LECTURE 3: RENORMALIZATION GROUPS (CONTINUED)

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3.1. Dynamical patterns of the renormalization group flow.

In the last lecture we saw that the renormalization group equation for the ϕ^4-theory in the one-loop approximation had the form $g_*' = Ag_*^2$. This equation has a fixed point at $g_* = 0$, which is attracting for $g_* < 0$ and repelling at $g_* > 0$, so it looks like

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The fact that this fixed point is repelling for $g_* > 0$ allowed us to show that ϕ^4 theory is not asymptotically free in the region $g_* > 0$.

Now consider what happens for an arbitrary renormalizable field theory with a single renormalizable coupling λ. Let us write down the renormalization group equation for λ based on some mass-independent finite renormalization prescription:

$$\mu\frac{d\lambda}{d\mu} = \beta(\lambda).$$

This equation always has a fixed point at $\lambda = 0$ (the free theory). The behavior of solutions of this equation near the fixed point can be determined by looking at the sign of the first nontrivial term of the Taylor expansion of $\beta(\lambda)$. There can be two patterns of behaviour.

1. An attracting, or ultraviolet (UV) stable fixed point ($\beta(\lambda)$ and λ have opposite signs for small λ):

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In this case, $\lambda(\mu)$ is driven to the fixed point $\lambda = 0$ in the UV limit $\mu \to \infty$, which implies asymptotic freedom. More precisely, if $\beta(\lambda) \sim A\lambda^{k+1}$, $\lambda \to 0$, then, as one finds by solving the renormalization group equation,

$$|\lambda(\mu)| \sim (|A|k)^{-1/k}|\ln(\mu/\mu_0)|^{-1/k}, \mu \to \infty. \tag{3.1}$$

However, in the limit $\mu \to 0$ $\lambda(\mu)$ is driven away from the fixed point.

2. A repelling, or infrared (IR) stable fixed point ($\beta(\lambda)$ has the same sign as λ for small λ).

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In this case $\lambda(\mu)$ is driven away from the fixed point $\lambda = 0$ in the UV limit $\mu \to \infty$, so there is no asymptotic freedom. However, in the IR limit $\mu \to 0$ the coupling $\lambda(\mu)$ approaches the point $\lambda = 0$, at the rate given by (3.1).

Now we face a question: what happens if $\lambda(\mu)$ is driven away from the fixed point? What will be its behavior when λ is not very small?

The answer to this question cannot be given within the framework of perturbation theory. If the field theory which we are considering actually exists non-perturbatively, then there could be two options:

Option 1. $\lambda(\mu)$ is driven to a fixed point $\lambda_0 \neq 0$:

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Option 2. $\lambda(\mu)$ is driven to infinity.

Remark. In general, it could happen that the differential equation $\lambda' = \beta(\lambda)$ has no nonzero solutions which are defined globally for all μ. (Example: $\beta(\lambda) = A\lambda^2$). This means, the coupling $\lambda(\mu)$ becomes infinite at some finite value of μ (e.g. $\lambda = \frac{\lambda_0}{1 - A\lambda_0 \ln(\mu/\mu_0)}$). This kind of singularity is called a renormalon. We will discuss this topic later in more detail. For now let us just say that the appearance of a renormalon does not necessarily mean that the theory does not exist but may just mean that we are doing perturbation expansion around a wrong vacuum state (see the next lecture).

Let us consider some examples.

Example 1: ϕ^4 theory. In this case, as we saw above, the fixed point $g = 0$ is UV stable for $g < 0$ and IR stable for $g > 0$, so we have asymptotic freedom in the unphysical region $g < 0$. For $g > 0$, there is no asymptotic freedom, and in the 1-loop approximation $\lambda(\mu)$ becomes infinite at finite μ: $\mu = \mu_0 e^{\frac{1}{A g_*^0}}$. Whether this singularity survives non-perturbatively, provided that ϕ^4-theory exists non-perturbatively – cannot be determined in perturbation theory.

Example 2: QED. In QED, the beta-function has the form

$$\beta(e) = \frac{e^3}{12\pi^2} + O(e^5), \tag{3.2}$$

where e is the dimensionless electric charge. This can be inferred from computing the 1-loop correction to the 3-point function $\langle \psi(x) A(y) \psi(z) \rangle$ where A is the electromagnetic field and ψ represents a charged particle, e.g. an electron (see also the solution of problem 2 in Witten's problem set 6). Therefore, the free fixed point $e = 0$ is IR stable in both regions $e < 0$, $e > 0$ (which are both physically meaningful). Thus QED is not asymptotically free.

