Lecture 5: Quantization of Yang-Mills fields

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This lecture treats Yang-Mills theory, an analogue of the field theory for electromagnetism (Example 2 of Lecture 4) in which the gauge group $C^\infty(\mathbf{R}^4, U(1))$ is replaced by $\mathcal{G} = C^\infty(\mathbf{R}^4, G)$ where G is a nonabelian compact Lie group. In this case Feynman discovered the disturbing fact that if one attempts to treat this theory following the approaches which worked in the case of the electromagnetic field, one obtains an S-matrix which is not unitary. Here we shall see that the reason for this is a difference of the geometry of the gauge orbits, which are linear in the electromagnetic case but nonlinear for the case of the nonabelian gauge field.

We begin much as in our treatment of electromagnetism in Lecture 4. As in electromagnetism, we have a field $A_\mu(\vec{x}, t)$ which is a vector representation of the Lorentz group (in other words, it is a one-form on $V = \mathbf{R}^4$); however each component A_0, A_1, A_2, A_3 now takes values in the Lie algebra Lie(G), so the one-form $A = A_\mu dx^\mu$ takes values in Lie(G). (More generally it could be taken to have values in an arbitrary unitary representation of G, but for simplicity we take the representation to be Lie(G) endowed with the adjoint representation.) An element g of the gauge group $\mathcal{G} = C^\infty(\mathbf{R}^4, G)$ acts on A by the transformation

$$A \mapsto A^g \overset{\text{def}}{=} gAg^{-1} + (\partial_\mu g)g^{-1} \tag{1}$$

The curvature is given by

$$F = dA + A \wedge A, \tag{2}$$

where if $A = \sum_a X_a \otimes \alpha_a$ for a basis X_a of Lie(G) and a collection of one-forms α_a, we define

$$A \wedge A = \sum_{a,b} [X_a, X_b] \alpha_a \wedge \alpha_b.$$

The curvature (which is a Lie(G)-valued two-form on $\mathbf{R}^4$) transforms under the gauge transformation g as

$$F \mapsto F^g = gFg^{-1}. \tag{3}$$

As in the abelian case, the Lagrangian is

$$\mathcal{L} = \frac{1}{\epsilon^2} \operatorname{Tr}(F^2), \tag{4}$$

where if $F = \sum_a X_a \otimes \beta_a$ for a collection of two-forms β_a, we define

$$\mathrm{Tr}\,(F^2) = \sum_{a,b} < X_a, X_b > \beta_a \wedge *\beta_b. \tag{5}$$

(A symmetric quadratic form on the space of two-forms on $\mathbf{R}^4$ is defined in the same way, and is also denoted by Tr .) Here, we have introduced an Ad-invariant inner product $< \cdot, \cdot >$ on Lie(G), which in the case of matrix groups is given by the trace Tr . We have also introduced a coupling constant ϵ.

This Lagrangian contains terms of the form $(dA)^2$ as well as terms cubic in A (of order 1 in derivatives) and terms quartic in A (with no derivatives). There are thus two types of vertices, a 3-valent vertex (corresponding to the term cubic in A) and a 4-valent vertex (corresponding to the term quartic in A). By studying the corresponding perturbation expansion, Feynman found that the S-matrix was not unitary.

To proceed further, we need to find the physical variables. We write the Lagrangian in a form that is first order in derivatives, by the device introduced in Lecture 4 where we regarded A and F as independent variables. Thus we see that the Lagrangian is (suppressing the overall multiplicative constant $1/\epsilon^2$ for the moment)

$$\mathcal{L} = \mathrm{Tr}\,(dA + A^2, F) - \frac{1}{2}\,\mathrm{Tr}\,F^2. \tag{6}$$

As in Lecture 4, we have ten variables

$$A_0, \quad A_k, \quad F_{0k} = E_k, \quad F_{ik} = \epsilon_{ikl} H_l.$$

Exactly as in Lecture 4, the variables A_k and E_k are canonically conjugate, while the F_{ik} (or equivalently the H_l) are excludable variables and A_0 is a Lagrange multiplier multiplying a constraint

$$\mathfrak{G} = \nabla_k E^k = \partial_k E^k + [A_k, E_k]. \tag{7}$$

As in Lecture 4 we obtain an equation of motion (*Gauss's Law*)

$$\mathfrak{G} = 0;$$

the difference from the case of abelian G (electromagnetism) is the presence in (7) of the quadratic term $[A_k, E_k]$.

