LECTURE 1: RENORMALIZATION GROUPS

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1.1. What is renormalization group?

In these lectures we will consider the method of renormalization group and its applications to quantum field theory.

Very generally speaking, the method of renormalization group is a method designed to describe how the dynamics of some system changes when we change the scale (distances, energies) at which we probe it.

Before describing more precisely what exactly this method does, recall two basic facts about physics:

1. Scale dependence. Physics is scale-dependent. For example, consider a fluid. At each scale of distances, we need a different theory to describe its behavior:

At ~ 1cm – classical continuum mechanics (Navier-Stokes equations)

At $\sim 10^{-5}$cm – theory of granular structure

At $\sim 10^{-8}$cm – theory of atom (nucleus + electronic cloud)

At $\sim 10^{-13}$cm – nuclear physics (nucleons)

At $\sim 10^{-13} - 10^{-18}$cm quantum chromodynamics (quarks)

At $\sim 10^{-33}$ string theory

At each scale, we have different degrees of freedom and different dynamics.

2. Decoupling. Physics at a larger scale (largely) decouples from the physics at a smaller scale. For example, to describe the behavior of fluid at the scale ~ 1cm, we don't need to know about the granular structure, nor about atoms or nucleons. The only things we need to know is the viscosity and the density of the fluid. Of course, these values can be computed from the physics at a smaller scale, but if we found them out in some way (for example, measurement), we can do without smaller scale theories at all. Similarly, if we want to describe atoms, we don't need to know anything about the nucleus except its mass and electric charge.

Thus, a theory at a larger scale remembers only finitely many parameters from the theories at smaller scales, and throws the rest of the details away. More precisely, when we pass from a smaller scale to a larger scale, we average over irrelevant degrees of freedom. Mathematically this means that they become integration variables and thus disappear in the answer.

This decoupling is the reason why we are able to do physics. If there was no decoupling, it would be necessary for Newton to know string theory to describe the motion of a viscous fluid.

Remark. This transition of information from smaller scale theories to larger scale theories through finitely many parameters is of course irreversible, since in such transition we forget (=integrate out) many irrelevant degrees of freedom. For

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example, it is possible to reconstruct the thermodynamics of gas from molecular theory (by averaging over all configurations), but it is impossible to reconstruct the behavior of molecules from the macroscopic behavior of gas itself.

The general aim of renormalization group method is to explain how this decoupling takes place and why exactly information is transmitted from scale to scale through finitely many parameters.

Remark 1. In quantum theory, decoupling of scales is not at all obvious. Indeed, because of the uncertainty principle, we have to work at all scales at a time. Renormalization group explains why decoupling survives in quantum theory.

Remark 2. In classical mechanics, there are 3 basic units of measurement (distance D, time T, mass M), and all others can be expressed through them. Thus, in classical mechanics we deal with three scales. In nonrelativistic quantum theory and in classical relativity there remains only two of them, as in the first case we can express M through T and D using the Planck constant, and in the second T can be expressed via D using the speed of light. Thus, in relativistic quantum theory we only have one scale – the scale of distances. Equivalently we can use the inverse scale – the scale of momenta. Thus we have:

SMALL distances, times = LARGE momenta, energies, masses.

1.2. The general scheme of the method of renormalization group.

As we already mentioned, in quantum physics in general, in particular in any quantum field theory, we must consider arbitrarily short scales. Thus, in order to understand the theory at a given scale, we should know something about its behavior at smaller scales. The theory is called renormalizable if all the information we need from smaller scales can be absorbed in finitely many parameters (renormalized masses and couplings). In this case, we can measure these parameters (or some functions of them), and thus have a complete picture of the theory. However, if the theory is not renormalizable, the number of parameters coming from smaller scales is infinite, so the theory is unmanageable. This is why we can successfully work only with renormalizable theories.

Now we make a general review of renormalization group theory. To think about renormalization group, one can use the following general picture, due to Wilson.

The central object in Wilson's picture is "the space of theories" S. In general, this object is extremely ill-defined. However, in special cases it may be possible to give a precise definition of S. In quantum field theory we can model this space as the space of all possible Lagrangians of certain types of fields.

