LECTURE 2: RENORMALIZATION GROUPS (CONTINUED)

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2.1. Finite renormalization.

As we saw in the previous lecture and in lectures of Witten and Gawedzki, renormalizable quantum field theories always come in families (which may even be infinitely-parametric in 2 dimensions), and we need to specify certain quantities in order to pinpoint a specific theory among such a family. The only ways to do this are to measure these quantities or to set their values artificially. But when we know them, we can (in principle) compute all other quantities in the theory as functions of them.

The procedure of picking a theory by setting its parameters to the results of measurements (or artificially) is called finite renormalization, as opposed to "infinite" renormalization which we discussed above. While infinite renormalization deals with removing divergences (infinities), finite renormalization adjusts the finite part of the coefficients in the Lagrangian (as functions of the cutoff Λ), which infinite renormalization leaves ambiguous.

Thus, a prescription of finite renormalization is a way to define a system of coordinates (or a parametrization) of the space of renormalizable theories of a given type.

It is clear that the number of quantities to be specified (i.e. the dimension of the space of renormalizable theories) equals to the number of coefficients of the Lagrangian of nonnegative dimension. However, these coefficients themselves are, in general, not suitable as coordinates on the space of theories, as they depend on the cutoff Λ and are infinite at $\Lambda \to \infty$. Therefore, people use other systems of coordinates.

For example, for the space Y of renormalizable theories of a scalar field, we could use the following parametrizations:

1. $T \to (\mathcal{G}_2(0), \frac{\partial \mathcal{G}_2}{\partial p^2}(0), \mathcal{G}_4(0,0,0))$, where $\mathcal{G}_i$ are connected correlation functions.

2. $T \to (\mathcal{G}_2(\mu^2), \frac{\partial \mathcal{G}_2}{\partial p^2}(\mu^2), \mathcal{G}_4(p_1,p_2,p_3))$, where $\mu \in M$ is some scale, and $p_1,p_2,p_3,p_4=-p_1-p_2-p_3$ are vertices of a regular tetrahedron such that $p_i^2=\mu^2$.

Notice that the the first recipe defines one fixed parametrization of the space of theories, while the second one is in fact a family of such parametrizations, depending on a scale $\mu \in M$. In the future, such families of parametrizations will be more important to us than fixed parametrizations.

Remark. The coordinates in both of the above parametrizations are not measurable quantities, since momenta of all particles which can be generated in a lab satisfy the condition $p^2=-m^2$, where m is the mass of the particle (people say that "p is on the mass shell"). In the case of ϕ^4 theory, it is possible to make up a system of coordinates purely out of measurable quantities. (Of course, here we make the (invalid) assumption that the ϕ^4 theory describes a real physical process.) Let us explain how to do this.

Suppose that we have an elementary particle ϕ which obeys a renormalizable ϕ^4-theory. We have seen that renormalizable ϕ^4-theories live in a 3-parametric family Y. For $T \in Y$, let σ_T be the residue of the 2-point function of T (in momentum space) at its pole. The dimensionless quantity σ_T is called the scale of ϕ in T.

Measurable quantities are masses of particles and scattering amplitudes, which are scale-of-ϕ-independent objects. So let us be interested only in scale-of-ϕ-independent characteristics of a given theory. In this case we can restrict our attention to the 2-parameter subfamily of theories $Y_1 \subset Y$ in which the scale of ϕ equals 1. This causes no loss of generality, since any theory $T \in Y$ can be normalized to a theory $T_1 \in Y_1$ by a transformation $\phi \to \alpha\phi$, in such a way that all scale-of-ϕ-independent quantities are unchanged.

Now our space of theories is 2-dimensional, so we need to measure 2 things in order to specify a concrete theory. One of the possible prescriptions is the following. The two measured things are

(i) The mass m of the field ϕ. Mathematically, it is determined from the condition that $-m^2$ is the pole of the 2-point function of the theory, regarded as a function of the square of momentum k^2 (in the Euclidean picture).

