Lecture 3: Comments on Scattering

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We first describe how to rewrite the expression for the S-matrix (equation (43) from Lecture 2)

$$\mathcal{S}'(\varphi_0) = \exp\left\{-i\lambda \int \mathcal{V}(\frac{\delta}{\delta\eta(x)})dx\right\} \exp\left\{i\int_V \eta(x)\varphi(x)dx + \frac{i}{2}\int_{V\times V} G(x-y)\eta(x)\eta(y)dxdy\right\}. \tag{1}$$

Here, the argument of $\mathcal{S}'$ is the solution of the free Klein-Gordon equation

$$\varphi_0(x) = \frac{1}{(2\pi)^{3/2}}\int \left(a^*(\vec{k})e^{ik\cdot x} + a(\vec{k})e^{-ik\cdot x}\right)\frac{d\vec{k}}{\sqrt{2k_0}}, \tag{2}$$

for $k_0 = \omega(\vec{k})$. We shall rewrite (1) as a path integral of the form

$$\mathcal{S}'(\varphi_0) = \int_\varphi \exp\Big\{i\int L(x)dx\Big\}\prod_x d\varphi(x), \tag{3}$$

where we integrate over fields φ tending asymptotically to certain asymptotic values φ_{in} (resp. φ_{out}) as $t \to -\infty$ (resp. $t \to +\infty$). Starting from the formula (3) one may derive the Feynman rules for an expansion of the S-matrix in terms of Feynman diagrams, where we expand the Lagrangian L about its stationary point φ_{cl}.

Our exposition closely follows Section 2.4 of [1], which is appended.

We now pass to several examples which illustrate the notion of renormalization. First we shall consider an instructive example which shows that the mass renormalization is not necessarily associated with infinities. Rather it is connected with the proper definition of the one particle states. As an example let us consider a Hamiltonian

$$H = H_0 + \lambda \sum_{n,m\geq 0} v_{n,m}. \tag{4}$$

Here H_0 is the free scalar field Hamiltonian

$$H_0 = \int \omega(\vec{k})a^*(\vec{k})a(\vec{k})d\vec{k}, \tag{5}$$

λ is a coupling constant and $v_{n,m}$ are polynomials in the creation and annihilation operators:

$$v_{n,m} = \int \mathcal{V}\Big((\vec{k})_n, (\vec{k}')_m\Big) a^*(\vec{k}_1)\ldots a^*(\vec{k}_n) a(\vec{k}'_1)\ldots a(\vec{k}'_m)\delta(\vec{k}_1+\ldots+\vec{k}_n-\vec{k}'_1-\ldots-\vec{k}'_m)(d\vec{k})_n(d\vec{k}')_m. \quad (6)$$

The delta function in (6) implies that the Hamiltonian commutes with the momentum operators:

$$[H, \vec{P}] = 0, \quad (7)$$

where

$$\vec{P} = \int \vec{k} a^*(\vec{k}) a(\vec{k}) d\vec{k}.$$

We shall make frequent use of the commutation relations

$$[a(\vec{k}), a^*(\vec{k}')] = \delta(\vec{k} - \vec{k}'), \quad (8)$$

$$[a(\vec{k}), a(\vec{k}')] = [a^*(\vec{k}), a^*(\vec{k}')] = 0. \quad (9)$$

We can consider that the v_{11} term is absent in (4); otherwise it is to be included into H_0 from the beginning.

Now if terms $v_{n,0}$ are present then the Fock vacuum Ω for which

$$a(\vec{k})\Omega = 0$$

is not an eigenvector of H. So the physical vacuum is distinct from Ω. Furthermore terms $v_{n,1}$, even in the absence of terms $v_{n,0}$, preclude the one particle states

$$\phi_1(\vec{k}) = a^*(\vec{k})\Omega$$

from being eigenvectors of H. On the other hand a Hamiltonian of the form (4) but summing only over n and m such that $n \geq 2$ and $m \geq 2$ has the same vacuum and one particle states as H_0. One can apply to it the usual formalism of scattering theory.