Wilson's procedure is the following. Let M be the set of scales of momenta. Mathematically, M is a principal homogeneous space (torsor) of $\mathbb{R}_+$. (This means, M is isomorphic to $\mathbb{R}_+$, but noncanonically; in particular, there is no distinguished element in M). Let $\Lambda, \mu \in M$ be scales of momenta, and $\mu < \Lambda$. For each theory $\mathcal{L} \in S$, one finds another theory, $R_{\Lambda\mu}\mathcal{L} \in S$ which is the effective theory, at the scale μ, for the original theory $\mathcal{L}$ at the scale Λ. Thus, we have a map $R_{\Lambda\mu} : S \to S$, and $R_{\Lambda_1\Lambda_2}R_{\Lambda_2\Lambda_3} = R_{\Lambda_1\Lambda_3}$. Thus, we have an action of the semigroup $\mathbb{R}_+^{\leq 1}$ on the space $S \times M$, by $\lambda \circ (\mathcal{L}, \Lambda) = (R_{\Lambda,\lambda\Lambda}\mathcal{L}, \lambda\Lambda)$. Abusing mathematical terminology, physicists call this semigroup the renormalization group.

1.3. Wilsonian scheme for the theory of a scalar field: a mathematical description (an interpretation of D.Gross' lectures by the preparers of these lecture notes).

Let us give a precise mathematical description of renormalization group in the special case of theories of a scalar bosonic field in 4 dimensions.

Let V be a 4-dimensional Euclidean space, and $\mathcal{S}(V)$ the space of Schwarz functions on V. By a homogeneous interaction of degree k we mean a continuous, homogeneous, Poincare invariant polynomial functional $J : \mathcal{S}(V) \to \mathbb{C}$. That is,

$$J(\phi) = \int K_J(x_1, ..., x_k)\phi(x_1)...\phi(x_k)dx_1...dx_k,$$

where K_J is a Poincare invariant tempered distribution.

Definition. *(i) A homogeneous interaction J is called local if K_J is supported on the diagonal.*

(ii) J is called quasilocal if K_J has rapid decay at infinity, i.e. if the Fourier transform $\tilde{K}_J$ of K_J has the form

$$\tilde{K}_J(p_1, ..., p_k) = \delta(p_1 + ... + p_k)\tilde{F}_J(p_1, ..., p_{k-1}),$$

where $\tilde{F}_J$ is a smooth function.

(iii) We say that a quasilocal interaction J is of order $\geq s$ if the function $\tilde{F}_J$ has order $\geq s$ at the origin. We will write $O(\nabla^s)$ for a quasilocal interaction of order $\geq s$.

It is clear a local interaction is the same thing as a differential polynomial in ϕ, and the order of a differential monomial is the number of derivatives in it. This statement can be generalized to the case of quasilocal interactions, as follows.

Proposition 1. *Let J be a quasilocal interaction of degree $k \geq 1$. Then for any $j \geq 0$ there exists a unique homogeneous differential operator D_j of order j on V^{k-1} with constant coefficients, such that for any $\phi \in \mathcal{S}(V)$, and any $s \geq 0$ the functional $J(\phi)$ is given by a Taylor formula*

$$J(\phi) = \sum_{j=0}^{s-1} \int_V D_j[\phi(x_1)...\phi(x_{k-1})]|_{x_1=...=x_{k-1}=x}\phi(x)dx + J^s(\phi), \ J^s(\phi) = O(\nabla^s).$$

Proof. We have

$$J(\phi) = \int_{(V^*)^(k-1)} \tilde{F}_J(p_1, ..., p_{k-1})\tilde{\phi}_1(p_1)...\tilde{\phi}(p_{k-1})\tilde{\phi}_k(-p_1 - ... - p_{k-1})dp_1...dp_{k-1},$$

where $\tilde{\phi}$ is the Fourier transform of ϕ. Since $\tilde{F}_J$ is smooth, it has a unique Taylor expansion

$$\tilde{F}_J(p_1, ..., p_{k-1}) = \sum_{j=0}^{s-1} D_j(p_1, ..., p_{k-1}) + O(|p|^s).$$

This implies the proposition.

Now let τ be a formal variable. We will work over the ring $\mathbb{R}[[\tau]]$.