So what happens in the UV limit? Solving the renormalization group equation in the 1-loop approximation, we get

$$e^2 = \frac{e_0^2}{1 - \frac{e_0^2}{6\pi^2} \ln(\mu/\mu_0)}. \tag{3.3}$$

Thus, if we trust the 1-loop approximation, the electric charge grows as $\mu \to \infty$, and becomes infinite at $\mu = \mu_0 \exp(6\pi^2/e_0^2)$ (the so-called Landau singularity).

One can wonder whether this singularity exists nonperturbatively. This is a rather academic question, since QED is not proved and generally not believed to exist nonperturbatively. However, let us estimate how big μ is. We have

$e_0^2/4\pi \sim 1/137$, and $\mu_0 \sim 1$MeV. So we get $\mu \sim e^{3\pi\times 137/2}\mu_0 \sim 10^{250}$MeV. Such energy is not contained in the whole observable part of the universe. So even if the singularity occured at μ, it does not concern us, since we are studying physics at much lower energy scales. This is the reason why, although QED is not asymptotically free, and its perturbation expansion is not valid at high energies, it gives a good approximation to reality in a certain range of energies, and can be used to make predictions which can be checked experimentally.

In formula (3.3), e_0 has the meaning of the observed (effective) charge when the measurements are made at an experimentally accessible scale of energies μ_0, while e has the meaning of the actual charge (i.e. the charge computed on the basis of a very precise theory which describes physics at a very high scale of energies μ). Passage from e to e_0 by moving down with the renormalization group flow from μ to μ_0 has the meaning of averaging out degrees of freedom between μ_0 and μ. The inequality $e_0 < e$ for $\mu_0 < \mu$, which follows from (3.3), has the meaning of *screening* of charges: the observed charge is less than the actual charge.

Remark. This screening phenomenon is known already in classical physics (screening of charges by medium): $e_0 = e/\varepsilon$, where $\varepsilon > 1$ is the dielectric permeability of the medium. To explain this, one should model the medium as a collection of very small dipoles. In the presence of a charged particle with charge e (say $e > 0$) they will align in such a way that the negative parts will face the particle. If we draw a sphere S centered at the particle, it will cut some of the dipoles in half, so that negative parts will be inside of the sphere and the positive parts outside. Now we can use Newton's theorem: the field of a uniformly charged sphere with charge q is zero inside of the sphere, and is the same as the field of a point charge q situated at the center, outside of the sphere. This theorem implies that the total contribution of the outside parts of the dipoles to the electric field at a point of S is zero, while the total contribution of the negative parts is the same as that of a negative point charge q at the center. So the effective value of the charge is $e - q$. (We should note that q depends on e, since the number of dipoles which will align depends on the strength of the electric field. In fact, q is proportional to e.)

Example 3: Nonabelian gauge theory in 4 dimensions. Before 1970-s, it was believed that all field theories are IR stable but not asymptotically free. However, in the 70-s it was found that pure non-abelian gauge theories are asymptotically free. Moreover, they remain asymptotically free even if fermions are added, as long as the number of fermions does not exceed a certain limit.

Namely, one can show that the β-function for a non-abelian gauge theory with gauge group G and fermions ψ_i. (a compact simple Lie group) has the following form in the 1-loop approximation:

$$\beta(g) = -\frac{1}{48\pi^2}(11C_{\mathfrak{g}} - 4\sum_i C_{V_i})g^3, \tag{3.4}$$

where $\mathfrak{g}$ is the Lie algebra of G, V_i are the representations of G where the fermions ψ_i live, and C_V is the Casimir C of $\mathfrak{g}$ acting in a representation V. Here the Casimir is normalized in such a way that $C_{\mathfrak{g}} = h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ ($h^\vee = N$ for $SU(N)$). The proof of formula (3.4) can be found it textbooks on QFT, for example in P.Ramond's textbook.

Formula (3.4) tells us that the pure gauge theory is asymptotically free, so that $g(\mu)$ is a decreasing function for small g (so-called "antiscreening"), and gives us the bound on the number of fermions which can be added without destruction of asymptotic freedom.