The canonical commutation relations are

$$\left\{ A^a_k(\vec{x}), E^k_b(\vec{y}) \right\} = \delta^a_l \delta^l_b \delta^{(3)}(\vec{x} - \vec{y}). \tag{8}$$

Here, a and b index a basis $\{X_a\}$ of the Lie algebra Lie(G) and we have decomposed $A_k = \sum_a A^a_k \otimes X_a$ for functions A^a_k which form a one-form A^a on $\mathbf{R}^3$.

We introduce a source term $\mathfrak{G}(\eta) = \int \mathrm{Tr}\,\mathfrak{G}(\vec{x})\eta(\vec{x})d\vec{x}$ (where $\eta : \mathbf{R}^3 \to \mathrm{Lie}(G)$ is an element of the Lie algebra of the gauge group). We find that the Poisson brackets of the source terms are

$$\{\mathfrak{G}(\eta_1), \mathfrak{G}(\eta_2)\} = \mathfrak{G}([\eta_1, \eta_2]), \tag{9}$$

in other words the map $\eta \mapsto \mathfrak{G}(\eta)$ (which may be regarded as the momentum map for the gauge group action) is a homomorphism of Lie algebras from the Lie algebra of the gauge group (equipped with the Lie bracket induced by $[,]$) to the Lie algebra of functions (equipped with the Poisson bracket). Equivalently, the constraints are a representation of the Lie algebra of the gauge group. The constraint $\mathfrak{G}$ generates gauge transformations through the Poisson bracket:

$$\{\mathfrak{G}(\eta), A_k(x)\} = \partial_k \eta + [A_k, \eta]. \tag{10}$$

The Hamiltonian is

$$\mathcal{H} = \frac{1}{2}(E^2 + H^2).$$

As in the case of electromagnetism, the variables F_{ik} are excludable. Likewise as in the case of electromagnetism, the constraint equations are first class: we must set $\mathfrak{G} = 0$ and also the Poisson brackets $\{\mathfrak{G}, f\}$ (for arbitrary functions f) must all be set to zero.

The observables the reduced system are thus functionals

$$\phi(A, E)|_{\mathfrak{G}=0} \tag{11}$$

subject to the condition

$$\{\mathfrak{G}, \phi\} = 0. \tag{12}$$

Reduction and solving constraints break the manifest Lorentz invariance. So if we wish to retain this invariance, we must learn how to describe dynamics (the functional integral) without explicitly solving the constraints. To understand how to treat a simpler case, let us return to a finite dimensional phase space Γ, and recall the treatment of this situation as at the beginning of Lecture 4. We assume we have a Lagrangian

$$l = \sum_{i=1}^{N} p_i dq_i - H + \sum_{\alpha=1}^{M} \lambda_\alpha \varphi^\alpha; \tag{13}$$

here we assume that $M < N$ (so that the dimension $2(N-M)$ of the reduced phase space is positive). To parametrize the physical phase space, we take

$$\tilde{\Gamma} = \{(p,q) \in \Gamma : \varphi^\alpha(p,q) = 0 \text{ for all } \alpha\}/\mathcal{G} \tag{14}$$

where $\mathcal{G}$ is the gauge group generated by the constraints φ^α through the Poisson bracket. (In the Yang-Mills example the action of the constraints via Poisson bracket actually does exponentiate to a group action, though this is not always true more generally.) To parametrize the physical degrees of freedom we introduce a slice

$$\chi_\beta(p,q) = 0, \quad \alpha = 1, \ldots, M \tag{15}$$

which intersects each orbit of $\mathcal{G}$ exactly once: the slice (15) is called a *subsidiary condition.* We choose the subsidiary conditions χ_α to satisfy

$$\{\chi_\alpha, \chi_\beta\} = 0. \tag{16}$$

In order to obtain a nondegenerate symplectic structure, we require also that

$$\det(\{\varphi_\alpha, \chi^\beta\}) \neq 0. \tag{17}$$

Imposing the subsidiary conditions (15) and also the constraint equations

$$\varphi_\alpha(p, q) = 0$$

we find a smaller phase space Γ^*. We may take local coordinates on the big phase space Γ given by