Let X be the space of all Lagrangians of ϕ with coefficients in $\mathbb{R}[[\tau]]$, invariant under $\phi \to -\phi$, of the form

$$\mathcal{L} = \int (\frac{1}{2}(a(\nabla\phi)^2 + m^2\phi^2) + I(\phi))d^4x, \tag{1.2}$$

where $I(\phi) = \sum_{k\geq 2} I_k(\phi)$ are interaction terms (I_k has degree k), such that

(i) $a, m^2 \in \mathbb{R}_+$ and $I_k = O(\tau)$ for all k;

(ii) For any $N \geq 0$ $I_k = O(\tau^N)$ for almost all k;

(iii) I_k are quasilocal.

Consider the space $S = X \times M$. The space S is the relevant space of theories in which Wilson's flow will be defined.

Let F_k be the space of smooth functions on the hyperplane $\sum_{i=1}^k v_i = 0$ in $(V^*)^{\oplus k}$ (we identify this hyperplane with $(V^*)^{(k-1)}$ by throwing out the last component v_k). Let $F = \prod_{i=1}^\infty F_{2i}$.

In the next few paragraphs we will define a map $\mathcal{G} : S \to F$, $\mathcal{G} = (\mathcal{G}_2, \mathcal{G}_4, ...)$. The value of $\mathcal{G}_k$ at $(\mathcal{L}, \Lambda)$ is called the k-point correlation function of the Lagrangian $\mathcal{L}$ at scale (cutoff) Λ.

Let $P(r)$ be a smooth function on $\mathbb{R}_+$ that equals 1 for $r \leq 1/2$, 0 for $r > 2$, and varies between 0 and 1 in between. Define the cutoff propagator (in momentum space) to be $\tilde{G}_\Lambda(p) = \frac{P(p^2/\Lambda^2)}{p^2+m^2}$.

Let $(\mathcal{L}, \Lambda) \in S$. By definition, the algorithm of computing $\mathcal{G}_k(\mathcal{L}, \Lambda)$ is as follows. $\mathcal{G}_k$ is the sum of amplitudes of all Feynman diagrams with k external edges. Computation of the amplitude is as usual (by Feynman rules), except that on each internal edge instead of the free propagator $1/(p^2+m^2)$ one should put the cutoff propagator $\tilde{G}_\Lambda(p)$. On external edges one should put the free propagator.

One can also define connected and 1-particle irreducible correlation functions, which are obtained by summing the amplitudes of connected and 1-particle irreducible diagrams, respectively (see Kazhdan's lectures).

For example, if $\Lambda = 0$, the cutoff propagator vanishes, so that connected correlation functions are the same as interactions.

Now let μ, Λ be arbitrary scales. Then one can make the following definition.

Definition. *A Lagrangian $\mathcal{L}_e$ of the form (1.2) is called an effective Lagrangian at the scale μ of a Lagrangian $\mathcal{L}$ at the scale Λ if $\mathcal{G}(\mathcal{L}_e, \mu) = \mathcal{G}(\mathcal{L}, \Lambda)$.*

Proposition 2. *For any $\mathcal{L} \in X$ and any μ, Λ there exists a unique effective Lagrangian $\mathcal{L}_e$ at the scale μ of $\mathcal{L}$ at the scale Λ.*

Proof. It is easy to check that a Lagrangian is uniquely determined by its correlation functions. This implies the uniqueness of $\mathcal{L}_e$, if it exists. The existence of $\mathcal{L}_e$ is proved by giving an explicit construction. The construction: the function $\tilde{F}_{J_k}$ for $\mathcal{L}_e$ is defined to be the sum of amplitudes of all connected Feynman diagrams with k external edges, computed for $\mathcal{L}$ with the propagator $\tilde{G}_{\Lambda\mu}(p) := \tilde{G}_\Lambda(p) - \tilde{G}_\mu(p)$. It is not difficult to check that $\mathcal{L}_e$ is an effective Lagrangian at μ for $\mathcal{L}$ at Λ.

We will denote the Lagrangian $\mathcal{L}_e$ by $R_{\Lambda\mu}\mathcal{L}$. This defines a map $R_{\Lambda\mu} : X \to X$ for any two scales Λ, μ. It is clear that $R_{\Lambda_1\Lambda_2}R_{\Lambda_2\Lambda_3} = R_{\Lambda_1\Lambda_3}$, and $R_{\Lambda\Lambda} = 1$.

Thus, we have a flow on the space S given by the formula $t(\mathcal{L},\Lambda)=(R_{\Lambda,t\Lambda}\mathcal{L},t\Lambda)$, $t\in\mathbb{R}^+$. This flow is called the renormalization group flow.

