Lecture 4: Singular Lagrangians

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In other lectures, considerable attention has been focused on how to construct the symplectic structure on the space of classical solutions, starting from the Lagrangian. In this lecture we will instead focus on the fixed time Hamiltonian formalism, and we will see that the transformation which maps the Lagrangian into the Hamiltonian is simply a change of variables. The lecture will chiefly treat the case of singular Lagrangians. Our aim is to present the first order formalism, in which the Lagrangian depends linearly on time derivatives of the dynamical variables.

In the usual Lagrangian formalism, we begin with the configuration space M: coordinates on M will be denoted q. Coordinates on the tangent bundle TM are then given by q and $\dot{q}$, and the Lagrangian is a function $\mathcal{L}(q, \dot{q})$. The action is then defined by

$$A = \int \mathcal{L}(q(t), \dot{q}(t))\, dt.$$

Applying the standard variational principle we see that the extrema of the action are given by the *Euler-Lagrange equation*

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0. \tag{1}$$

This is ordinarily a system of equations of *second order* since the Euler-Lagrange equation involves the term $\ddot{q}\partial^2\mathcal{L}/\partial\dot{q}\partial\dot{q}$. The Lagrangian is said to be *nondegenerate* if the Hessian $\partial^2\mathcal{L}/\partial\dot{q}\partial\dot{q}$ is invertible.

In the Hamiltonian formalism, on the other hand, we work with $T^*M \times \mathbf{R}$ (where T^*M is the cotangent bundle of configuration space, or the phase space). On this there is a natural 1-form

$$pdq - H(p, q)dt, \tag{2}$$

where we have introduced (q, p) as natural coordinates on T^*M. The corresponding Lagrangian is written as

$$l = p\dot{q} - H(p, q), \tag{3}$$

and the action is

$$A = \int (p\dot{q} - H(p, q))dt.$$

The Euler-Lagrange equation applied to A then gives Hamilton's equations:

$$\dot{p} = -\frac{\partial H}{\partial q}, \qquad \dot{q} = \frac{\partial H}{\partial p}. \tag{4}$$

Note that the Lagrangian l is linear in time derivatives, and the Hessian of l is identically zero. Thus our system is of first order. In the standard fashion, we have converted a system of second order differential equations into a system of first order differential equations, at the expense of increasing the number of variables. To explicitly see how the Hamiltonian formalism is derived from the Lagrangian formalism, we introduce $v = \dot{q}$ and treat q and v as independent variables. The Lagrangian then depends on q, v, $\dot{q}$ and $\dot{v}$, and has the following form:

$$l = \frac{\partial \mathcal{L}}{\partial v}(\dot{q} - v) + \mathcal{L}(q, v). \tag{5}$$

Comparing with (3) one finds that the Hamiltonian is given by

$$H = v\frac{\partial \mathcal{L}}{\partial v} - \mathcal{L},$$

and we obtain the Hamiltonian formalism by performing the Legendre transformation

$$(q, v) \mapsto (q, p)$$

where we have defined

$$p = \frac{\partial \mathcal{L}}{\partial v}.$$

On a general manifold Γ with coordinates ξ, we would write the Lagrangian as

$$l = \sum_a f_a(\xi)\dot{\xi}^a - \phi(\xi). \tag{6}$$

The one-form on $\Gamma \times \mathbf{R}$ corresponding to (2) would be written as

$$\sum_a f_a(\xi) d\xi^a - \phi dt = \omega - \phi dt,$$

and it gives rise to a two-form $\Omega = d\omega$. (Here d denotes the de Rham differential on Γ, rather than on $\Gamma \times \mathbf{R}$.) The equation of motion becomes

$$\Omega\dot{\xi} = \frac{\partial \phi}{\partial \xi}.$$

The condition that the Lagrangian is nonsingular is equivalent to the condition that the two-form Ω is invertible: for this to hold, it is necessary that Γ be even dimensional. For important Lagrangians in physics (for instance the Yang-Mills Lagrangian and the Lagrangian for electrodynamics) the two-form Ω is in fact not invertible. Let us investigate this situation in more detail.