For example, consider QCD (quantum chromodynamics), which is believed to describe strong interactions. In QCD, $G = SU(3)$, and $V_i = \mathbb{C}^3$, so $C_{\mathfrak{g}} = 3$ and $C_{V_i} = 1/2$, so we get

$$\beta(g) = -\frac{1}{48\pi^2}(33 - 2N_f)g^3, \tag{3.4}$$

where N_f is the number of fermions. So, in order for this theory to be asymptotically free, we have to have $N_f \leq 16$.

Asymptotic freedom of QCD can be checked experimentally. One of the possible experiments is described in a simplified way in the appendix to Witten's Lecture 4.

3.2. Are there any asymptotically free theories without nonabelian gauge fields?

Example 3 in the previous section raises the question whether there can be an asymptotically free theory without nonabelian gauge fields. The answer is the following.

In two dimensions, there are many asymptotically free theories without nonabelian gauge fields. For example, the 2-dimensional theory with N fermions and interaction $\sum_j (\bar{\psi}_j \psi_j)^2$, which we will consider in the next lecture, is asymptotically free.

In four dimensions, this is not the case. Namely, there is the following result.

Coleman-Gross Theorem. *Let $\mathcal{L}(\phi_i, \psi_j)$ be a Poincare invariant, renormalizable Lagrangian of scalar fields ϕ_k and spinor fields ψ_j:*

$$\mathcal{L} = \int d^4x \left(\frac{1}{2} \sum_k (\nabla \phi_k)^2 + \sum_j (\psi_j, (iD + M_j)\psi_j) + V(\phi) + Y(\phi, \psi) \right), \tag{3.5}$$

where $V(\phi)$ is a quartic polynomial, and $Y(\phi, \psi)$ are Yukawa interaction terms of the form $\phi\psi\psi$. Suppose that the function $V(\phi)$ is bounded from below. Then the quantum field theory defined by $\mathcal{L}$ is not asymptotically free.

It is clear that this theorem generalizes to the case where $\mathcal{L}$ is allowed to contain Abelian gauge fields: in example 2 we saw that this would make the theory even "less" asymptotically free.

The condition that $V(\phi)$ is bounded from below cannot be dropped: we have seen that the ϕ^4 theory with $g < 0$ is asymptotically free. However, one can show that such a theory cannot exist nonperturbatively. Namely, one can show that if such a theory had existed, its spectrum of energy would not be bounded from below, which contradicts the existence of the theory (the energy positivity axiom would be violated).

Let us show this for the case of ϕ^4 theory. We consider the ϕ^4 theory with negative coupling constant g. In this case, the classical potentail $V_0(\phi) = \frac{g\phi^4}{4!} + \frac{m^2}{2}\phi^2$ is not bounded from below. Apriori, it does not imply that the theory makes no sense, since this problem could be cured in the quantum theory. In order to see what happens in the quantum theory, we should compute the effective potential, $V_{\mathrm{eff}}(\phi)$, which represents the effective classical theory for the quantum theory defined by $V_0(\phi)$. If this potential is not bounded from below, we can conclude that the theory does not exist quantum-mechanically.

As we know, the effective classical action is computed as a sum of amplitudes of 1-particle irreducible Feynman diagrams (see the end of Kazhdan's lecture on Feynman graphs):

$$S_{\mathrm{eff}}(\phi) = \sum_{k\geq 1} \int_{\sum p_i = 0} \Gamma_{2k}(p_1, ..., p_{2k})\tilde{\phi}(p_1)...\tilde{\phi}(p_{2k})dp, \tag{3.6}$$

where Γ_{2k} are 1-particle irreducible correlations functions, and $\tilde{\phi}$ denotes the Fourier transform of ϕ. By the definition, the effective potential $V_{\mathrm{eff}}(\phi)$ is the function of one variable defined by the formula

$$V_{\mathrm{eff}}(\phi) = S_{\mathrm{eff}}(\phi), \tag{3.7}$$

where ϕ is a constant function. That is,

$$V_{\mathrm{eff}}(\phi) = \sum_{k\geq 2} \Gamma_{2k}(0)\phi^{2k}. \tag{3.8}$$

In general, it is not easy to see whether this function is bounded from below or not. However, the fact that the ϕ^4 theory with $g < 0$ is asymptotically free allows to figure this out. The demonstration that $V_{eff}(\phi)$ is not bounded from below is contained in Weinberg's textbook in Chapter 18, Section 18.2. This unboundedness implies that at large scales of ϕ there exist states of arbitrarily low energy, i.e. that the corresponding quantum theory in fact does not exist.