Remark 1. It is easy to see that if a Lagrangian $\mathcal{L}$ is free (i.e. $I_k=0$) then $R_{\Lambda\mu}\mathcal{L}=\mathcal{L}$, since in this case we have no internal vertices and no nontrivial diagrams.

Remark 2. In principle, instead of the described "smeared cutoff", we could use the most obvious cutoff, i.e. define a cutoff propagator $\tilde{G}^*_\Lambda(p):=\frac{1_\Lambda}{p^2+m^2}$, where 1_Λ is the characteristic function of the ball $|p|<\Lambda$. In the language of functional integral, this means the following: the k-point correlation function (in position space) is obtained by integration of $\phi(x_1)...\phi(x_n)$ not over all fields $\phi(x)$, but only over fields whose Fourier tranform is supported in the region $|p|<\Lambda$. Similarly, the effective Lagrangian $\mathcal{L}_e$ at μ for $\mathcal{L}$ at Λ is given (for $\mu<\Lambda$) by the formula

$$e^{-\mathcal{L}_e(\phi)}=\int e^{-\mathcal{L}(\phi+\eta)}D\eta \tag{1.3}$$

where ϕ is a field whose Fourier transform is supported in the ball $|p|\le\mu$, and the functional integral is taken over all fields η whose Fourier transform is supported in $\mu\le|p|\le\Lambda$.

This picture has a nice physical interpretation: in order to compute $R_{\Lambda\mu}$, we integrate out degrees of freedom – the amplitudes of harmonics with frequencies between Λ and μ.

Remark 3. The map $R_{\Lambda\mu}$, although it is formally defined (by Feynman rules) for any Λ,μ, makes physical sense only for $\Lambda>\mu$. This is well illustrated by Remark 2, as there $R_{\Lambda\mu}$ by the definition reduces to integrating out degrees of freedom. In fact, beyond perturbation theory the map $R_{\Lambda\mu}$ is usually not defined for $\Lambda<\mu$.

Remark 4. Unfortunately, the sharp cutoff $\tilde{G}^*_\Lambda$ defined in Remark 2 is not quite satisfactory, since the corresponding renormalization group flow on the space of Lagrangians does not preserve quasilocality (quasilocality is necessary for formulation of Theorem 4). Indeed, let us consider a Lagrangian

$$\mathcal{L}=\int(\frac{1}{2}(a(\nabla\phi)^2+m^2\phi^2)+\frac{g}{4!}\phi^4)d^4x, \tag{1.4}$$

where $g\sim\tau$. It is easy to see that in the order τ^2 we will get a correction to the quadratic term of the form

$$\int\phi(x)\phi(y)(G^*_\Lambda(x-y)-G^*_\mu(x-y))dxdy,$$

where $G^*_\Lambda(x)$ is the cutoff propagator in position space. Since the cutoff propagator in momentum space is not smooth, this term is not quasilocal.

This is why it is better to use the smooth cutoff $\tilde{G}_\Lambda$ instead of the sharp cutoff $\tilde{G}^*_\Lambda$.

Now we will describe the main property of the renormalization group. Very roughly, this property is that the operator $R_{\Lambda\mu}$ for small μ/Λ makes all non-renormalizable terms in the Lagrangian irrelevant, so in the limit $\Lambda\to\infty$ only renormalizable terms matter. If there is only finitely many renormalizable terms (which is usually the case), this means that the renormalization group flow has a finite dimensional attractor in the space of theories – the space of renormalizable, or critical theories. This property is the reason why we actually never have to consider non-renormalizable theories in the perturbative setting.

Let us describe this property more carefully. First recall that in the theory of a scalar field, the only renormalizable Lagrangians are of the form (1.4). Let us denote the set of all renormalizable Lagrangians by X_r.

Let $T_0(X)$ be the tangent space to X at any point (all of them are naturally isomorphic).

For any $\mathcal{L} \in T_0(X)$, denote by $P_s(\mathcal{L})$ the part of the Lagrangian $\mathcal{L}$ whose coeffients have scaling dimension $\leq -s$. That is, if $\mathcal{L}(\phi) = \sum_k J_{2k}(\phi)$, then by definition $P_s(\mathcal{L})(\phi) = \sum J_{2k}^{s-2k+4}(\phi)$, where J^l for $l > 0$ was defined in Proposition 1, and by convention for nonpositive l $J^l(\phi) = J(\phi)$.