We shall show (in the framework of formal series expansions) that it is possible to find a unitary transformation $R = e^{iQ}$, where Q is given by an expression

$$Q = Q_1 + Q_1^* + Q_2 + Q_2^*$$

and

$$Q_1 = \sum_n Q_{n,0}, \qquad Q_1^* = \sum_m \bar{Q}_{0,m}$$

$$Q_2 = \sum_n Q_{n,1}, \qquad Q_2^* = \sum_m \bar{Q}_{1,m}$$

with terms of the type $(n, 0)$, $(0, m)$, $(n, 1)$ and $(1, m)$, and corresponding coefficient functions $q((\vec{k})_n)$, $\bar{q}((\vec{k}')_m)$, $q((\vec{k})_n, \vec{k}')$, and $\bar{q}(\vec{k}, (\vec{k}')_m)$, such that after the conjugation

$$\tilde{H} = RHR^{-1}$$

all unwanted terms can be cancelled. Of course, each Q is a power series in λ:

$$Q = \sum \lambda^j Q^j$$

Alternatively we can say that in a new representation of canonical variables

$$b^{\#} = Ra^{\#} R^{-1}$$

the Hamiltonian H has the same vacuum and one particle states as a free hamiltonian $\tilde{H}_0$, which however is different from H_0.

We shall define $\tilde{H}^{(j)} = RHR^{-1} + O(\lambda^{j+1})$, solving in such a way that all the $\tilde{H}^{(j)}$ only include terms with $n \geq 2$ and $m \geq 2$. We see that

$$e^{iQ} H e^{-iQ} = H_0 + \lambda(i[Q^1, H_0] + \sum_{n,m} v_{n,m}) + \lambda^2(-[Q^2, H_0] + i[Q^1, \sum_{n,m} v_{n,m}]) + \dots \qquad (10)$$

We shall decompose this equation into its components at every order in λ, successively cancelling the terms $v_{n,0}$, $v_{0,m}$, $v_{n,1}$ and $v_{1,m}$.

To leading order in λ, we need to solve

$$i[Q^1, H_0] + \sum_{n\geq 0} \sum_{m\geq 0} v_{n,m} = \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} v^{(1)}_{n,m}, \qquad (11)$$

with some $v^{(1)}_{n,m}$ (in other words we arrange to cancel the terms $v_{n,m}$ with $n \leq 1$ or $m \leq 1$). We shall show how to choose Q^1 to satisfy (11); then to second order in λ the equation (10) implies

$$-[Q^2, H_0] + i[Q^1, \sum_{n\geq 0} \sum_{m\geq 0} v_{n,m}] = \sum_{n\geq 2} \sum_{m\geq 2} v^{(2)}_{n,m} \qquad (12)$$

for some $v^{(2)}_{n,m}$ with $n \geq 2$ and $m \geq 2$. It will then be necessary to solve (12) for Q^2, and so on; this process enables us successively to cancel all the terms $v_{n,1}, v_{1,m}, v_{n,0}, v_{0,m}$.

We now describe how to solve equation (11) for Q^1: exactly the same procedure will then suffice to solve (12) for Q^2, and so on. Since (11) expresses Q^1 as a linear function of $\sum_{n,m} v_{n,m}$, it suffices to solve the equations

$$[Q^1_1, H_0] = iv_{n,0}, \qquad (13)$$

$$[(Q^1_1)^*, H_0] = iv_{0,m}, \qquad (14)$$

$$[Q^1_2, H_0] = iv_{n,1}, \qquad (15)$$

$$[(Q^1_2)^*, H_0] = iv_{1,m} \qquad (16)$$

for individual values of n and m.

Notice that for any operator $r_{n,m}$ of type n, m,

$$[r_{n,m}, H_0] = \sum_{n,m} \int d\vec{k}_1 \dots d\vec{k}_n d\vec{k}'_1 \dots d\vec{k}'_m \delta(\vec{k}_1 + \dots + \vec{k}_n - \vec{k}'_1 - \dots - \vec{k}'_m) \times$$

$$r_{n,m}((\vec{k})_n,(\vec{k}')_m)a^*(\vec{k}_1)\dots a^*(\vec{k}_n)a(\vec{k}'_1)\dots a(\vec{k}'_m)(\sum_j \omega(\vec{k}_j) - \sum_j \omega(\vec{k}'_j)),$$

so that the equations (13 - 16) are solved simply by dividing the coefficient functions $v_{n,0}$, $v_{0,m}$, $v_{n,1}$, $v_{1,m}$ by factors $\sum_j \omega(\vec{k}_j)$, $-\sum_j(\omega(\vec{k}'_j)$, $\sum_j \omega(\vec{k}_j) - \omega(\vec{k}'_1)$, $\omega(\vec{k}_1) - \sum_j \omega(\vec{k}'_j)$ respectively. Assuming that the one-particle energy is *convex*, in other words that

$$\omega(\vec{k}_1) + \omega(\vec{k}_2) \geq \omega(\vec{k}_1 + \vec{k}_2)$$

(which means that a particle cannot decay into two or more particles while conserving momentum) we see that the factors above are nonvanishing and there is no difficulty in dividing by them.