Thus, we have seen that a sensible 4-dimensional quantum field theory which is asymptotically free must contain nonabelian gauge fields. But theories which are not asymptotically free are usually not believed to exist nonperturbatively. Does this mean that theories without nonabelian gauge fields, like QED, do not make any physical sense? (This would really be very strange, since QED was used to make some predictions, which were in a very good agreement with experiment).

The answer of course is no. Even though QED may not exist nonperturbatively as a separate theory, its Lagrangian may be a part of a bigger Lagrangian, which contains other fields (in particular, nonabelian gauge fields), and define a theory which is asymptotically free and perhaps exists nonperturbatively. However, at low energies, when considering electromagnetic interactions, we can ignore all the additional terms in the Lagrangian, and use the Lagrangian of QED.

In fact, the real situation is the following. There is the so called "Standard model", which is a field theory with many different fields, and includes QED as a subtheory. This theory is believed to exist nonperturbatively, and to describe all interactions except gravity in the real world. Now, if we want to compute the scattering data for two electrons in this theory, the answer will be of the order e^2, where e is the charge of the electron, and corrections due to terms in the Lagrangian not contained in the Lagrangian of QED are of order e^4 and higher (see appendix to Witten's lecture 4). Therefore, at low energies, where e is small, QED in the 1-loop approximation is still valid.

3.3. Renormalization group equations with many couplings.

So far we have considered the renormalization group flow only for theories with one critical coupling. Similarly one can define the renormalization group flow in the case when there are many such couplings, $g_1, ..., g_n$. We will assume that the renormalization group flow is defined by a mass independent family of renormalization prescriptions π_μ parametrized by $\mu \in M$ (for example, dimensional regularization – minimal subtraction). In this case, subcritical couplings do not enter in the renormalization group equation for g_i, so it has the form

$$\mu \frac{dg_i}{\partial \mu} = \beta_i(g_1, ..., g_n). \tag{3.9}$$

In general, we know nothing about the flow defined by (3.9), except that the point $g_i = 0$ (the free theory) is a fixed point of this flow. However, there is the following conjecture, which says that this flow behaves as a gradient flow.

Conjecture 3.1. *There exists a function $F(g_1, ..., g_n)$ such that its total derivative with respect to μ is positive at any nonsingular point of the flow (3.9).*

Remark. Here we, of course, assume that the theory we are considering exists nonperturbatively.

Conjecture 3.1 prohibits various complicated dynamical patterns which could occur in a multidimensional dynamical system (limiting cycles, ergodicity, attractors, etc.), and implies that any trajectory is either driven to a fixed point or to infinity.

The intuition behind conjecture 3.1 is the Wilsonian point of view that the (backward) renormalization group flow comes from erasing degrees of freedom. The meaning of the function F is "the measure of the number of degrees of freedom". Unfortunately, it is unknown how make mathematical sense of this idea in general, so conjecture 3.1 remains open. It is proved, however, in many special cases, for example, in 2-dimensional field theories.

Remark. In 2-dimensional theories with bosonic fields the space of couplings is infinite-dimensional, since the classical dimension of elementary bosonic fields vanishes (couplings are arbitrary functions). Therefore, the renormalization group equations are equations in the space of functions.

3.4. The renormalization group equation for composite operators.

Suppose we are given a renormalizable field theory T given by some Lagrangian $\mathcal{L}$. Recall from Witten's lecture 3 that every local operator $\mathcal{O}(x)$ in the classical field theory defined by $\mathcal{L}$ can be quantized (non-canonically), and that the space $A_l^0(T)$ of classical operators of scaling dimension $\leq l$ admits a canonical quantization, which we will denote by $A_l(T)$.

Recall that for any operator $\mathcal{O} \in A_l(T)$, the correlation functions of $\mathcal{O}$ are defined to be the ε-coefficients of the correlation functions of the Lagrangian

$$\mathcal{L}_\varepsilon = \mathcal{L} + \varepsilon \mathcal{O}(y), \tag{3.10}$$

where $\varepsilon^2 = 0$.

Let $Z = \cup_{l\geq 0} Z_l$ be a subspace in the space of differential polynomials of fields (with complex coefficients), invariant under the orthogonal group, and the groups G_1, G_2 of scaling the values of fields and scalings of spacetime, and such that the natural map $Z_l \to A_l^0(T)$ is an isomorphism for all theories T of a given type. We will call V the set of representative differential polynomials, and will always represent a classical local functional by its unique preimage in Z_l. Abusing language, we will call elements of Z "operators".