We will call a Lagrangian $\mathcal{L} \in T_0(X)$ purely non-renormalizable if $\mathcal{L} = P_1(\mathcal{L})$. It is clear that $\mathcal{L}$ is purely nonrenormalizable if and only if $\tilde{F}_{J_2}(0) = 0$, $\frac{\partial \tilde{F}_{J_2}}{\partial p^2}|_{p=0} = 0$, and $\tilde{F}_{J_4}(0,0,0) = 0$. Denote by X_n the set of all purely non-remormalizable Lagrangians.

It is obvious that any Lagrangian can be written uniquely as a sum $\mathcal{L} = \mathcal{L}_r + \mathcal{L}_n$, where $\mathcal{L}_r \in X_r$, $\mathcal{L}_n \in X_n$. Thus, $X = X_r \times X_n$.

Now fix an element $\mathcal{L} \in X_r$, and consider the intersection $(\mathcal{L}+X_n) \cap R_{\Lambda\mu} X_r \subset X$.

Lemma 3. *The intersection $(\mathcal{L} + X_n) \cap R_{\Lambda\mu} X_r$ consists of 1 point.*

Proof. The Lemma is obvious modulo τ, and in higher orders in τ it is established by induction (the implicit function theorem).

Let $B_{\Lambda\mu}(\mathcal{L})$ be the unique point in $(\mathcal{L} + X_n) \cap R_{\Lambda\mu} X_r$.

For any $\nu \in \mathbb{R}^+$, define the linear rescaling map $Q_\nu : T_0(X) \to T_0(X)$ by $Q_\nu(\mathcal{L})(\phi(x)) = \mathcal{L}(\nu^{-1}\phi(\nu^{-1}x))$. For example, $Q_\nu(\int \phi^6(x) d^4x) = \nu^{-2} \int \phi^6(x) d^4x$.

The following theorem represents the main properties of the renormalization group.

Theorem 4.

(i) (Convergence) There exists a limit $B_\mu(\mathcal{L}) = \lim_{\Lambda\to\infty} B_{\Lambda\mu}(\mathcal{L})$. That is, all interactions converge (in the weak sense).

(ii) (Rate of convergence) The rate of this convergence is determined by the formula

$$P_s(B_{\Lambda\mu}(\mathcal{L})) - P_s(B_\mu(\mathcal{L})) = O(\Lambda^{-s+\varepsilon}), \Lambda \to \infty.$$

for any $s \geq 1$ and $\varepsilon > 0$.

(iii) (Stability) Let $Y_r(\mu) = B_\mu(X_r)$. Let $\Delta\mathcal{L} \in X_n$, Then there set $(\mathcal{L} + X_n) \cap R_{\Lambda\mu}(X_r + Q_{\Lambda/\mu}(\Delta\mathcal{L}))$ consists of only one point, $B_{\Lambda\mu}^{\Delta\mathcal{L}}(\mathcal{L})$. Moreover, there exists a limit $\lim_{\Lambda\to\infty} B_{\Lambda\mu}^{\Delta L}(\mathcal{L}) = B_\mu^{\Delta\mathcal{L}}(\mathcal{L})$, which belongs to $Y_r(\mu)$.

(iv) (Set convergence) Let $X(s) \subset X$ be the set of Lagrangians which are polynomials of degree $\leq s$ in ϕ. Then for $s \geq 4$ $Y_r(\mu) = \lim_{\Lambda\to\infty} R_{\Lambda\mu}(X_s)$.

The proof of this theorem will be partially discussed later. It is essentially contained in the paper by J. Polchinski "Renormalization and effective Lagrangians", Nucl. Phys. B231(1984) 269-295.

Theorem 4 shows that in our case the renormalization group flow has an attractor – a three-dimensional family of trajectories Y_r, which can be parametrized by the values of critical and subcritical coefficients in the Lagrangian. Thus, the only things we (effectively) inherit from smaller scales are the values of these 3 parameters.

We see that the space $Y_r(\mu)$ is defined completely canonically, and $R_{\mu_1\mu_2}Y_r(\mu_1) = Y_r(\mu_2)$. The space $Y_r(\mu)$ has the meaning of the space of exact (renormalized) quantum field theories, i.e. those which are obtained by perturbative renormalization as described in Witten's lectures. These are the only theories that we will care about.