Invoking the Darboux theorem, on Γ we write the coordinates ξ as (p, q, z) in such a way that the one-form ω becomes

$$\omega = pdq + d\theta.$$

The total derivative $d\theta$ may be discarded, and after this the one-form corresponding to the Lagrangian becomes

$$l = pdq - \phi(p, q, z)dt,$$

where $\frac{\partial \phi}{\partial z} = 0$. The variables z may be divided into "excludable" variables (which will ultimately be discarded) and others which enter linearly. After eliminating the excludable variables, the Lagrangian becomes

$$l = pdq - \phi(p, q)dt - \sum_{\alpha} \lambda^{\alpha} \varphi_{\alpha}(p, q). \tag{7}$$

Here, the λ^{α} play the role of *Lagrange multipliers* while the $\varphi_{\alpha}(p, q)$ are constraints. Among the equations of motion we now obtain the constraints

$$\varphi_{\alpha}(p, q) = 0,$$

so we may substitute $p = p(\eta)$, $q = q(\eta)$ where η are coordinates on the manifold

$$\Gamma^{l} = \{(p, q) : \varphi_{\alpha}(p, q) = 0\}.$$

We thus obtain a new phase space Γ^{l} on which the excludable variables and the Lagrange multipliers have been eliminated, so that the variables z no longer appear and we are left with a subspace parametrized by the variables (p, q) where the constraints $\varphi_{\alpha}(p, q) = 0$ have been imposed. The new phase space Γ^{l} may have a nondegenerate symplectic form, and then our Hamiltonian reduction is done. If not, we repeat the above process until Lagrange multipliers no longer appear, in which case we have reduced to a phase space with a nondegenerate symplectic form.

Explicit solution of the constraints could be complicated, so it is instructive to understand what properties of the singular Lagrangian (7) guarantee that it will become nonsingular after the constraints are imposed. There are two important instances of this. We can distinguish two different types of constraints. A collection of constraints $\{\varphi_a\}$ are called *first class* if the Poisson brackets of all constraints vanish on the zero locus of the constraints: in other words, if

$$\{\varphi_a, \varphi_b\}|_{\varphi=0} = 0. \tag{8}$$

We claim that if $\{\phi, \varphi_a\}|_{\varphi=0} = 0$ for all a, then in the next step of the procedure one obtains a proper nondegenerate Lagrangian. More precisely, the submanifold $\Gamma|_{\varphi=0}$ may be fibered so that the base is symplectic and the fibers are isotropic with respect to the Poisson bracket $\{\cdot, \cdot\}$. One may summarize the procedure by saying that "first class constraints kill twice": when one restricts to the zero locus of the constraints φ_a, one must also set to zero the variables which are canonically conjugate to the φ_a, in other words one must set to zero all Poisson brackets with the φ_a. If there are N constraint equations φ_a, $a = 1, \ldots, N$, then imposing the constraints reduces the dimension of the phase space by $2N$.

The second type of constraints φ_a are those for which $\det\{\varphi_a, \varphi_b\} \neq 0$: a system of constraints for which this condition holds is called a system of *second class constraints*. If a Lagrangian gives rise to a system of second class constraints, the next step of reduction leads to a nondegenerate Lagrangian. The number of ϕ_a must be $2N$ and the dimension of the phase space is then reduced by N through imposition of the constraints. Constrained quantization was originally considered by Dirac [1].

Example 1: the free scalar field

Given a field $\varphi(t,\vec{x})$, and the free field Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2, \tag{9}$$

canonical quantization normally specifies that one should introduce the canonically conjugate momentum $\pi(t,\vec{x}) = \partial_0\varphi(t,\vec{x})$, where $x_0 = t$ and $(x_1,x_2,x_3) = \vec{x}$. The points $\vec{x}$ in space should be thought of as labelling the field. Notice that the momentum π is not Lorentz invariant.