$$q = (\chi^\alpha, q^*)$$

and

$$p = (p^\alpha, p^*);$$

here p^* and q^* are the physical variables, while p^α and χ^α must be excluded. Let us write the Liouville measure as

$$dp^* dq^* = \delta(\chi)\delta(\varphi)\det(\{\varphi, \chi\}) dp dq; \tag{18}$$

if we were able to solve

$$\varphi(p, q) = 0 \Longleftrightarrow p^\alpha = p^\alpha(p^*, q^*) \tag{19}$$

this would correspond to the transformation of the delta function under a change of variables,

$$\delta(\varphi) = (\partial\varphi/\partial p)\delta(p - p(p^*, q^*)) \tag{20}$$

which would give rise to the determinant in (18).

We additionally make use of the fact that

$$\delta(\varphi) = \int e^{i\lambda\varphi} d\lambda,$$

so that we can omit the delta function $\delta(\varphi)$ if the action already contains a term $\sum_\alpha \lambda_\alpha \varphi^\alpha$. Thus the functional integral giving rise to the evolution operator is

$$\int \exp \frac{i}{\hbar} \int (p\dot{q} - H - \lambda\varphi) dt \prod_t \frac{dp(t) dq(t) d\lambda(t)}{2\pi\hbar} \delta(\chi) \det(\{\chi, \varphi\}). \tag{21}$$

To pass to the S-matrix only the *physical* degrees of freedom are fixed at the asymptotic values of t, while the Lagrange multiplier and the values of the constraints are allowed to vary freely. Returning to the Yang-Mills case, we get the path integral

$$\int \exp \left\{ \frac{i}{\hbar} \int_{\mathbf{R}^4} ((\partial_0 A)E - \frac{1}{2}(E^2 + H^2) - A_0(\nabla \cdot E)) dx \right\} \prod_t \prod_{\vec{x}} \delta(\partial_k A_k)(\det M_C) dA_\mu dE_k, \tag{22}$$

where as our subsidiary condition we have taken the *Coulomb gauge* condition,

$$\partial_k A_k = 0, \tag{23}$$

and have introduced the operator M_C on Lie(G)-valued functions on $\mathbf{R}^3$, defined by

$$M_C \eta = -\partial_k \partial_k \eta - [A_k, \eta]. \tag{24}$$

The only objects appearing in (22) which are not Lorentz invariant are $\det M_C$ and $\partial_k A_k$.

We can explicitly integrate in (22) over the variables E to get

$$\int \exp\left\{\frac{i}{\hbar}\int_{\mathbf{R}^4}(F_{\mu\nu})^2)dx\right\}\prod_x dA_\mu(x)\prod_t\prod_{\vec{x}}\delta(\partial_k A_k)\det M_C(t). \tag{25}$$

We interpret this expression as an integral over $\mathcal{A}/\mathcal{G}$, where $\mathcal{A}$ is a space of connections and $\mathcal{G}$ is a gauge group. Our subsidiary condition (23) is the choice of a point on the orbit of $\mathcal{G}$: the determinant $\det M_C$ gives the correct measure.

To check that we have indeed obtained the correct measure on the space of $\mathcal{G}$ orbits reduced from the measure $\prod_x\prod_\mu dA_\mu(x)$, it suffices to check that

$$\int \delta(\partial_k A_k^g)\prod_x\prod_{g\in\mathcal{G}} dg_x|_{\partial_k A_k=0} = \frac{1}{\det M_C}; \tag{26}$$

here dg_x is the Haar measure on G at the point g_x. We can linearize the action of $\mathcal{G}$ so that infinitesimally we obtain

$$\int \delta(M_C\eta)\prod_x d\eta_x = \frac{1}{\det M_C}$$

where we have defined the operator M_C by (24). After this understanding we can proceed to choose a different subsidiary condition, in particular a manifestly Lorentz invariant one

$$\partial_\mu A_\mu = c, \tag{27}$$

where c is an arbitrary scalar function. The reduced measure is given by

$$\det M_L \prod_x \delta(\partial_\mu A_\mu - c(x))\prod dA_\mu(x)$$

where $\det M_L$ is obtained by averaging the δ-function

$$\frac{1}{\det M_L} = \int \delta(\partial_\mu A_\mu^g - c(x))\prod_x dg(x).$$

Here, we have defined the operator M_L (compare the definition (24) of M_C) by

$$M_L\eta = \Box\eta + \partial_\mu[A_\mu,\eta]. \tag{28}$$

The path integral now becomes

$$\int \exp\left\{\frac{i}{\hbar}\int(F_{\mu\nu}^2 dx)\right\}\prod_x\delta(\partial_\mu A_\mu - c)(\det M_L)\prod_x dA_\mu(x). \tag{29}$$

