Lecture 2:
The Harmonic Oscillator and Free Fields

Ludwig Faddeev

17 October 1996

The treatment given here closely follows Sections 2.2 and 2.3 of [1].

We revisit the functional integral formalism in holomorphic quantization used to describe the harmonic oscillator. We make use of operators $\hat{a}$ and $\hat{a^*}$ satisfying $[\hat{a}, \hat{a^*}] = 1$. States are represented by analytic functions $\phi(a^*)$ of one complex variable a^*; operators are represented by integral kernels $A(a^{*\prime\prime}, a')$ depending on two *independent* complex variables $\hat{a^*}''$ and a'. The operator whose integral kernel is A may equivalently be represented by its *normal symbol*

$$K(a^{*\prime\prime}, a') = e^{-a^{*\prime\prime}a'} A(a^{*\prime\prime}, a'). \tag{1}$$

We compose operators by writing the integral kernel of the composition of the operators with integral kernels A and B as

$$AB(a^{*\prime\prime}, a') = \int_{\mathbf{R}^2} A(a^{*\prime\prime}, a) e^{-a^* a} B(a^*, a') \frac{da^* da}{2\pi i}. \tag{2}$$

The operator corresponding to the normal symbol is written as

$$\mathcal{K} = K(\hat{a^*}, \hat{a}) \tag{3}$$

where the ordering of the variables has been changed in such a way that all starred variables appear to the left of all unstarred variables.

The evolution operator $U_\triangle = e^{-iH\triangle}$ was written

$$U_\triangle(a^{*\prime\prime}, a') = \exp(-a^{*\prime\prime}a' - ih(a^{*\prime\prime}, a')\triangle).$$

We pause to point out that we have been regarding $\phi(a^*)$ (which represents a state) as an analytic *function* of one complex variable. In geometric quantization it is actually a *section* of a line bundle L over the phase space $\mathbf{R}^2$. We denote the section of the line bundle as

$$\tilde{\phi}(a^*, a) = e^{-\frac{1}{2}a^* a} \phi(a^*); \tag{4}$$

the object $\tilde{\phi}$ (the "dressed" analytic function) is a section of a line bundle L over the phase space $\mathbf{R}^2$. The dressed kernel $\tilde{A}(a^{*\prime\prime}, a')$ is a section of the pullback of two copies of L over $\mathbf{R}^2 \times \mathbf{R}^2$:

$$\tilde{A}(a^{*\prime\prime}, a') = e^{-\frac{1}{2}a^{*\prime\prime}a''} A(a^{*\prime\prime}, a') e^{-\frac{1}{2}a^{*\prime}a'}. \tag{5}$$

Dressed kernels are composed by integrating over $\mathbf{R}^2$. One may also recover the ordinary kernel from the dressed kernel by integration:

$$A(a^{*\prime\prime}, a') = \int e^{a^{*\prime\prime}a - \frac{1}{2}a^*a} \tilde{A}(a^*, a|b^*, b) e^{-\frac{1}{2}b^*b} e^{b^*a'} \frac{da^*da}{2\pi i}\frac{db^*db}{2\pi i}. \tag{6}$$

The time evolution operator becomes

$$U(a^{*\prime\prime}, a') = \int e^{a^{*\prime\prime}a_N - \frac{1}{2}a_N^*a_N} e^{a_N^*a_{N-1} - a_{N-1}^*a_N + \ldots - a_1^*a_1 + a_1^*a_0} \times \tag{7}$$

$$e^{-i[h(a_N^*, a_{N-1}) + \ldots h(a_1^*, a_0)]\triangle} \prod_{k=0}^{N} e^{-\frac{1}{2}a_0^*a_0 + a_0^*a'} \frac{da_k^*da_k}{2\pi i}$$

$$= \int \exp\Big\{a^{*\prime\prime}a(t'') - \frac{1}{2}a^*(t'')a(t'') + i\int_{t'}^{t''}\Big[\frac{\dot{a}^*a - a^*\dot{a}}{2i} - h(a^*, a)\Big\} dt$$

$$-\frac{1}{2}a^*(t')a(t') + a^*(t')a'\Big\} \prod_{t' \le t \le t''} \frac{da^*(t)da(t)}{2\pi i}. \tag{8}$$