We may alternatively replace the Lagrangian (9) (which is second order in time derivatives) by a first order Lagrangian: we do this by introducing a vector field φ_μ (in other words a collection of four fields $(\varphi_0,\varphi_1,\varphi_2,\varphi_3)$ which transform according to the standard action of the Lorentz group on $\mathbf{R}^4$) and write

$$l = (\partial_\mu\varphi)\varphi^\mu - \frac{1}{2}(\varphi^\mu)^2. \tag{10}$$

Of course under the substitution $\varphi_\mu = \partial\varphi/\partial x^\mu$ the Lagrangian in (10) reduces to that in (9): however if we regard φ_μ as independent of φ, (10) is first order in time derivatives. (We have introduced the Einstein summation convention, summing over pairs of repeated indices μ, where $\mu = \{0,k\}$ and $k = 1,2,3$ correspond to space coordinates while $\mu = 0$ corresponds to the time coordinate.) We may rewrite (10) as

$$l = (\partial_0\varphi)\varphi^0 - (\partial_k\varphi)\varphi^k - \frac{1}{2}\varphi_0^2 + \frac{1}{2}\varphi_k^2. \tag{11}$$

Under the substitution $\pi = \varphi_0$, and solving for φ_k, $\varphi_k = \partial_k\varphi$, we obtain

$$\mathcal{L} = (\partial^0\varphi)\pi - \frac{1}{2}(\pi^2 + (\partial_k\varphi)^2), \tag{12}$$

which indeed shows that π and φ are canonically conjugate and that the Hamiltonian H has the usual form

$$H = \frac{1}{2}(\pi^2 + (\partial_k\varphi)^2.)$$

The variables φ_k can be eliminated because they enter (10) quadratically and without time derivatives.

Example 2: electromagnetism

In this example it is important that spacetime has dimension 4. We start with a vector field $A_\mu(x)$ (the photon field), which should be thought of as a 1-form or a connection:

$$A = \sum_\mu A_\mu dx^\mu.$$

The associated curvature is

$$F = dA,$$

or as a two-form $F_{\mu\nu} dx^\mu \wedge dx^\nu$, where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{13}$$

The Lagrangian is

$$\mathcal{L} = \int |F|^2, \tag{14}$$

which is second order in time derivatives. A trick due to J. Schwinger enables us to replace it with a Lagrangian which is first order in time derivatives: this is to regard A and F as independent variables. By analogy with our procedure in Example 1, we write

$$\mathcal{L} = (\partial_\mu A_\nu - \partial_\nu A_\mu) F^{\mu\nu} - \frac{1}{2} F^2_{\mu\nu}. \tag{15}$$

Of course (15) reduces to (14) by restoring the functional dependence (13) of F on A. However the Lagrangian in (15) is linear in time derivatives of the fields, and also introduces some constraints.

We rewrite (15) as

$$\mathcal{L} = (\partial_0 A_k) F^{0k} + A_0 (\partial_k F^{0k}) - F^{ik}(\partial_i A_k - \partial_k A_i) - \frac{1}{2}(F^{0k})^2 + \frac{1}{2}(F^{ik})^2. \tag{16}$$

Here, $F^{0k} = E^k$ are the components of the *electric field* $\vec{E}$, while A_k are the components of the *vector potential*. Up to the Hodge star operator $*$ which converts two-forms on $\mathbf{R}^3$ into one-forms, the quantity F^{ik} is the *magnetic field* $\vec{H}$: we substitute

$$\vec{H} = \nabla \times \vec{A}, \tag{17}$$

or $H_i = \epsilon_{ijk} \nabla_j A_k$ in terms of the totally antisymmetric tensor ϵ in three indices which represents the Hodge star operator in $\mathbf{R}^3$. Thus the Lagrangian becomes

$$\mathcal{L} = (\partial_0 A_k) E^k - \frac{1}{2}(\vec{E}^2 - \vec{H}^2) + A_0 \partial_k E^k, \tag{18}$$

where we already excluded the independent variables $\vec{H}$, so that $\vec{H}$ in (18) is given by (17). On the other hand, A_k and E^k are canonically conjugate variables (appearing in a term which is first order in time derivatives), and A_0 is a Lagrange multiplier multiplying the constraint