(ii) The effective coupling g – the scattering amplitude for two particles at the some point (p_1, p_2, p_3, p_4) "on shell", i.e. with $p_i^2 = -m^2$ and $\sum p_i = 0$. One of the possible choices is to take p_i to be vertices of a regular tetrahedron lying on some 3-dimensional subspace of the dual spacetime V^* (it is clear that all such configurations can be obtained from each other by rotation). The effective coupling measures how strong the interaction is.

However, as we will see in the future, one often needs to consider parametrizations of the space of theories in which parameters are not necessarily measurable or have any physical meaning. An example of such physically meaningless, but very useful parametrization is the dimensional regularization prescription, which is described in the next section.

2.2. The dimensional regularization prescription of finite renormalization.

Now let us describe a prescription of finite renormalization which uses dimensional regularization.

Recall from Lecture 1 that given a renormalizable ϕ^4-Lagrangian, we can compute its correlation functions using dimensional regularization and minimal subtraction. At first sight it seems that the answer is completely canonical and does not depend on anything. However, after careful consideration it becomes obvious that the answer depends on the choice of unit of length. This happens because the dimensions of couplings in the Lagrangian, in general, depend on the dimension D of the spacetime, so that they should be regarded not as functions but rather as sections of some line bundle over the space of values of D. In order to work with functions, we should trivialize this bundle.

The correct way of computing correlation functions with dimensional regularization is, in general, the following. We should choose a scale of momentum μ, and for any coupling (or mass) κ in the Lagrangian find the dimensionless quantity κ_d such that $\kappa = \kappa_d \mu^{\dim(\kappa,d)}$, where $\dim(\kappa, D)$ is the dimension of κ when dimension of spacetime is D (this is some function of the form $aD + b$), and d is the physical dimension of the spacetime. Now we should replace all masses and couplings κ^i in the Lagrangian with the expressions $\kappa^i(D) = \kappa^i_d \mu^{\dim(\kappa,D)}$, and perform dimensional regularization – minimal subtraction with this new Lagrangian, as described in Section 1.6 of Lecture 1.

Remark. One should remember that the Feynman integrals used to define correlation functions will have d-dependent dimensions, so they have to be multiplied by appropriate powers of μ to make their dimensions constant, before the subtraction procedure is applied.

It is clear that the answer (correlation functions) heavily depend on the choice of μ. The reason is that some couplings κ will nontrivially depend on D (for fixed κ_d), and their first derivatives by D (which contain $\ln \mu$) will influence the regular part of the correlation functions at $D = d$.

For example, in the case of ϕ^4-theory, we have a Lagrangian of the form

$$\mathcal{L} = \int d^4x \left(\frac{a}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4\right), \tag{2.1}$$

and $m(D) = m_d \mu, a(D) = a_d, g(D) = g_d \mu^{4-D}$.

Now, the dimensional regularization prescription of finite renormalization is: for any $\mu \in M$, $T \to (m, a, g)$, where (m, a, g) are the parameters of the ϕ^4-Lagrangian of the form 2.1 which gives the correlation functions of T if renormalized with the help of dimensional regularization, using the scale μ.

The parameters m, a, g have the meaning of "the effective mass at scale μ", "the inverse of the effective scale of ϕ at μ", and "the effective coupling at μ" (we will later explain why).

2.3. Scale-dependence of finite renormalization prescriptions.

In physics we are interested in the behavior of a system at different scales, and how it changes when going from scale to scale. Therefore, we are interested in scale-dependent parametrizations of the space of theories, such that at each scale μ the corresponding parametrization represents in some way the behaviour of the system at μ. Among the prescriptions 1,2 given in Section 2.1, only 2 satisfies this property, while 1 is scale-independent.

The dimensional regularization prescription also satisfies this property. That is, given a theory $T \in Y$, the parameters of the Lagrangian $\mathcal{L}_\mu(T)$ which reproduces the correlation functions of T if renormalized by dimensional regularization using μ represent the behavior of the theory at scale μ. This is not immediately obvious, since the dimesional regularization procedure is formal and has no physical meaning.

The reason is that for large μ the mass can be neglected, and thus the function Γ_4 is a function of p_i/μ, which does not explicitly depend on μ. Therefore, the coupling $g(\mu)$, which is formally obtained from the dimensional regularization procedure, characterizes the magnitude of Γ_4 at $p_i \sim \mu$.