This formula is better than (41) in Lecture I. All integrations here are done over a real plane R^2, in which $a^*(+)$ and $a(+)$ are complex coordinates. If one takes the explicit Hamiltonian $h(a^*, a) = \omega a^*a$, this integral is Gaussian: we may compute it by "completing the square", or equivalently observing that the integrand is an inhomogeneous quadratic and so the integral may be obtained by replacing the value of the integrand by its value at the (complex) critical point of the integrand. Explicitly, if A is a matrix with determinant 1 we have

$$\int \exp\{\frac{1}{2}(Az, z) + (b, z)\}dz = e^{-\frac{1}{2}(A^{-1}b, b)}. \tag{9}$$

The critical points of the integrand (8) are the solutions of the equation of motion

$$\dot{a}^* - i\omega a^* = 0, \quad a^*(t'') = a^{*\prime\prime}$$

$$\dot{a} + i\omega a = 0, \quad a(t') = a'.$$

We obtain as in Lecture 1

$$U_0(a^{*\prime\prime}, a'; t'' - t') = \exp(a^{*\prime\prime}a'e^{i\omega(t''-t')}), \tag{10}$$

which implies

$$U_0(t'')\phi(a^*) = \phi(e^{-i\omega t'}a^*).$$

Now let us consider a more complicated example: we perturb the Hamiltonian by adding a term involving an external force $\eta(t)$ (which should be thought of as a real valued function), which depends on the time t. If we introduce

$$q = \frac{1}{\sqrt{2\omega}}(a^* + a),$$

the new hamiltonian takes the form

$$H = \omega a^*a + \eta(t)q(t)$$

and is explicitly time dependent. The equations of motion for a and a^* acquire corrections:

$$\dot{a}^* - i\omega a^* - i\eta(t) = 0, \tag{11}$$

$$\dot{a}^* + i\omega a^* + i\eta(t) = 0. \tag{12}$$

The time evolution operator now depends on t'' and t' and not only on their difference: we obtain

$$U(a^*, a; t'', t') = \exp\Big\{a^* a e^{-i\omega(t''-t')} - i a^* e^{-i\omega t''} \int_{t'}^{t''} e^{i\omega s} \frac{\eta(s)}{\sqrt{2\omega}} ds - i a e^{i\omega t'} \int_{t'}^{t''} e^{-i\omega s} \frac{\eta(s)}{\sqrt{2\omega}} ds$$
$$- \int_{t=t'}^{t''} \frac{\eta(t)}{\sqrt{2\omega}} \int_{s=t}^{t''} e^{-i\omega t} e^{i\omega s} \frac{\eta(s)}{\sqrt{2\omega}} dt ds\Big\}, \tag{13}$$

(where from now on the primes on the variables a and a^* will be suppressed). We note that the expression for U contains both terms linear in η and terms quadratic in η.

Suppose $\eta = 0$ for t outside some interval $[\tau_1, \tau_2]$ (in other words the external force is turned on at time τ_1 and turned off at time τ_2). We would like to study how the motion of the system deviates from its motion in the absence of an external force: this is described by the normal symbol for the *transition operator* T, defined as follows.

$$T(a^*, a) = \lim_{t''\to\infty,\ t'\to-\infty} U_0(t'')^{-1} U(t'', t') U_0(t'). \tag{14}$$

Recalling the expression for U_0 given in (10), the transition operator is obtained from the expression (13) for U by removing the initial term $a^* a e^{-i\omega(t''-t')}$ from the argument of the exponential, and also removing the factors $e^{-i\omega t''}$ and $e^{i\omega t'}$ from the second and third terms in the argument of the exponential in (13). Notice that the integration variable s in (13) is subject to the constraint $s < t$; this enables us to rewrite the formula for the normal symbol of T in the form

$$T(a^*, a)_{\text{symb}} = \exp\left\{-i \int_{-\infty}^{\infty} q(t)\eta(t) dt + \frac{i}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-i\omega|t-s|}}{2i\omega} \eta(t)\eta(s) dt ds\right\}. \tag{15}$$

To obtain this formula we have symmetrized over t and s.