$$G = \nabla \cdot E = \partial_k E^k. \tag{19}$$

The constraint $G = 0$ is called *Gauss's law*: it tells us that the divergence of the electric field is zero. This is a first class constraint, $\{G(\vec{x}), G(\vec{y})\} = 0$, and we have also

$$\{G, \vec{E}\} = \{G, \vec{H}\} = 0. \tag{20}$$

For any function λ on $\mathbf{R}^3$ we have

$$\{A_k, \int G(\vec{x}) \lambda(\vec{x}) d\vec{x}\} = \partial_k \lambda(\vec{x});$$

equivalently G is a generator of $U(1)$ gauge transformations: the gauge group $\mathcal{G}$ is the group of maps from $\mathbf{R}^3$ to $U(1)$, and its Lie algebra is the group of maps from $\mathbf{R}^3$ to $\mathbf{R}$. An element $\lambda \in \mathrm{Lie}(\mathcal{G})$ sends A_k to $A_k + \partial_k \lambda$. The imposition of Gauss's law may be viewed as symplectic reduction with respect to the action of the $U(1)$ gauge group $\mathcal{G}$.

Since the constraint G is of first class, it reduces the degrees of freedom from three functions $A_k(\vec{x})$ to two. Thus (in physical language) light has two polarizations.

The above treats the situation of the electromagnetic field in the absence of sources. If on the other hand the electric field interacts with a charged field J^μ, Gauss's law becomes

$$G = \nabla \cdot E + J_0(\vec{x}) \tag{21}$$

where J_0 is the *charge density*. If we define

$$Q = \int_{\mathbf{R}^3} J_0 dx$$

if we express Q as a flux of the electromagnetic field via a surface in $\mathbf{R}^3$ (for instance a sphere $S^2(R)$ of radius R) we obtain

$$Q = \lim_{R \to \infty} \int_{S^2(R)} \epsilon_{ikj} E^k dS_{ij}.$$

Thus the equation (21) leads to *Coulomb's law*, according to which the electromagnetic force from a point charge at distance r is proportional to $1/r^2$.

Example 4: Gravitation

In this example we introduce a metric $g^{\mu\nu}$ with the associated *affine connection* Γ^k_{ij} and the scalar curvature (we omit indices for clarity)

$$R = \partial\Gamma + \Gamma^2.$$

To write a Lagrangian that is first order in time derivatives, we take g and Γ as independent variables: more precisely we write

$$h^{\mu\nu} = \sqrt{g} g^{\mu\nu},$$

and write the Lagrangian as

$$\mathcal{L} = h(\partial\Gamma + \Gamma^2). \tag{22}$$

The connection Γ contains first order time derivatives of h, so $\partial\Gamma$ contains second order time derivatives of h. This may be altered by integrating by parts, so that we obtain

$$\mathcal{L} = -\Gamma\partial h + h\Gamma^2. \tag{23}$$

This Lagrangian is not manifestly covariant, however it is this form which is to be used in the so called *asymptotically flat* case.

The definition of asymptotic flatness can be given on different levels of sophistication; we shall use the most naïve but practical way of introducing the admissible coordinates. In such coordinates we have in the vicinity of space infinity

$$g = g_0 + O(\frac{1}{r}),$$

$$\partial g = O(\frac{1}{r^2})$$

$$\Gamma = O(\frac{1}{r^2})$$

where g_0 is the Minkowski metric. An admissible change of variables is of the form

$$X_\mu \to \Lambda_{\mu v} X_r + a_\mu + b_\mu(x), \tag{24}$$

where $\Lambda_{\mu v}$ is a Lorentz rotation, a_μ defines a translation and

$$b_\mu(X) = O\left(\frac{1}{r}\right), \quad \partial b = O\left(\frac{1}{r^2}\right).$$

In these coordinates the density (23) is of order $O\left(\frac{1}{r^4}\right)$ and the action

$$A = \int \mathcal{L}\, dx$$

is invariant with respect to the change of coordinates (24).