Let us illustrate this by considering an example.

Given a ϕ^4-Lagrangian $\mathcal{L}$ of the form (2.1), its 1-particle irreducible 4-point function, computed by dimensional regularization using the scale μ, in the 1-loop approximation equals

$$(2.2)\quad \Gamma_4(p_i,\mu,m,a,g) = -g + \frac{g^2}{a^2}[f((p_1+p_2)^2) + f((p_1+p_3)^2) + f((p_1+p_4)^2)],$$

where $f(k^2)$ is the amplitude of the corresponding 1-loop diagram. This amplitude was computed in Gawedzki lecture 3 (page 12; the formula for $\hat{I}_{4,ren}(k)$), and the answer is

$$(2.3)\qquad f(k^2) = -\frac{A}{6}\int_0^1 \ln(\alpha(1-\alpha)\frac{k^2}{\mu^2} + \frac{m^2}{a\mu^2})d\alpha + const,$$

where A is some positive numerical constant.

For large μ the term $m^2/a\mu^2$ can be neglected, and we get

$$(2.4)\qquad f(k^2) \sim -\frac{A}{6}\int_0^1 \ln(\alpha(1-\alpha)\frac{k^2}{\mu^2})d\alpha + const = -\frac{A}{6}\ln(k^2/\mu^2) + const,$$

In particular, if p_1, p_2, p_3, p_4 are vertices of a regular terrahedron such that $p_i^2 = \mu^2$ (a point representing the scale μ), then for large μ in the 1-loop approximation we get

$$\Gamma_4 = -g + \frac{g^2}{a^2}C,$$

where C is some numerical constant independent of μ. In particular, the scale-of-ϕ-independent quantity $\frac{\Gamma_4}{a^2}$ has the form

$$\frac{\Gamma_4}{a^2} = -g_* + g_*^2 C, g_* = g/a^2$$

If g_* is small, $\Gamma_4/a^2 \sim -g_*$. So the "effective coupling constant" g_* does indeed characterize the intensity of interaction at the scale μ.

2.4. The renormalization group flow corresponding to a scale-dependent renormalization prescription.

Given a scale-dependent finite renormalization prescription, we can define the renormalization group flow in the space of parameters, which is given by transition functions from the coordinates at one scale to coordinates at another scale. This is done as follows.

Let Y denote the space of renormalizable theories of a single scalar field ϕ. For any scale $\mu \in M$ we have an injective map $\pi_\mu : Y \to \tilde{P}$, where $\tilde{P}$ is a fixed (μ-independent) space of parameters. We will assume that the image $P_\mu = \mathrm{Im}\pi_\mu$ is independent of μ. Denote this image by P.

Now we can fix a theory T and look at the dynamics of $\pi_\mu(T)$ as μ changes. To do this, for any $\mu_1, \mu_2 \in M$ define a transformation $R_{\mu_1\mu_2} : P \to P$ by $R_{\mu_1\mu_2} = \pi_{\mu_2} \circ \pi_{\mu_1}^{-1}$. This allows us to define a flow on $P \times M$: $t(z,\mu) = (R_{\mu,t\mu}z, t\mu)$. The flowlines of this flow are of the form $(\pi_\mu(T),\mu)$, where $T \in Y$ is a fixed theory. This flow is called the renormalization group flow associated to the corresponding family π_μ of renormalization prescriptions.

It is convenient to represent the renormalization group flow as a vector field. Let $\pi_\mu(T) = (m_\mu(T), a_\mu(T), g_\mu(T))$, where $m = m_\mu, a = a_\mu, g = g_\mu$ are the effective mass, inverse scale, and coupling at μ for some renormalization prescription. Then the vector field of renormalization group looks like

$$W = \mu\frac{\partial}{\partial\mu} + \tilde{\gamma} m\frac{\partial}{\partial m} + \tilde{\beta}\frac{\partial}{\partial g} + \tilde{\delta} a\frac{\partial}{\partial a}, \tag{2.5}$$

where $\tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ are functions on $P \times M$.