Remark. The need to introduce Z is caused by the fact that in any field theory, there are nontrivial relations between differential monomials (field equations), and therefore each local functional has many representations as a differential polynomial of elementary fields.

Now we will define the renormalization group flow on the space of operators. We will use the notations of Lecture 2 and Witten's lecture 3.

Let Y be the space of renormalizable theories of the corresponding type, and Y_l be the space of Lagrangians of the form (3.10), where $\mathcal{O}$ is of dimension $\leq l$. The space Y_l is a vector bundle over Y: $Y_l \to Y$, whose fiber at a point $T \in Y$ is the space $A_l(T)$ of operators of dimension $\leq l$ for the theory T.

Let $\pi_\mu : Y \to P$ be a renormalization prescription. Let $P_l = P \times Z_l$ be the total space of the trivial bundle over P with fiber Z_l, and and $\pi_\mu^l : Y_l \to P_l$ be an isomophism of vector bundles which descends to $\pi_\mu : Y \to P$. π_μ^l is called an extension of the renormalization prescription π_μ to operators of dimension $\leq l$ (this extension is not, in general, defined by π_μ, and has to be defined additionally; below we will give one of such definitions). Let $R_{\mu_1\mu_2}^l = \pi_{\mu_2} \circ \pi_{\mu_1}^{-1} : P_l \to P_l$. The transformation $R_{\mu_1\mu_2}^l$ is called the remormalization group transformation for composite operators for the renormalization prescription π_μ^l. It defines a flow on $P_l \times M$ by $t(z,\mu) = (R_{\mu,t\mu} z, \mu)$.

As before, it is convenient to represent the renormalization group flow as a vector field. Denote this vector field on P_l by W_l. It is clear that W_l projects to W under the projection $P_l \to P$, and preserves the structure of a vector bundle on P_l. Therefore,

$$W_l(z,\mu) = W(z,\mu) + \tilde{\gamma}_l(z,\mu), \tag{3.11}$$

where $\tilde{\gamma}_l : P \times M \to \mathrm{End} Z_l$ is an operator-valued function (regarded as a vertical vector field on P_l).

If $\mathcal{O}_i$ is a basis of Z_l, homogeneous with respect to both G_1, G_2, and $\mathcal{O} = \sum f_i \mathcal{O}_i$, then the renormalization group equations can be written explicitly as

$$\mu \frac{df_i(z,\mu)}{d\mu} = \sum_j \tilde{\gamma}_l^{ij}(z,\mu) f_j(z,\mu), \tag{3.12}$$

where $\tilde{\gamma}_l^{ij}$ is the matrix of the operator $\tilde{\gamma}_l$ in the basis $\mathcal{O}_i$, and $z = z(\mu)$ is a solution of the renormalization group equation for parameters of the theory.

Now we should explain how to extend the map π_μ to π_μ^l. As usual, there are many ways to do it. However, if π_μ is the dimensional regularization prescription, there is the following natural way of extending π_μ to π_μ^l.

We will consider the ϕ^4 theory. Let $T \in Y$, $\pi_\mu(T) = \mathcal{L}$ be the Lagrangian which describes T at scale μ (in the sense of dimensional regularization) and $\mathcal{O} \in A_l(T)$.

Proposition 3.2. *There exists a unique differential polynomial $F_{\mathcal{O}} \in Z_l$ such that the correlation functions of the operator $\mathcal{O}$ in the theory T are equal to the ε-coefficients of the correlation functions of the Lagrangian $\mathcal{L}_\varepsilon = \mathcal{L} + \varepsilon F_{\mathcal{O}}$ computed using dimensional regularization and minimal subtraction at scale μ (see Section 2.2 of Lecture 2).*

Proof. The statement follows from the fact that classical field equations survive quantization, which was explained (for ϕ^4 theory), and checked in the simplest case in Witten's lecture 3, Section 3.7.

Now we define the map π_μ^l by the formula $\pi_\mu^l(T, \mathcal{O}) = (\pi_\mu(T), F_{\mathcal{O}})$. As was explained above, this defines a renormalization group flow on the space of operators. Moreover, since our renormalization prescription is mass-independent, the function $\tilde{\gamma}_l(z, \mu)$ does not explicitly depend on μ.

