Exam

(1) Prove via path integrals the character-valued index theorem for the Dirac operator. That is, let M be a spin manifold and g an isometry of M with a lift to the spin bundles. Calculate

$$\operatorname{Tr} g(-1)^F \tag{1}$$

via path integrals.

That is, write down the path integral for sections of a suitable M bundle over a circle. Taking the circumference of the circle to be small, compute the integral explicitly and obtain the Atiyah-Bott formula for (1).

(2) Compute in ϕ^4 theory in four dimensions the anomalous dimension of the operator ϕ^2, in one-loop approximation.

(3) Determine the asymptotic behavior of perturbation theory in soft or cutoff scalar field theories, as follows. (Fermions and gauge fields introduce different behavior, not to mention other types of theories such as sigma models.)

(a) Consider the integral

$$Z(g) = \int_{\infty}^{\infty} d\phi \, \exp(-\tfrac{1}{2}\phi^2 - g\phi^4/4!). \tag{2}$$

Expand in an asymptotic series in g

$$Z(g) = \sum_k a_k g^k \tag{3}$$

and determine the large k behavior of a_k. Hint: write an integral representation for a_k and find a method of computing the integral for large k, for instance, Where is the maximum of the integrand?

(b) Interpret the answer to (a) as a statement about the number of graphs of a particular kind, weighted according to their automorphism groups.

(c) Now consider the path integral

$$Z(g) = \int D\phi \, e^{-L} \tag{4}$$

in a scalar field theory in n dimensions, with

$$L = \int d^n x \left(\tfrac{1}{2}|d\phi|^2 + \tfrac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4. \right). \tag{5}$$

Write down, again, an integral representation for the coefficients a_k in the asymptotic expansion of $Z(g)$ and determine the large k behavior, at least at a physical level of rigor, by finding the maximum of the integrand.

You should get a non-linear differential equation, and you should be able to prove that it has a spherically symmetric solution; it is believed that this solution does determine the large k behavior of the a_k in a cutoff ϕ^4 theory or in dimensions below four. In four dimensions, the much more complicated renormalization, which you've presumably ignored in the computation above, changes things (L must be redefined by incorporation of counterterms and the expansion in powers of g is not made as simply as you've presumably done above).

(4) Consider in two dimensions a theory of n fermi fields ψ^i, $i = 1, \ldots, n$. Each of the ψ^i is the sum of a section of S_+ and a section of S_-, those being the two spin bundles, which we recall are real if the two-dimensional world has Lorentz signature. (But you will actually compute with Euclidean signature.)

The Lagrangian is

$$L = n \int d^2x \left(\tfrac{1}{2} \sum_i (\psi^i, D\psi^i) + \tfrac{1}{2} m_0 (\psi^i, \psi^i) - \frac{g_0}{8} \left(\sum_i (\psi^i, \psi^i) \right)^2 \right). \tag{6}$$

Notice the crucial factor of n in front of the Lagrangian. Here D is the Dirac operator. In what follows, you can think of the fermion propagator as being

$$\frac{1}{\Gamma \cdot p + i m_0}. \tag{7}$$

Expressions such as (ψ, ψ) (which physicists write as $\bar{\psi}\psi$) are the only possible invariants of the given kind.

Consider the operator $\Sigma = \frac{1}{n} \sum_i (\psi^i, \psi^i)$.

(a) Consider the problem of computing the one point function $\langle \Sigma \rangle$ (which is an abbreviation for the vacuum expectation value $\langle \Omega | \Sigma | \Omega \rangle$ in perturbation theory. In k-loop order, which Feynman diagrams survive as $n \to \infty$?

(b) Write down an integral equation that formally sums all the Feynman diagram contributions to $\langle \Sigma \rangle$ that survive for $n \to \infty$.

(c) The integral equation in (b) needs some regularization and renormalization. Do this and obtain the renormalized equation obtained as the cutoff goes to infinity.

(d) If one had set $m_0 = 0$ in (6), one would have an extra $\mathbf{Z}_2$ symmetry that is broken by the mass. What is this? Is the operator Σ even or odd under this symmetry?

The renormalization in (c) can be done so that one of the parameters, a renormalized mass m, is odd under the $\mathbf{Z}_2$, while the other parameters are even. Hopefully you've done so!