This very simple example demonstrates the main principle of the method of renormalization group.

1.4. Applications of renormalization theory to phase transitions.

As we mentioned, one of the main applications of renormalization group in quantum field theory is that it allows to absorb all short-scale information in finitely many constants. However, there are other applications as well. One of them is to the theory of phase transitions in many-body systems (statistical mechanics). For example, consider a magnetic chain on a lattice L. Each vertex v of the lattice has a spin $s(v)$, 1 or -1. The Lagrangian of the theory, for example, can be

$$\mathcal{L} = \frac{c}{T} \sum_{v--v'} (s(v) - s(v'))^2,$$

where T is the temperature, and $v - -v'$ means thet v is a neighbor of v'. For low T the correlation is very strong, so there are large (macroscopic) clusters of aligned spin (magnetism) For high T the spins are almost uncorrelated, so there are only very small clusters of aligned spins (no magnetism). The transition from one situation to the other is happening at some fixed temperature T_c called the critical temperature. At this temperature we expect to see clusters of all sizes, and the theory to be scale invariant (conformal). Hence this theory should correspond to a "fixed point" (=an attracting trajectory) of the renormalization group flow on the relevant space of theories. We may try to find this trajectory if we have some description of the relevant space of theories and of the renormalization group flow, and thus understand the behaviour of the magnet at the critical temperature.

Remark. In general (for example, in the theory of a scalar field), the renormalization group does not have a attracting trajectory, but has an attracting finite-dimensional family of trajectories – the space of critical (renormalizable) theories. In this case we have to make finitely many measurements, in order to determine at which point of this space we are. For instance, in the theory of a scalar field we would have to measure three parameters, which determine a point on the 3-dimensional variety Y_r. All other physical quantities can then be computed, i.e. expressed via the results of these measurements.

1.5. Reminder of renormalization theory.

Before we discuss how renormalization group works in quantum field theory, we recall the basics of classical renormalization theory. Since there were other lectures on this subject, we will do it briefly.

In quantum field theory, we consider Lagrangians depending on certain kinds of fields (scalar fields, fermionic fields, gauge fields). For example, we can consider a Lagrangian of the form (1.1). We are interested in correlation functions of the form $\langle T(\phi_1(x_1)...\phi_N(x_N))\rangle$, where T means time ordering. It is usually easier to compute the Fourier transform of this function. Feynman rules allow us to do it perturbatively, i.e. to obtain the answer in the form of a power series in coupling constants. This answer comes with two problems:

1. Every term of the series is given by a divergent integral.

2. Even if all terms are convergent, the series is divergent in any neighborhood of zero.

Renormalization theory helps to overcome the first problem. The second problem has to do with the (difficult) question whether the theory actually exists nonperturbatively. If it does, this problem can be solved by regarding the series expansion not as a Taylor expansion but as an asymptotic expansion of a certain function.

Renormalization theory works in two steps: 1) regularization; 2) renormalization.

Regularization is introducing some kind of cutoff, which makes the integral convergent. After regularization, the integral is convergent, but depends on the regularization parameter Λ. If Λ tends to ∞ (i.e. when the cutoff is removed), the integral tends to infinity. This problem is removed by the second step – renormalization. Namely, we find a way to change the parameters in the Lagrangian in such a Λ-dependent way that when Λ and the parameters in the Lagrangian are varied simultaneously, with $\Lambda \to \infty$, the integral converges to a finite limit. This limit is then proclaimed to be the value of the integral.

The most naive cutoff scheme is suppressing large momenta by changing the propagator, for example $\frac{1}{k^2+m^2} \to \frac{1_{|k|\le\Lambda}}{k^2+m^2}$ (where 1_A is the characteristic function of a set A), or $\frac{1}{k^2+m^2} \to \frac{1}{k^2+m^2}\left(\frac{\Lambda^2}{k^2+\Lambda^2}\right)^l$. This scheme ha s some nice features (e.g. it can be applied in a non-perturbative setting), but is not always good enough (for example, in gauge theory), as it does not preserve all symmetries. Namely, it preserves Lorenz symmetry but does not preserve gauge symmetries or conformal symmetries. So if we use this scheme, we have no gauge or conformal symmetry for each finite Λ, and may not recover this symmetry even in the limit.