Remark. The mass independence of the dimensional regularization prescription implies that the map $\tilde{\gamma}_l(z)$ respects not only the filtration of the space Z_l of operators by classical scaling dimension, but also the grading by scaling dimension. That is, operators mix only with operators of the same dimension. This may be false for a mass-dependent prescription.

Let us now consider the example of the ϕ^4 theory. In this case, equation (3.12) has the form

$$\mu \frac{df_i}{d\mu} = \sum_j \tilde{\gamma}_l^{ij}(g_*(\mu), a(\mu)) f_j \tag{3.13}$$

(there is no dependence on m_* due to mass independence of the renormalization prescription). Let Δ_i be the degree of $\mathcal{O}_i$ with respect to scaling of ϕ. It is clear from the scale-of-ϕ-invariance of (3.13) that $\gamma_l^{ij}(g_*, a) = \hat{\gamma}_l^{ij}(g_*) a^{(\Delta_i - \Delta_j)/2}$. Therefore, if we conjugate equation (3.13) by $a^{-D_\phi/2}$, where D_ϕ is the scale-of-ϕ-degree operator, and introduce functions $h_i = f_i a^{-\Delta_i/2}$, we will get the following equations:

$$\mu \frac{dh_i}{d\mu} = \sum_j \gamma_l^{ij}(g_*(\mu)) h_j, \tag{3.14}$$

where

$$\gamma_l = \hat{\gamma}_l - \frac{D_\phi \delta}{2}. \tag{3.15}$$

The operator γ_l is already a function only of one variable g_*.

3.5. Anomalous dimension.

Definition. The operator γ_l is called the operator of anomalous dimensions at g_*, for operators of degree $\leq l$. The matrix $\gamma_l^{ij}(g_*)$ is called the matrix of anomalous dimensions at g_*.

For the simplest operators one can define scalar-valued anomalous dimension. Namely, Let $F \in Z_l[a^{\pm 1/2}]$ be a generalized eigenvector of $\gamma_l(g_*)$ with eigenvalue $\gamma_F(g_*)$ for all g_*. In this case we will say that $\gamma_F(g_*)$ is the anomalous dimension of the operator F at g_*. In general, any quantity X depending on μ and obeying the equation $\mu\frac{dX}{d\mu}=\gamma_X(g_*(\mu))X$, where γ_X is a scalar function, will be said to have anomalous dimension γ_X.

Now let us explain the name "anomalous dimension". Let F be a (real) operator (with a-dependent coefficients) of engineering dimension d_F, scale-of-ϕ degree 0, and scalar anomalous dimension $\gamma_F(g_*)$. For simplicity we assume that it is a true (not only generalized) eigenvector of γ_l. Let

$$\langle F(x)F(0)\rangle = G_2^F(x^2,\mu,m_*,a,g_*) \tag{3.16}$$

be the 2-point function of F in position space. Using the renormalization group flow, we conclude that

$$\mu\frac{dG_2^F(x^2,\mu,m_*(\mu),a(\mu),g_*(\mu))}{d\mu} = 2\gamma_F(g_*(\mu))G_2^F(x^2,\mu,m_*(\mu),a(\mu),g_*(\mu)). \tag{3.17}$$

But the engineering dimension of G_2^F is $2d_F$, so (3.17) can be rewritten in the form

$$\begin{aligned} &s\frac{d}{ds}G_2^F((x/s)^2,\mu_0,m_*(s\mu_0),a(s\mu_0),g_*(s\mu_0)) = \\ &2(d_F+\gamma_F(g_*(s\mu_0)))G_2^F((x/s)^2,\mu_0,m_*(s\mu_0),a(s\mu_0),g_*(s\mu_0)). \end{aligned} \tag{3.18}$$

Integrating this equation, we get

$$\begin{aligned} &G_2^F((x/s)^2,\mu_0,m_*(s\mu_0),a(s\mu_0),g_*(s\mu_0)) = \\ &s^{2(d_F+\int_{\mu_0}^{s\mu_0}\gamma_F(g_*(r\mu_0))d\ln r)}G_2^F(x^2,\mu_0,m_*(\mu_0),a(\mu_0),g_*(\mu_0)). \end{aligned} \tag{3.19}$$

If m_*,a,g_* did not depend on μ for large μ, we would get

$$G_2^F(x^2)\sim|x|^{-2(d_F+\gamma_F)}, \tag{3.20}$$

which shows that the operator F has "quantum scaling dimension" $d_F+\gamma_F$. Thus, anomalous dimension is just the difference between the quantum and the classical scaling dimension.