If m is zero, how many solutions are there to the integral equation found in (c)? Are they invariant under the $\mathbf{Z}_2$?

(5) QCD is an $SU(3)$ gauge theory with fermions (quarks) which describes the nuclear forces. For this problem you only need to know a few facts. For instance, the theory contains a "current" operator, which is simply a Lorentz vector J^i (that is a local operator of spin one) that is conserved, $d_i J^i = 0$ (or $d * J = 0$), has zero anomalous dimension (we'll learn why later), and has the property that its two point function needs an additive c-number renormalization.

The renormalized two point function of J is therefore the limit as a cutoff Λ goes to infinity of

$$\int d^4x e^{ik\cdot x}\langle J^i(x)J^j(0)\rangle \quad - \quad A(\Lambda/\mu, g)(k^i k^j - \delta^{ij} k\cdot k). \tag{8}$$

Here g is the (dimensionless) coupling constant, Λ the cutoff, μ a mass that must be introduced in renormalization, and A is a function of the indicated arguments which must be "subtracted" to make the two point function finite. Also, the theory is asymptotically free and for present purposes you can ignore quark bare masses and assume that there are no parameters in the theory except g, μ, and Λ (and of course Λ is eventually eliminated by taking it to infinity). There is no multiplicative renormalization of J in (8) because the anomalous dimension vanishes. Because J is conserved, the expression in (8) is of the form

$$(k^i k^j - \delta^{ij} k^2) B(k^2, g, \mu) \tag{9}$$

where Λ is omitted as we want to consider the limit $\Lambda \to \infty$.

I suppose it also helps to know that J is built in terms of quarks as $J = \bar{q}\Gamma^i q$, so that the one loop approximation to $\langle JJ\rangle$ is given by a one loop diagram with a fermion loop that we computed in homework. (For those who are getting a paper and not internet copy of this, the loop is drawn below at the end of problem 7.) But all you really need to know is that in the free field theory, that is if $g = 0$, $A = a\ln(\Lambda/\mu)$, where a is a constant, independent of g.

(a) Write down the renormalization group equation obeyed by B.

(b) Use this equation, together with the fact that QCD is asymptotically free, to determine the large k^2 behvavior of B.

This large k^2 behavior, after including some additional theoretical wrinkles to put it in a form that can be compared with experiment, is one of the classic tests of QCD.

(6) Let $\mathcal{O}(x)$ be a local operator which annihilates the vacuum:

$$\mathcal{O}(x)|\Omega\rangle = 0 \tag{10}$$

Show at least at a physical level of rigor that $\mathcal{O}(x) = 0$.

Now extend this slightly: suppose that for some distinct points x and y $\langle\Omega|\mathcal{O}^*(y)\mathcal{O}(x)|\Omega\rangle = 0$. ($\mathcal{O}^*$ is the hermitian adjoint of $\mathcal{O}$.) Prove that $\mathcal{O} = 0$.

(7) Prove that a $U(1)$ gauge theory in four dimensions (without magnetic monopoles) cannot be conformally invariant unless it is free.

For present purposes, a $U(1)$ gauge theory is any theory in four dimensions that has a two-form valued local field F which obeys the Bianchi identity

$$dF = 0. \tag{11}$$

Now in four-dimensions with Euclidean signature we can decompose $F = F_+ + F_-$ where F_+ and F_- are selfdual and antiselfdual.

(a) Work in four-dimensional (flat) Euclidean space $\mathbf{R}^4$ and show that conformal invariance implies vanishing of the two point function $\langle F_+(x)R_-(y)\rangle$ for $x \neq y$.

(Conformal invariance means that the correlation functions can be extended to be defined for points in the one-point compactification $\mathbf{S}^4$ of $\mathbf{R}^4$ and that when so extended the correlation functions are invariant under the group of conformal motions of $\mathbf{S}^4$, which is $SO(5,1)$. Use $SO(5,1)$ to prove vanishing of $\langle F_+(x)F_-(y)\rangle$.)

(b) Given this, use the Bianchi identity to prove that $\langle d^*F(x)d^*F(y)\rangle = 0$.

(c) To conclude the argument, use the result of (6) to deduce that $d^*F = 0$, which (together with the Bianchi identity $dF = 0$) implies that F is a free field.