It is clear that the renormalization group commutes with the group G_1 of rescalings of ϕ: $a \to ta, m \to t^{1/2}m, g \to t^2 g, \mu \to \mu$, as well as with the group G_2 of changing the units of measurement: $a \to a, m \to tm, g \to g, \mu \to t\mu$. The invariants of these actions are $g_* = g/a^2$ and $m_* = m/\mu\sqrt{a}$. This shows that $\tilde{\gamma} = \hat{\gamma}(g_*, m_*)$, $\tilde{\beta} = \hat{\beta}(g_*, \mu_*)a^2$, and $\tilde{\delta} = \delta(g_*, m_*)a$.

It is convenient to rewrite W in the coordinates μ, g_*, m_*, a. The result is

$$W = \mu\frac{\partial}{\partial\mu} + (\gamma - 1)m_*\frac{\partial}{\partial m_*} + \beta\frac{\partial}{\partial g_*} + \delta a\frac{\partial}{\partial a}, \tag{2.6}$$

where $\gamma := \hat{\gamma} - \delta/2$, and $\beta = \hat{\beta} - 2g_*\delta$.

Using this formula, we can write down a differential equation for correlation functions. Let $\Gamma_j(p_i, \mu, m_*, a, g_*)$ be the j-point function of the theory with parameters m_*, a, g_* computed at scale μ. Since the "engineering" (i.e. classical) dimension of Γ_j is $4 - j$, we have

$$\left(\mu\frac{\partial}{\partial\mu} + s\frac{\partial}{\partial s} + j - 4\right)\Gamma_j(sp_i, \mu, m_*, a, g_*) = 0. \tag{2.7}$$

Subtracting from (2.7) the equation $W\Gamma_j = 0$, we get

$$\left(s\frac{\partial}{\partial s} - (\gamma - 1)m_*\frac{\partial}{\partial m_*} - \beta\frac{\partial}{\partial g_*} - \delta a\frac{\partial}{\partial a} + j - 4\right)\Gamma_j(sp_i, \mu, m_*, a, g_*) = 0. \tag{2.8}$$

This equation allows one to study the dynamics of Γ_j under the scaling of external momenta.

Now consider the renormalization group flow in terms of the coordinates g_*, m_*, a. According to formula (2.6), integral curves of this flow are given by the formulas $g_* = g_*(\mu), m_* = m_*(\mu), a = a(\mu)$, where

$$g_*' = \beta(g_*, m_*), m_*' = (\gamma(g_*, m_*) - 1)m_*, a' = \delta(g_*, m_*)a. \tag{2.9}$$

and prime denotes the derivative with respect to $\ln\mu$. From these formulas it is clear that a expresses trivially through g_*, μ_*:

$$a(\mu) = a_0 e^{\int_{\mu_0}^{\mu} \delta(g_*(s), m_*(s)) d\ln s}. \tag{2.10}$$

However, equations for g_* and m_* cannot be easily solved, even if β and γ are known, unless β and γ have some special form.

However, suppose that the vector field W has a third one-parameter symmetry group: $\mu \to \mu, a \to a, m_* \to tm_*, g_* \to g_*$ (mass rescaling). In this case we will call our renormalization prescription mass independent. Then β, γ, δ are independent on m_*, and equations (2.9) are easily solved. Namely, $g_*(\mu)$ is represented implicitly by

$$\frac{\mu}{\mu_0} = e^{\int_{g_*(\mu_0)}^{g_*(\mu)} \frac{dg}{\beta(g)}}, \tag{2.11}$$

and m_* is expressed via g_* by

$$m_*(\mu) = \frac{\mu_0}{\mu} m_*(\mu_0) e^{\int_{\mu_0}^{\mu} \gamma(g_*(s)) d\ln s}. \tag{2.12}$$

Unfortunately, most renormalization prescriptions are mass-dependent. This is the case, for example, for prescription 2 from Section 2.1. However, this prescription is “asymptotically mass-independent”, i.e. almost mass independent at high momenta. Indeed, the correlation functions, which are used as parameters in this prescription, do not significantly depend on m at high momenta. Therefore, although β and γ depend on m_*, there exist limits $\beta(g_*) := \lim_{m_* \to 0} \beta(g_*, m_*)$, $\gamma(g_*) := \lim_{m_* \to 0} \gamma(g_*, m_*)$, so that formulas (2.11), (2.12) become valid for very large μ_0, μ, and, in particular, are applicable for studying the asymptotics of solutions at $\mu \to \infty$.