Here,

$$q(t) = \frac{1}{\sqrt{2\omega}} (a^* e^{i\omega t} + a e^{-i\omega t}) \tag{16}$$

is a solution of the free equation of motion

$$\ddot{q} + \omega^2 q = 0,$$

or

$$Dq = 0.$$

The operator D is defined by

$$D = \frac{d^2}{dt^2} + \omega^2.$$

The kernel

$$G(t,s) = G(t-s) = \frac{e^{-i\omega|t-s|}}{2i\omega}$$

is the *Green's function* for the operator D, or the fundamental solution to the equation

$$\left(\frac{d^2}{dt^2} + \omega^2\right)G(t,s) = \delta(t-s). \tag{17}$$

Since G depends only on the difference $t-s$, we shall from now on regard G as a function of one variable t. We may describe it in terms of its Fourier transform as

$$G(t) = \lim_{\varepsilon\to 0+} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikt}}{-k^2+\omega^2-i\epsilon}dk, \tag{18}$$

which is often denoted as

$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikt}}{-k^2+\omega^2-i0}dk.$$

The integral over k should be completed to a contour integral and '$-i0$' tells us how we have moved the poles in the integrand (which were on the real axis). The concrete choice of Green's function corresponds to the domain of definition $\mathcal{O}$ of the operator D: $\mathcal{O}$ consists of functions $f(t)$ which are asymptotic to $ae^{i\omega t}$ as $t\to-\infty$ and to $be^{-i\omega t}$ as $t\to\infty$ (where a and b are constants).

We now pass from quantum mechanics to quantum field theory, replacing $\mathbf{R}$ by Minkowski space V of dimension $d+1$ with signature $(+,-,\ldots,-)$ (in other words d space dimensions and one time dimension; sometimes this will be denoted as $V = V' \oplus \mathbf{R}$, where we have denoted d-dimensional space by V'.) We shall usually index the coordinates of V by $\mu = 0,\ldots,d$ while the coordinates of space V' are indexed by $j=1,\ldots,d$. Thus a point x in V is denoted $x = (x^0, \vec{x})$; its coordinates are $x^\mu = (x^0, x^1, \ldots, x^d)$.

Let us list the formulas from quantum mechanics which we shall generalize:

$$h = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 \;= \omega a^* a \tag{19}$$

$$\ddot{q} + \omega^2 q = 0. \tag{20}$$

$$[\hat{a}, \hat{a^*}] = 1 \tag{21}$$

$$\phi(a^*) = \frac{1}{\sqrt{n!}}(a^*)^n,\ H\phi = n\omega\phi \tag{22}$$

$$q = \frac{a^*+a}{\sqrt{2\omega}} \tag{23}$$

$$T(a^*,a) = \exp\left\{-i\int q(t)\eta(t)dt + \frac{i}{2}\int G(t-s)\eta(t)\eta(s)dtds\right\}. \tag{24}$$

The analogue of the coordinate $q(t)$ is a scalar field (denoted by φ) which depends on $x \in V$. The field φ satisfies the classical equation of motion, which is the *Klein-Gordon equation*

$$\Box\varphi + m^2\varphi = 0, \tag{25}$$

where we have introduced

$$\Box = \partial_\mu \partial^\mu = \partial_0^2 - \triangle.$$

To see the analogy with (20) we introduce the operator $\Omega^2 = -\triangle + m^2$, so that (25) becomes

$$\ddot{\varphi} + \Omega^2 \varphi = 0.$$

A free field φ should be thought of as a collection of oscillators with different frequencies, which are eigenvalues of the operator Ω: Ω^2 can be diagonalized by Fourier transforms and we get for the eigenfrequencies

$$\omega^2 = \vec{k}^2 + m^2, \tag{26}$$

or

$$\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}.$$

The space coordinate (or equivalently its Fourier transform, the momentum coordinate $\vec{k}$) labels the oscillators. Quantization now is evident. In the holomorphic quantization the basis elements, denoted by $\Phi_n(a^*)$, are given by

$$\Phi_n(a^*) = \prod_{j=1}^{n} a^*(\vec{k}_1) \ldots a^*(\vec{k}_n); \tag{27}$$

at times we shall use the notation $(\vec{k})_n$ to denote the n-tuple of *unordered* points $(\vec{k}_1, \ldots, \vec{k}_n)$ in $(V')^*$. The Hilbert space corresponding to n particles is the space of *symmetric* functions of n momentum variables. Proper states are given by

$$\int f((\vec{k})_n) a^*(\vec{k}_1) \ldots a^*(\vec{k}_n) d\vec{k}_1 \ldots d\vec{k}_n. \tag{28}$$

The Hamiltonian becomes

$$H = \int \omega(\vec{k}) \hat{a}^*(\vec{k}) \hat{a}(\vec{k}) d\vec{k}. \tag{29}$$