There is a more ingenious regularization scheme, which is called dimensional regularization. This scheme preserves all symmetries, but it is only valid in the perturbation expansion. Now we will describe what dimensional regularization is.

1.6. Dimensional regularization.

The idea of dimensional regularization is to compute the integral corresponding to each Feynman graph in the "space" of dimension D, where D is a generic complex number. If the graph is divergent, the answer turns out to be a meromorphic function of D, usually having a pole at D equal to the physical dimension d. The renormalization techique usually used with this kind of regularization is minimal subtraction: we subtract the pole part (this is called "minimal subtraction") and proclaim the rest (evaluated at $D = d$) to be the value of the integral.

Let us now describe the algorithm of dimensional regularization. For simplicity we assume that our theory includes only scalar massive fields.

Suppose we have a Feynman graph G with N external edges, and momenta $k_1, ..., k_M$ at them ($k_1 + ... + k_M = 0$). Let $q_1, ..., q_{N-M}$ be the loop variables over which we are integrating. To uniformize the notation, we set $q_{N-M+i} = k_i$. Then the amplitude of the graph is (up to a factor) given by the formula

$$\Sigma^G(q_{N-M+1},...,q_N) = \int_{V^{N-M}} g(q)dq_1...dq_{N-M},\ g(q) = \frac{Q(q)}{\prod_j (l_j(q)^2+m_j^2)}, \tag{1.5}$$

where Q is a polynomial, $q=(q_1,...,q_N)$ and $l_j : V^N \to V$ are given by linear forms on $\mathbb{R}^N$.

Unfortunately, integral (1.5) is divergent, so we will explain how to regularize it using dimensional regularization. As any regularization scheme, dimensional regularization works inductively in the number of loops (or internal vertices) of the graph. So we assume that the coefficients in the Lagrangian have been adjusted at the previous step of the process in such a way that all proper subgraphs of G are convergent.

In order to regularize the integral corresponding to G, we pretend that it exists and, using the formula $\int_0^\infty e^{-at}dt = a^{-1}$, transform it to the form

$$\Sigma^G = \int Q(q)\left[\int_{t_j\in[0,\infty)} e^{-\sum t_j l_j(q)^2 - \sum t_j m_j^2} dt\right] dq =$$
$$\int_{t_j\in[0,\infty)} e^{-\sum t_j m_j^2}\left[\int Q(q) e^{-\sum t_j l_j(q)^2} dq\right] dt. \tag{1.7}$$

The q-integral in brackets in the last expression is a Gaussian integral, so it can be computed according to the standard rules. Namely, from Poincare invariance it is clear that the answer is of the form $\psi(t,A,d)\det(B(t))^{-d/2}$, where $B(t)$ is a quadratic form on $\mathbb{R}^{N-M}$ such that $B(t)_{ij} = (\sum t_s l_s^2)_{ij}$, $A=(a_{ij})=(k_i\cdot k_j)$, and ψ is a polynomial in d.

Thus we get

$$\Sigma^G = \Sigma^G(A) = \int_{t_j\in[0,\infty)} e^{-\sum t_j m_j^2}\psi(t,A,d)(\det B(t))^{-d/2}dt. \tag{1.8}$$

Of course, this integral is also divergent. However, since it does not contain integration with respect to q, so we can replace d in it with a generic complex number D. It is easy to see that the obtained integral $\Sigma^G_D(A)$ is convergent for $\mathrm{Re}D << 0$, and one can prove that it continues meromorphically to the whole complex plane (this is a nontrivial theorem).

Definition. $\Sigma^G(A)$ is the value of the regular part of $\Sigma^G_D(A)$ at $D=d$. That is,

$$\Sigma^G(A) = \frac{1}{2\pi i}\oint_{|D-d|=\varepsilon} \Sigma^G_D(A)d\ln(D-d). \tag{1.9}$$

Now, if the analytic continuation of integral (1.8) is singular at $D=d$, we should adjust the Lagrangian by adding to it a counterterm $C_G(D)$ (infinite at $D=d$) in such a way that the singular part cancels and the regular part remains unchanged. After this we can go to the next step of the process.

Remark. The only divergences that produce a pole at $D=d$ are logarithmic divergences. Pure power divergences produce poles at smaller values of D, and do not produce a singularity at $D=d$.

For a more detailed description of the general theory of integration in a D-dimension space, where D is a complex number, see the notes "On integration in a space with complex dimension" by Pavel Etingof.