mathcal{O}^*(y)\mathcal{O}(x)|\Omega\rangle = 0$. ($\mathcal{O}^*$ is the hermitian adjoint of $\mathcal{O}$.) Prove that $\mathcal{O} = 0$.

Solution:

We recall that in a quantum field theory the correlation function (better distribution) $F(x_1, \dots, x_n) = \langle\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots\mathcal{O}_n(x_n)\rangle$ is initially defined for $x_1, \dots, x_n$ distinct points in Minkowski space V. It is the boundary value of an *analytic* function on a connected domain $\mathcal{D} \subset (V_{\mathbb{C}})^{\times n}$ where $\mathcal{D}$ includes:

(1) The subspace of $E^{\times n}$ consisting of n distinct points where E is Euclidean space; and

(2) an open set in $V^{\times n}$ consisting of n spacelike separated points.

(The precise description of some admissible open sets in (2) is given in the lemma of Jost.) The restriction of F to either of the two sets described determines F.

Now suppose $\mathcal{O}$ is an operator which annihilates the vacuum: $\mathcal{O}(x)|\Omega\rangle = 0$. We would like to show that $\mathcal{O} = 0$, which means that any correlation function involving $\mathcal{O}$ vanishes identically. For this we restrict the correlation function to a set of the form (2) above, and then

$$\begin{aligned}
\langle\mathcal{O}_1(x_1)\cdots\mathcal{O}(x)\cdots\mathcal{O}_n(x_n)\rangle &= \langle\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\mathcal{O}(x)\rangle \\
&= \langle\Omega|\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\mathcal{O}(x)|\Omega\rangle \\
&= 0.
\end{aligned}$$

For the second assertion, we suppose $\langle\overline{\mathcal{O}(y)}\mathcal{O}(x)\rangle = 0$ for distinct points x, y in Euclidean space; we want to show that any correlation function with $\mathcal{O}$ vanishes (in Euclidean space). Let π denote the perpendicular bisecting hyperplane to the segments joining x and y, and denote by $\overline{p}$ the reflection of p in π. Thus $y = \overline{x}$. Now let

$$\begin{aligned}
\mathcal{O}' &= \mathcal{O}_1(p_1)\cdots\mathcal{O}_n(p_n), \\
\overline{\mathcal{O}'} &= \overline{\mathcal{O}_1(\overline{p}_1)}\cdots\overline{\mathcal{O}_n(\overline{p}_n)},
\end{aligned}$$

denote some product of operators. We must show $\langle \mathcal{O}(x)\mathcal{O}' \rangle = 0$. Consider

$$\langle (\overline{\mathcal{O}(\overline{x})} + \lambda \mathcal{O}')\ (\mathcal{O}(x) + \lambda \overline{\mathcal{O}'}) \rangle.$$

By reflection positivity this is nonnegative. Set

$$\lambda = \frac{-\mathrm{Re}\langle \mathcal{O}(x)\mathcal{O}' \rangle}{\langle \mathcal{O}'\overline{\mathcal{O}'} \rangle}$$

to conclude

$$(\mathrm{Re}\langle \mathcal{O}(x)\mathcal{O}' \rangle^2 \leq \langle \overline{\mathcal{O}(\overline{x})}\mathcal{O}(x) \rangle\ \langle \mathcal{O}'\overline{\mathcal{O}'} \rangle = 0.$$

Thus $\mathrm{Re}\langle \mathcal{O}(x)\mathcal{O}' \rangle = 0$. The arguments applied to $\sqrt{-1}\mathcal{O}'$ show the imaginary part vanishes as well.