The final step is to average over $c(x)$. Ultimately the expression (29) becomes

$$\int \exp\left\{\frac{i}{\hbar}\int_{\mathbf{R}^4}(F_{\mu\nu})^2 dx\right\}\prod_x\delta(\partial_\mu A_\mu(x) - c)(\det M_L)\prod dA_\mu(x)\prod_c e^{i\alpha\int c^2 dx} \tag{30}$$

$$= \int \exp\left\{\frac{i}{\hbar}\int_{\mathbf{R}^4}(F_{\mu\nu})^2 + \alpha(\partial_\mu A_\mu)^2)dx\right\}\prod_x(\det M_C)\prod dA_\mu(x). \tag{31}$$

Here, α is an arbitrary constant. The integrand in (30) is not local (because the determinant of the operator M_C is not local): to transform it into a local expression we introduce Grassmann variables $c = (c_1, \ldots, c_r)$ and $\bar{c} = (\bar{c}_1, \ldots, \bar{c}_r)$, in other words odd variables in a Grassmann algebra, satisfying the anticommutation relations

$$c_i \bar{c}_j + \bar{c}_j c_i = 0,$$

$$c_i^2 = \bar{c}_i^{\,2} = 0$$

and recall the identity

$$\det A = \int e^{\bar{c}\mathfrak{A}c} dc^* dc \tag{32}$$

where A is an endomorphism of a vector space of dimension r and we have introduced the corresponding endomorphism of a vector space of dimension $2r$,

$$\mathfrak{A} = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}. \tag{33}$$

We now replace the Grassmann variables c and $\bar{c}$ (which took values in a finite dimensional vector space) by Grassmann variables c and $\bar{c}$ taking values in the infinite dimensional space of functions on $\mathbf{R}^4$. Rewriting the determinant in (30) by formally applying (32), (30) is transformed into the form

$$\int \exp\left\{ \frac{i}{\hbar\epsilon^2} \operatorname{Tr} \int_{\mathbf{R}^4} \Big((F_{\mu\nu})^2 + \alpha(\partial_\mu A^\mu)^2 + (\partial_\mu \bar{c}, \partial_\mu c + [A_\mu, c] \Big) dx \right\} \prod dA_\mu dc d\bar{c}. \tag{34}$$

We have now transformed the path integral into an integral of the form $\int \exp(i\text{Action})$, where the action is local and has a nondegenerate quadratic form, and we integrate over all fields (which are assumed to satisfy appropriate boundary conditions). Here c and $\bar{c}$ are fields which should be regarded as carrying charge (because of the terms mixing A_μ, c and $\bar{c}$) and which should be regarded as fermionic (because they are odd elements in a superalgebra or Grassmann algebra). The expression (34) is the generating functional from which the S-matrix of Yang-Mills theory may be extracted. There are still the 3-valent and 4-valent vertices in which all the lines emanating from the vertex correspond to the propagator of A. Additionally we now have another 3-valent vertex with one A propagator, one c propagator and one $\bar{c}$ propagator: the c and $\bar{c}$ propagators may be distinguished by writing an arrow on the c propagator pointing toward the vertex and an arrow on the $\bar{c}$ propagator pointing away from the vertex.

Near a classical solution of the equations of motion for the Yang-Mills fields, the correction terms entering the perturbation expansion are proportional to $F_{\mu\nu}^2$ (recall that a term of the form $F_{\mu\nu}^2$ appeared in our original Lagrangian). The term $\frac{1}{\epsilon^2}(F_{\mu\nu}^2)$ in the Lagrangian is renormalized to

$$\left(\frac{1}{\epsilon^2} + a\ln(\Lambda)\right) F_{\mu\nu}^2$$

for an appropriate cutoff parameter Λ and a suitable constant a. We notice that a must be negative in order for the renormalization procedure to succeed: this property is referred to in the literature as "asymptotic freedom".