The state $\Phi((\vec{k})_n)$ is a generalized eigenstate for the Hamiltonian

$$H\Phi((\vec{k})_n) = \left(\sum_{j=1}^{n} \omega(\vec{k}_j) \right) \phi((\vec{k})_n). \tag{30}$$

and the momentum operator $\vec{P}$:

$$\vec{P}\Phi((\vec{k})_n) = \left(\sum_{j=1}^{n} \vec{k}_j \right) \Phi((\vec{k})_n). \tag{31}$$

Thus we obtain a description of the spectrum in terms of particles. In a one particle subspace, the components of the momentum operator form a complete set of commuting operators, and the Hamiltonian is a function $\omega(\vec{k})$ of them.

In field theory one requires multiparticle states in order to obtain scattering, and one must also allow the possibility of creating and annihilating particles. A Hilbert space accommodating all possible numbers of particles and on which the creation and annihilation operators may act is the *Fock space* $\mathcal{H}_{\text{Fock}}$, which should be thought of as a linear combination of all the n-particle states, or as the result of tensoring together a collection of Hilbert spaces representing one particle systems and symmetrizing over the action of the permutation group.

If we introduce the variable $\pi(\vec{x})$ which is canonically conjugate to the field operator $\varphi(\vec{x})$, the free field Hamiltonian becomes (compare with (19)

$$H_0 = \frac{1}{2}\int_{V'}\left(\pi^2(\vec{x}) + (\nabla\varphi, \nabla\varphi) + m^2\varphi^2(\vec{x})\right) d\vec{x}. \tag{32}$$

The initial data analogous to (23) is

$$\phi(\vec{x}) = \frac{1}{(2\pi)^{d/2}}\int \left(a^*(\vec{k})e^{-i(\vec{k},\vec{x})} + a(\vec{k})e^{i(\vec{k},\vec{x})}\right)\frac{d\vec{k}}{\sqrt{2\omega}}. \tag{33}$$

We may perturb the Hamiltonian H_0 by adding a term:

$$H = H_0 + \int \eta(\vec{x},t)\varphi(\vec{x},t)d\vec{x}.$$

The normal symbol of the corresponding transition operator is then (by analogy with (24))

$$T_\eta(a^*(\vec{k}), a(\vec{k})) = \exp\left\{-i\int_V \eta(x)\varphi(x)dx + \frac{i}{2}\int_{V\times V} G(x-y)\eta(x)\eta(y)dxdy\right\}, \tag{34}$$

where the Green's function satisfies

$$(\Box + m^2)G(x,y) = \delta^{d+1}(x-y); \tag{35}$$

it is given in terms of its Fourier transform by

$$G(x) = \frac{1}{(2\pi)^{d+1}}\int_{V^*}\frac{e^{-i(k,x)}}{k^2 - m^2 + i0}dk. \tag{36}$$

The field operator φ generalizing (16) is

$$\varphi(\vec{x},t) = \frac{1}{(2\pi)^{d/2}}\int\left\{a^*(\vec{k})e^{-i(\vec{k},\vec{x})}e^{i\omega t} + a(\vec{k})e^{i(\vec{k},\vec{x})}e^{-i\omega t}\right\}\frac{d\vec{k}}{\sqrt{2\omega}}. \tag{37}$$

Let us now consider what happens when we perturb the Hamiltonian by adding, not a term corresponding to an external force, but a term corresponding to an interaction between particles. We suppose for example that

$$H = H_0 + \lambda\int_{V'}\mathcal{V}(\varphi(\vec{x}))d\vec{x}, \tag{38}$$

where $\mathcal{V}(\varphi)$ is a polynomial in φ in which all terms are higher than quadratic; we could for example take

$$\mathcal{V}(\varphi) = \frac{\varphi^3}{3!}. \tag{39}$$

We shall expand in terms of powers of the *coupling constant* λ, which is assumed small. The term $\mathcal{V}$ is independent of the time; it should be regarded as analogous to a potential.