This “asymptotic mass independence” is characteristic of all prescriptions π_μ which are based on correlation functions at μ. But all of them are mass dependent at finite μ.

Nevertheless, mass independent renormalization prescriptions exist. The most convenient of them is the dimensional regularization prescription.

Let us explain why the dimensional regularization prescription is mass-independent. We will work in the spacetime dimension D (a generic complex number), with $\epsilon = 4 - D$. For a renormalizable Lagrangian $\mathcal{L}$ of the form (2.1) with parameters m, a, g, let $\Gamma_i^\epsilon(p_j, m, a, g)$ be the correlation functions of $\mathcal{L}$ in dimension D, computed by plain integration in D dimensions as described in Lecture 1, and $(\Gamma_i^{\epsilon,sub}(p_j, \mu, m, a, g)$ be the correlation functions in dimension D, computed using scale μ, with pole parts subtracted. The functions Γ_i^ϵ are singular at $\epsilon = 0$, while $\Gamma_i^{\epsilon,sub}$ are regular at $\epsilon = 0$, and the functions $\Gamma_i := \lim_{\epsilon \to 0} \Gamma_i^{\epsilon,sub}$ are the true correlation functions of the theory, computed at scale μ as in Section 2.2.

Let m^b, a^b, g^b be the bare parameters, defined by the condition

$$\Gamma_i^\epsilon(p_j, m^b, a^b, g^b) = \Gamma_i^{\epsilon,sub}(p_j, \mu, m, a, g)$$

(b stands for “bare”). These bare parameters are functions of renormalized parameters, ϵ, and μ, of the form

$$m^b = m^b(m, a, g, \epsilon, \mu), a^b = a^b(m, a, g, \epsilon, \mu), g^b = g^b(m, a, g, \epsilon, \mu).$$

From dimensional analysis (using the groups G_1 and G_2) it follows that the functions m^b, a^b, g^b are defined by the equations

$$a^b = aZ_a(m_*, g_*, \epsilon), m^b(a^b)^{-1/2} = \mu m_* Z_m(m_*, g_*, \epsilon), g^b(a^b)^{-2} = \mu^\epsilon g_* Z_g(m_*, g_*, \epsilon),$$

where

$$Z_i(m_*, g_*, \epsilon) = 1 + \sum_{k\geq 1} Z_i^k(g_*, m_*)\epsilon^{-k}.$$

(no positive powers of ϵ, as we used minimal subtraction).

The key point now is that Z_i do not actually depend on m_*. Indeed, the functions Z_i are sums of counterterms introduced in the process of renormalization. These counterterms at each step of the process are just sums of pole parts (in ϵ) of amplitudes of all diagrams with a given number of loops. These pole parts are determined purely by the asymptotic expansion of the amplitudes at $|p_i| \to \infty$. From this one can deduce that these pole parts are independent on m. Thus, $Z_i = Z_i(g_*, \epsilon)$.

Let W_ϵ be the vector field of the form

$$W_\epsilon = \mu\frac{\partial}{\partial\mu} + (\gamma_\epsilon - 1)m_*\frac{\partial}{\partial m_*} + \beta_\epsilon\frac{\partial}{\partial g_*} + \delta_\epsilon a\frac{\partial}{\partial a},$$

which preserves functions $\Gamma_i^{\epsilon,sub}(p_j, \mu, m, a, g)$. It is clear that such W_ϵ exists and is unique, and $\lim_{\varepsilon\to 0} W_\varepsilon = W$. On the other hand, using the definition of $\Gamma_i^{\epsilon,sub}$, we get that W_ε is defined by the condition $W_\epsilon m_*^b = 0, W_\epsilon a^b = 0$, $W_\epsilon g_*^b = 0$. Writing these equations in components, we get

$$\beta_\epsilon Z_m' + \gamma_\epsilon Z_m = 0,\ \beta_\epsilon Z_a' + \delta_\epsilon Z_a = 0,\ \epsilon g_* + \beta_\epsilon\frac{(g_* Z_g)'}{g_* Z_g} = 0$$