However this method cannot be generalized to cancel terms $v_{n,m}$ where both $n \geq 2$ and $m \geq 2$, for then the factor appearing in the integral for $[Q_1, H_0]$ involves $\sum_{j=1}^{n} \omega(\vec{k}_j) - \sum_{l=1}^{m} \omega(\vec{k}'_l)$ which will in general become zero for some values of $\vec{k}_j$ and $\vec{k}'_l$, so it is no longer possible to divide through by this factor in order to solve the equation.

The equations of higher order are solved analogously. As a result we achieve our goal of cancelling the unwanted terms.

In the course of this procedure a nonvanishing vacuum energy (constant terms in the expansion) appears, and the one particle energy $\omega(\vec{k})$ changes. In particular, in the relativistic case this amounts to the renormalization of mass: in the new Hamiltonian $\tilde{H}$, the function $\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$ which multiplies $a(\vec{k})a^*(\vec{k})$ in the Hamiltonian H is replaced by $\omega_{ren}(\vec{k}) = \sqrt{\vec{k}^2 + M^2}$ for an appropriate value of M. What we have done may thus be described as *mass renormalization.*

Our second example of renormalization is "charge renormalization". Its origin can be illustrated by a simple example. We begin with a Hamiltonian

$$H = -\frac{1}{2}\triangle + \lambda\delta^3(x),$$

where $\delta^3(x)$ is the Dirac delta distribution. After Fourier transforming the equation $H\hat{\psi}(x) = E\hat{\psi}(x)$ (in position space) becomes

$$\vec{p}^2\psi(\vec{p}) + \lambda \int \psi(\vec{p})d^3\vec{p} = \vec{k}^2\psi(\vec{p}) \tag{17}$$

where the eigenvalue of the energy is $E = \vec{k}^2$. One may rewrite this as

$$\psi(\vec{p}) = \delta^3(\vec{p} - \vec{k}) - \frac{\lambda}{\vec{p}^2 - \vec{k}^2 - i0} \int \psi(\vec{p}')d\vec{p}'. \tag{18}$$

Putting $t = \int \psi(\vec{p}')d\vec{p}'$, we obtain integrating both sides over $\vec{p}$

$$t = 1 - \lambda t \int \frac{d\vec{p}}{\vec{p}^2 - \vec{k}^2 - i0}. \tag{19}$$

The integral diverges; to regularize it we introduce a cutoff L and integrate only over $|p| \leq L$. We find

$$\int_{|p|\leq L} \frac{d\vec{p}}{|p|^2 - |\vec{k}|^2 - i0} = 4\pi \int_0^L \frac{|p|^2 d|p|}{|p|^2 - |k|^2 - i0}. \tag{20}$$

This leads to

$$\begin{aligned} t &= 1 - \lambda t \int_{|p|\leq L} \frac{d\vec{p}}{|p|^2 - k^2 - i0} \\ &= 1 - \lambda t \Big(4\pi L + k^2 \int \frac{d|p|}{|p|^2 - k^2 - i0} \Big). \end{aligned} \tag{21}$$

This may be rewritten as

$$\lambda t = \frac{\lambda}{1 + \lambda(4\pi L + f(\vec{k}))},$$

where

$$f(\vec{k}) = 4\pi(|\vec{k}|^2) \int_0^\infty \frac{d|\vec{p}|}{|\vec{p}|^2 - |\vec{k}|^2 - i0} = 2\pi^2 i |\vec{k}|.$$

We may rewrite this equation in terms of a "renormalized" parameter λ_{ren}, by writing

$$\lambda t = \frac{\lambda_{ren}}{1 + \lambda_{ren} f(\vec{k})}$$

where

$$\lambda_{ren} = \frac{\lambda}{1 + 4\pi\lambda L}$$

or

$$\frac{1}{\lambda_{ren}} = \frac{1}{\lambda} + 4\pi L. \tag{22}$$

We shall now choose λ_{ren} (depending on L and λ) in such a way that λ_{ren} may remain finite as L tends to infinity. It is clear from (22) that this is only possible if λ is *negative* (in other words it describes an attracting force, not a repelling one): then one may allow λ to depend on L in such a way that $\frac{1}{\lambda}$ tends to $-\infty$ as L tends to ∞, while the value of the renormalized constant $\lambda_{\rm ren}$ remains finite.

Renormalization is discussed further in E. Witten's lectures (October 24 and October 31).

References

[1] L.D. Faddeev, A.A. Slavnov, *Gauge Fields: An Introduction to Quantum Theory*, Addison-Wesley (Frontiers in Physics vol. 83), (second edition), 1991.