We shall compute the *scattering operator* or *S-matrix*

$$\mathcal{S}' = \lim_{t''\to\infty,\ t'\to-\infty} e^{iH_0t''}e^{-iH(t''-t')}e^{-iH_0t'}. \tag{40}$$

Repeating the previous formalism we obtain a functional integral for the time evolution operator where the integrand contains a factor

$$\exp\left\{-i\lambda\int\mathcal{V}(\varphi(x))dx\right\}. \tag{41}$$

This factor is contained in the integrand for the time evolution operator U in the path integral which is the generalization to $d+1$ dimensions of the integral given in (8). We now use the identity

$$\exp\left\{-i\lambda\int\mathcal{V}(\varphi(x))dx\right\} = \exp\left\{-i\lambda\int\mathcal{V}(\frac{1}{i}(\frac{\delta}{\delta\eta(x)})dx\right\}\exp\left\{i\int\varphi(x)\eta(x)dx\right\}|_{\eta=0} \tag{42}$$

to enable us to use our formula (34) for the transition operator to extract a formula for the S-matrix.

We replace integration of functions involving $\mathcal{V}(\varphi(x))$ by differentiation with respect to η of the formula (15) for the transition operator. The purpose of this device is to replace integration (which involves the fields φ in a nonlinear manner) by differentiation with respect to the parameter $\eta(x)$, which pairs linearly with $\varphi(x)$ The formula (34) for T_η may be thought of as a *generating functional* for the S-matrix when one perturbs the Hamiltonian by adding a potential term involving terms of higher than quadratic order in φ.

We obtain

$$\mathcal{S}'(a^*,a) = \exp\left\{-i\lambda\int\mathcal{V}(\frac{\delta}{\delta\eta(x)})dx\right\}\exp\left\{i\int_V\eta(x)\varphi(x)dx + \frac{i}{2}\int_{V\times V}G(x-y)\eta(x)\eta(y)dxdy\right\}. \tag{43}$$

We are applying $\delta/\delta\eta(x)$ to the exponential of an expression which contains one term linear in η and another term quadratic in η. One application of $\delta/\delta\eta(x)$ produces terms of the form $\varphi(x)$ or of the form $\int_V G(x-y)\eta(y)dy$. Differentiating twice with respect to η produces factors of $G(x-y)$. We may represent the combinatorics of the resulting expression for the S-matrix diagrammatically: this is the formalism of *Feynman diagrams*. To obtain the term of order λ^N in the formula for the S-matrix one should proceed as follows. We restrict for the moment to the potential $\mathcal{V}(\varphi) = \varphi^3/3!$ The factor $\mathcal{V}(\varphi)$ is represented by a vertex with three edges emanating from it (*trivalent* vertex) , which represent the three derivatives $\delta/\delta\eta(x)$ applied to the expression (34). Each vertex carries a factor of the coupling constant λ, and the j-th vertex is labelled by a coordinate $x_j \in V$. The combination $(\delta/\delta\eta)^2$ yields a term $G(x_j - x_k)$; diagrammatically this is represented by a line connecting the j-th and k-th vertices. (Here, j need not be distinct from k, so the line could start and end at the same vertex.)

Diagrammatically the term of order λ^N in the asymptotic expansion of $\mathcal{S}'$ is obtained by the following procedure:

1. Draw all diagrams with N trivalent vertices. Label the j-th vertex by the coordinate $x_j \in V$.

2. A line connecting the j-th and k-th vertices (*internal* line) carries a factor $G(x_j - x_k)$ (the *propagator*).

3. A line emanating from the j-th vertex but not connected to any other vertex carries a factor $\varphi(x_j)$.

4. The expression thus obtained must be multiplied by $(-i\lambda)^N/N!(3!)^N$ and integrated over the $x_1, \ldots, x_N$.

We note that many of the diagrams thus obtained will differ from each other only by permutations of the integration variables $x_1, \ldots, x_N$ or of the edges leaving any given vertex; these permutations are compensated by the factor $1/(3!)^N N!$ One obtains the same answer by summing only over one representative of each equivalence class under such diagrams and omitting the factor $1/(3!)^N N!$, provided one divides by the automorphism group of the diagram (the group of relabellings of the variables $x_1, \ldots, x_N$ and permutations of the lines leaving each vertex which leaves the diagram invariant).

Of course if the potential $\lambda\varphi^3/3!$ were replaced by a different order polynomial in φ, for example $\lambda\varphi^4/4!$, one would obtain an analogous formula involving vertices with a larger number of edges (in this case four) emanating from them.

We note that one must include contributions from disconnected diagrams, and also from diagrams where an edge starts and ends at the same vertex.

References

[1] L.D. Faddeev, A.A. Slavnov, *Gauge Fields: An Introduction to Quantum Theory*, Addison-Wesley (Frontiers in Physics vol. 83), (second edition), 1991.