(prime denotes the derivative by g_*). From these equations we find:

$$\beta_\epsilon = -\frac{\varepsilon g_*^2 Z_g}{(g_* Z_g)'},\ \gamma_\epsilon = -\beta_\epsilon\frac{Z_m'}{Z_m}, \delta_\epsilon = -\beta_\epsilon\frac{Z_a'}{Z_a}.$$

Using the facts that Z_i has a Laurent expansion in inverse powers of ϵ, and the fact that β, γ, δ are by definition analytic at $\varepsilon = 0$, we get the following formulas:

$$\beta_\epsilon(g_*) = -g_*\epsilon + g_*^2(Z_g^1)',\ \gamma_\epsilon(g_*) = g_*(Z_m^1)', \delta_\epsilon(g_*) = g_*(Z_a^1)'$$

In the limit $\varepsilon \to 0$, these formulas give

$$\beta(g_*) = g_*^2(Z_g^1)',\ \gamma(g_*) = g_*(Z_m^1)', \delta(g_*) = g_*(Z_a^1)'$$

The last formula shows that β, γ, δ depend only on g_*, and gives a simple and useful rule of evaluating them.

In particular we see that although counterterms in dimensional regularization can have poles of arbitrary order, the contribution to the renormalization group vector field comes only from residues of these poles. Thus, in principle the renormalization group vector field should be more elementary than the complete system of counterterms. This is one of the reasons why it is useful to write down the renormalization group equation.

We have seen that the asymptotic properties of our field theory at $\mu \to \infty$ depend on the properties of the function $\beta(g_*)$, since this function determines the dynamics of the coupling $g_*(\mu)$. This function is called the beta-function. Later we will see that the sign of this function near $g_* = 0$ is especially important.

2.5. Computation of the renormalization group flow in the 1-loop approximation.

In this section we will compute the vector field W in the 1-loop approximation. Let $m_* = m_*(\mu)$, $g_* = g_*(\mu)$, $a = a(\mu)$ be an integral curve of the flow. We will express the derivatives m'_*, g'_*, a' from the equations $\mu\frac{\partial}{\partial\mu}\Gamma_2(p^2, m_*, a, g_*, \mu) = 0$, $\mu\frac{\partial}{\partial\mu}\Gamma_4(p_i, m_*, a, g_*, \mu) = 0$.

First of all, the 1-loop correction to the 2-point function can be absorbed in mass renormalization: $m(\mu) = m_0(1 + b\frac{g}{a^2}\ln(\mu/\mu_0)) + O(g^2)$, where b is a numerical constant. Thus, $a' = 0, m'_* = -m_*(1 + bg_*)$.

In order to compute g'_*, we differentiate equation (2.2) with respect to $\ln\mu$. This yields, modulo g_*^3, the following expression:

$$g'_* = Ag_*^2. \tag{2.13}$$

Thus, we get in the one-loop approximation: $\beta(g_*) = Ag_*^2, A > 0; \gamma(g_*) = bg_*; \delta(g_*) = 0$.

Remark. A more careful calculation gives: $A = \frac{3}{16\pi^2}, b = \frac{1}{32\pi^2}$.

Now consider the renormalization group equation (2.13). The general solution of this equation is

$$g_* = \frac{g_*^0}{1 - Ag_*^0\ln(\mu/\mu_0)}. \tag{2.14}$$

Thus, we have found an approximation to the function $g_*(\mu)$ based on the 1-loop approximation to the β-function.

Now we face the following fundamental questions. In what respect is expression (2.14), obtained from the 1-loop approximation to the beta-function, any better than the expression

$$g_* = g_*^0 + A(g_*^0)^2\ln(\mu/\mu_0), \tag{2.15}$$

obtained directly from (2.2)? (Nothe that (2.15) coincides with (2.14) modulo $(g_*^0)^3$!) In what sense does (2.14) better represent "reality" than (2.15)? In other words, why did we care to introduce renormalization group, beta-function, and so on, instead of just getting (2.15) directly?