Inspection of the density (23) cubic in the variables (h, Γ), shows that out of $10+40 = 50$ variables, the Γ^μ_{00} enter linearly and 26 other components Γ enter quadratically without time derivatives. So the Lagrangian (23) takes the form (7), however the constraints do not commute with the corresponding ϕ. So we must solve them explicitly. Fortunately this is possible and altogether solving the constraints and eliminating the excludable variables allows one to express 30 components of Γ via the metric $h^{\mu\nu}$ and 6 components Γ^0_{ik}. Let

$$\begin{aligned} q^{ik} &= h^{00}h^{ik} - h^{in}h^{nk} \\ \Pi_{ik} &= \frac{\Gamma^0_{ik}}{h^{00}} \end{aligned}$$

be a (density of weight $+2$) of the first quadratic form and (a density of weight -1) of the second quadratic form of the surface $x_0 = 0$ embedded into space-time. After solving for Γ, mentioned above, the Lagrangian (23) reduces to the form

$$\Pi_{ik}\partial_0 q^{ik} + \frac{1}{h^{00}} C_0(\Pi, q) + \frac{h^{0k}}{h^{00}} C_k(\Pi, q) + \partial_i \partial_k q^{ik}$$

where C_0 and C_k are some functionals of Π, q and derivatives. The term $\partial_i\partial_k q^{ik}$ cannot be discarded in spite of the fact that it is a pure divergence as its flux through infinity does not vanish. Indeed this flux

$$\mathcal{M} = \lim_{R\to\infty} \int\limits_{S_R} \epsilon_{k\ell m} \partial_{ik} q^{ik} dS^{\ell n}$$

defines the full energy $\mathcal{M}$ of the gravitational field and a very nontrivial theorem states that $\mathcal{M}$ is positive unless the metric is not flat.

Four constraints C_0, C_k are of the first class; let $\eta_k(x)$, $\eta_0(x)$ be the vector field and the function on the space slice. We put

$$\begin{aligned} C_0(\eta_0) &= \int C_0(\Pi(x), q(x))\eta_0(x) d^3x \\ C(\eta) &= \int C_k(\Pi(x), q(x))\eta^k(x) d^3x \end{aligned}$$

to get the brackets

$$\{C(\eta''), C(\eta')\} = C([\eta^n, \eta'])$$

where [,] is the commutator of vector fields. This means that $C(\eta)$ are generators of diffeomorphisms of our space slice. Furthermore

$$\{C(\eta), C_0(\eta_0)\} = C_0(\eta\eta_0)$$

where $\eta\eta_n$ is the result of the action of the vector field η on the function η_0. Finally

$$\{C_0(\eta_0''), C_0(\eta_0')\} = C(\hat{\eta}),$$

where the vector field $\hat{\eta}$ is given by

$$\hat{\eta}^n = q^{in}(\partial_i \eta_0'' \eta_0' - \eta_0'' \partial_i \eta_0')$$

What about the Hamiltonian? Taking into account the asymptotic condition, it is more natural to consider $\left(\frac{1}{h^{0n}} + 1\right)$ rather than $1/h^{nm}$ to be a lagrangian multiplier; thus, the Hamiltonian

$$H = C_0 + \partial_i \partial_n q^{in}$$

differs from the constraint C_0 only by a term which is a total divergence. The explicit expression for C_0 shows that H is quadratic in Π and first derivatives of q, like all other Hamiltonians in previous examples. The 4 constraints kill 4 degrees of freedom from the 6 in q^{ik}, so gravitons (like photons above) have two polarizations.

The reduction above is associated with the names of Dirac and Arnowitt-Deser-Misner [ADM]. The explicit forms of all expressions above in the coordinates q^{ik}, Π_{ik} can be found in my survey [2].

Example 4: the Dirac equation

In this lecture, lack of time prevents us from treating the case of the Dirac equation, in which the field is a spin 1/2 fermion, and the Lagrangian is first order in time derivatives from the beginning and is nondegenerate.

References

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[3] L.D. Faddeev, A.A. Slavnov, *Gauge Fields: An Introduction to Quantum Theory*, Addison-Wesley (Frontiers in Physics vol. 83), (second edition), 1991.