In general, formula (3.20) does not hold, because there is a nontrivial renormalization group flow on the space of parameters. If this flow does not have an UV stable fixed point, there is usually no hope to derive a scaling law of the form (3.20). However, if the flow has an UV stable fixed point g_*^f, and $\gamma_F(g_*^f)=\gamma_F$, then up to logarithmic factors formula (3.20) is true.

Furthermore, if there is an UV stable fixed point g_*^f, one can replace matrix anomalous dimensions by scalar ones. Namely, one should bring the matrix $\gamma_l(g_*^f)$ to Jordan normal form. The generalized eigenvectors of $\gamma_l(g_*^f)$ with eigenvalue ν are then called "operators of pure anomalous dimension ν".

Unfortunately, the only type of an UV stable fixed point that we can study successfully in perturbation theory is the fixed point $g_*^f=0$ for an asymptotically free theory. In this case, the anomalous dimensions $\gamma_F(g_*^f)$ in the above sense vanish, because $\gamma_l(0)=0$. However, for g_* close to zero anomalous dimensions are nonzero, and the knowledge of them helps get a very sharp UV asymptotics of operators and correlation functions. The most remarkable thing is that in order to compute the exact asymptotics of operators, up to multiplication by a constant, it is enough to know anomalous dimensions only in the first order approximation, which are obtained by an easy explicit calculation. Let us show how this works.

We will consider ϕ^4 theory with $g < 0$ (never mind that it does not exist non-perturbatively; the results will apply to any asymptotically free theory).

Let F be a scale-of-ϕ-independent (i.e. G_1-nvariant) eigenvector of γ_l with eigenvalue $\gamma_F(g_*)$, and

$$\gamma_F(g_*) = \gamma_F^1 g_* + O(g_*^2), \tag{3.21}$$

We will assume that F is scale of ϕ-independent We have seen above that because of asymptotic freedom for large μ one has $g_*(e^t\mu_0) = -\frac{16\pi^2}{3t} + o(t^{-2+\alpha})$, for any $\alpha > 0$. Therefore, using the fact that a is constant modulo g_*^2, we obtain from formula (3.19):

$$G_2^F(x^2) \sim \mathrm{const}|x|^{-2d_F}|\ln|x||^{32\pi^2\gamma_F^1/3}. \tag{3.22}$$

This remarkable formula is actually the exact asymptotics (!!!), except that in order to compute the constant on the right hand side we of course have to know the function γ_F to all orders. However, if we are not interested in the constant, we only need to know the first order approximation.

Fromula (3.22) can be generalized to the case of an arbitrary correlation function. Namely, if $F_1, ..., F_n$ are any scale-of-ϕ-independent operators which are eigenvectors of the operator γ_l^1, with eigenvalues $\gamma_{F_1}^1, ..., \gamma_{F_1}^n$, then

$$\langle F_1(e^{-t}x_1)...F_n(e^{-t}x_n)\rangle \sim C(x_1, ..., x_n)e^{-t(\sum d_{F_i})}t^{16\pi^2(\sum \gamma_{F_i}^1)/3}. \tag{3.23}$$

In order to compute the anomalous dimension of an operator, it is enough to study the dynamics of its 2-point function. Let us show how to do it in examples.

Examples. 1. $F = \phi$. The anomalous dimension of ϕ is zero in the 1-loop approximation, since there is no 1-loop contributions to the 2-point function at $x \neq 0$. However, in the 2-loop approximation, computing the only nontrivial 2-loop diagram, one gets a nontrivial correction. Of course, this correction does not enter in formula (3.23), so that correlation functions scale asymptotically by the naive classical rule.

In fact, it is easy to see that

$$\gamma_\phi = -\delta/2 \tag{3.24}$$

This is true because by the definition $\hat{\gamma}_\phi = 0$, since Schwinger functions do not change along the renormalization group flow.