The answer is the following. Assume that ϕ^4 theory exists in the non-perturbative setting (it is not believed to be the case, but never mind; you can think instead of another field theory). Then formula (2.15) has a chance to be valid only when $g_*^0\ln(\mu/\mu_0)$ is small, since the series for g_* will contain terms of the form $(g_*^0)^{k+1}\ln(\mu/\mu_0)^k$. On the other hand, formula (2.13) is valid for small g_*, since it is the beginning of the Taylor expansion of β. So in the region where g_* is small but $g_*^0\ln(\mu/\mu_0)$ is not (for example, $Ag_*^0\ln(\mu/\mu_0) = 1/2, g_*^0 << 1$), formula (2.14) is reliable but (2.15) is not.

2.6. Asymptotic freedom.

So far we have always worked with quantum field theories in the perturbative setting, primarily because this was the only setting in which we could define and study them. However, physics lives over $\mathbb{R}$ rather than $\mathbb{R}[[\tau]]$, so we should investigate when we can use perturbative results to judge about the actual process. One necessary condition is clear: all couplings should be small, so that expansions in formal series with respect to them have a chance to make sense.

In particular, we can ask whether we can trust the rules of perturbative renormalization, which were discussed in the previous lecture and in Witten's lectures. As you remember, these rules tell how to handle divergences arising at high momenta (ultraviolet divergences). Thus, in order to trust the theory of perturbative renormalization we should be sure that interactions are small at high momenta.

A (Euclidean) quantum field theory (in the non-perturbative sense; for example, in the sense of Wightman axioms) is called asymptotically free if its interactions are small at high momenta. That is, a theory is asymptotically free if its correlation functions in momentum space are asymptotically (at high momenta) close to those of a free theory. Thus, in order to use perturbative renormalization legitimately, we should be sure that our theory is asymptotically free.

Of course, in the perturbative setting we cannot even show that a quantum field theory exists, much less that it is asymptotically free. It is therefore a pleasant surprise that if we assume that a theory with a given perturbation expansion exists, its asymptotic freedom can be checked perturbatively, usually already in the 1-loop approximation. Later we will discuss it in more detail, but now we will just demonstrate it for ϕ^4-theory.

We will only consider renormalizable theories, i.e. points of the space Y. It is clear that such a theory $T \in Y$ is free (i.e. quadratic) if the effective coupling g_* vanishes. Thus, the condition for asymptotic freedom is that the coupling g_* vanishes asymptotically, at high momenta. That means, $\lim_{\mu\to\infty} g_*(\mu) = 0$.

It is clear that we cannot see whether $g_*(\mu)$ approaches zero or not by looking at $g_*(\mu)$ in the perturbation expansion. This is clear from the example of the function $f(\mu) = \frac{1}{1+\tau\ln(\mu/\mu_0)}$, which vanishes at infinity for any finite τ, but diverges at infinity if reduced modulo τ^N for each N. What we can see, however, is whether, as μ increases, the value of $g_*(\mu)$ increases or decreases.

Let us do this for ϕ^4 theory. In this case, the physically meaningful region of values of g_* is $g_* > 0$. So we should find out whether g_*' is positive or negative when g_* is in this region.

Now it is very easy for us to answer this question. Namely, if g_* is small, we can trust formula (2.13): $g_*' = Ag_*^2$. This means that g_* increases, i.e. gets away from zero. Of course, we cannot claim on the basis of (2.13) that g_* increases all the time (remember that (2.13) is only valid for small g_*), but what we can definitely guarantee on the basis of (2.13) is that $\lim_{\mu\to\infty} g_*(\mu) \neq 0$. Thus **the ϕ^4 theory is not asymptotically free**, and its perturbation expansion is not valid at high momenta. This is a non-perturbative result which we obtained using only perturbation theory.

Thus, renormalization group gives us a subtle way of partial summation of the perturbation series, allowing us to extend the region of validity of perturbation theory, and thus pushing us closer toward the world of non-perturbative field theories.