2. $F = \phi^2$. In this case, analyzing the function $\langle\phi^2(x)\phi^2(0)\rangle$, which was computed in Witten's lecture 3, it is easy to find that in the 1-loop approximation

$$\gamma_{\phi^2}(g_*) = -\frac{g_*}{16\pi^2} + O(g_*^2). \tag{3.25}$$

In fact, it is easy to see that $\gamma_{\phi^2} = -2\gamma - \delta$. This follows from the fact that the anomalous dimension of the Lagrangian by definition vanishes. Therefore, the anomalous dimension of $m^2\phi^2/2$ vanishes, because it occurs in the Lagrangian, and cannot mix with other terms. Thus, $\gamma_{\phi^2} = -2\gamma_m$. Since $m = m_* a^{1/2}\mu$, $\gamma_m = \gamma_{m_*} + \gamma_a/2 = \gamma + \delta/2$, as desired.

Since we know that $\gamma = \frac{g_*}{32\pi^2} + O(g_*^2)$, we obtain another proof of (3.25).

Formula (3.25) allows us to write down the asymptotics of the 2-point function of ϕ^2 in the asymptotically free case. Indeed, using (3.22), we get

$$G_2^{\phi^2}(x^2) \sim \mathrm{const}|x|^{-4}|\ln|x||^{-2/3}. \tag{3.26}$$

Finally, let us discuss to what extent the notion of anomalous dimension is canonical. The problem is, we have made a lot of choices (renormalization prescription, the space Z), and of course the anomalous dimension operator γ_l will depend on them. So what is choice independent? The answer: 1) if g_*^f is a nonzero UV stable fixed point, then the eigenvalues of $\gamma_*^f(g_*)$ are invariantly defined, and they are anomalous dimensions of fields in the quantum theory corresponding to g_*^f; 2) the eigenvalues of the first order coeffients of γ_l at $g_* = 0$ are also invariantly defined; in the asymptotically free case they give the logarithmic part of the short-distance asymptotics of the correlation functions.

3.6. The canonical part of the β-function.

Suppose we have a quantum field theory with a single renormalizable coupling λ. In this case, we saw that the renormalization group flow is defined by the equation

$$\mu\frac{d\lambda}{d\mu} = \beta(\lambda), \tag{3.27}$$

where $\beta(\lambda) = A\lambda^{k+1} + ...$ is the beta-function. Clearly, we can redefine the coupling λ by using a change of variable $\lambda_1 = f(\lambda)$. This redefinition should be trivial for small values of the coupling in order to make physical sense, so we should impose the condition $f'(0) = 1$; that is, $f(\lambda) = \lambda + a_2\lambda^2 + ...$ Of course, under this transformation the beta-function will change. So to what extent is the beta-function defined canonically?

This question reduces to the following mathematical problem from the theory of "normal forms".

Problem. Let S_A^k be the set of all formal vector fields of the form $A\lambda^{k+1}\frac{d}{d\lambda} + ...$ with real coefficients ($k > 0$), and G be the group of formal diffeomorphisms of the form $\lambda \to f(\lambda)$, $f'(0) = 1$. Compute the quotient S_A^k/G, where G acts by conjugation. In other words, find the normal form of a vector field $\beta(\lambda)\frac{d}{d\lambda}$, where β is as above.

This problem is easily solved by recursion, and the answer is the following.

Answer. $S_A^k/G = \{(A\lambda^{k+1} + B\lambda^{2k+1})\frac{d}{d\lambda}, B \in \mathbb{R}\}$. In particular, any vector field of the form $\beta(\lambda)\frac{d}{d\lambda}$ as above can be reduced by conjugation to the normal form $(A\lambda^{k+1} + B\lambda^{2k+1})\frac{d}{d\lambda}$, while different normal forms are not equivalent.

Thus, we see that it is not only the coefficient A that is defined canonically in the beta-function, but also the coefficient B. In particular, in ϕ^4-theory, the beta-function has the form $A\lambda^2 + B\lambda^3 + ...$, and in QED the form $A\lambda^3 + B\lambda^5 + ...$, where A, B are defined canonically.

Let us explain the meaning of the coefficient B on the example of ϕ^4 theory. Consider this equation in the region $\lambda < 0$, where we have asymptotic freedom. Solving the equation $\lambda' = A\lambda^2 + B\lambda^3 + \ldots$, we get

$$\lambda(t) = \frac{1}{At} + \frac{B}{A^2 t^2} \ln t + O(1/t^2), \tag{3.28}$$

where $t = \ln(\mu/\mu_0)$. Thus, the coefficient B encodes the second term of the asymptotic expansion of $\lambda(t)$, which characterizes the singularity of λ, as a function of $z = 1/t$, at $z = 0$.