LECTURE 4: RENORMALIZATION GROUPS (CONTINUED)

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2.1. Dynamical mass generation.

In the previous three lectures we developed a general formalism of the method of renormalization group, derived the renormalization group equations, and considered its applications. In particular, we considered the notion of asymptotic freedom. In this lecture, we will actually consider an example of a good quantum theory which exists nonperturbatively and is asymptotically free. But before this let us finish the discussion of general properties of the renormalization group equation.

Consider a theory given by a Lagrangian which classically has a symmetry prohibiting masses (e.g. QED with a chiral fermion with no antichiral partner; such a theory has a chiral symmetry which prohibits the mass of a fermion). Let us ask the question: can a physical mass be dynamically generated in the corresponding quantum theory? Of course, this can only happen if the prohibiting symmetry is broken quantum mechanically.

Let m_p be the physical mass of the particle in this theory. That is, m_p is the pole of the 2-point function in momentum space.

Since the Lagrangian can have no mass term, there is no bare masses. Therefore, it is clear from counting of dimensions that the physical mass of the particle is given by

$$m_p = \mu f(g), \tag{4.1}$$

where f is a scalar function and g is a dimensionless coupling.

As we made clear in the previous lectures, the main property of the renormalization group vector field on the space $P \times M$ (where M is the set of momentum scales, and P the space of parameters) is that any physical parameters – actual (not bare!) masses of particles, scattering amplitudes – are CONSTANT along the trajectories of this vector field. Thus, the renormalization group equation says

$$(\mu\partial_\mu + \beta\partial_g)\mu f(g) = 0, \tag{4.2}$$

which yields

$$m_p = C\mu e^{-\int_{g_0}^{g} \frac{dx}{\beta(x)}}. \tag{4.3}$$

Consider the behavior of this function for weak coupling: $g \to 0$. As usual, this behavior is determined from the behaviour of the beta-function, which can be obtained from a 1-loop calculation. For example, assume that in our theory $\beta(g) \sim \pm g^3$ (e.g. QED,QCD). Then

$$m_p \sim \tilde{C}\mu e^{\mp 1/g^2}. \tag{4.4}$$

Thus, for an asymptotically free theory we have $m_p \sim \mu e^{-1/g^2}$. This is a reasonable behavior, since $m_p \to 0$ as $g \to 0$, so the theory is massless in the UV limit, as it should be. Also, if Λ is a cutoff and g_Λ the coupling at Λ, we have $m_p \sim \Lambda e^{-1/g_\Lambda^2}$. In particular, $g_\Lambda^2 \sim 1/\ln\Lambda$ as $\Lambda \to \infty$, which can also be seen from the renormalization group equation.

Note that (4.4) goes to 0 as $g \to 0$ faster than any polynomial; so it is a nonperturbative correction; in particular, all perturbative (Feynman diagram) approximations to $m_p(g)$ vanish. This is not surprising, since the Feynman diagrams are written from the Lagrangian in which there is no room for mass corrections.

Of course, it is possible that the proportionality coefficient in (4.4) vanishes, in which case $m_p = 0$ even in the nonperturbative quantum theory. This can happen for example if the prohibiting symmetry is unbroken (e.g. supersymmetry).

However, if the theory is not asymptotically free, but IR free, we get $m_p \sim \mu e^{1/g^2}$, which is absurd. Thus, in this case, mass generation is not possible.

Finally, let us explain why theories which are not asymptotically free are usually not believed to exist as well defined nonperturbative quantum theories. For this purpose consider the dependence of the bare coupling g_0 of the cutoff ε (in the dimensional regularization – minimal subtraction scheme); this dependence can be derived from the renormalization group equation. Namely, we have (see Lecture 2)

$$g_0(\varepsilon, g) = g e^{\int_0^g \frac{\beta(x)dx}{x(\varepsilon x/2 - \beta(x))}}, \tag{4.5}$$

where $\varepsilon = 4 - d$.

Let $\beta(x) = \pm x^2 + \ldots$. Then formula (4.5) can be written in the form

$$g_0 = g + \sum_{m=1}^{\infty} g^{m+1} P_m(1/\varepsilon), \tag{4.6}$$

where P_m is a polynomial of degree m with zero free term. This expression yields counterterms in all orders of perturbation theory.

Let us see, however, whether (4.5) and (4.6) make sense nonperturbatively. If the sign in the beta-function is minus (asymptotic freedom), then the integral in (4.5) is well defined, at least for small enough g, since the denominator is positive. However, if the sign is plus (no asymptotic freedom), then the denominator vanishes for $x \sim \varepsilon$, so if ε is small then the integral gets a singularity even for small x. This is usually regarded as a sign that the theory does not exist, i.e. that ist perturbation expansion does not correspond to any nonperturbative theory. In particular, the lack of asymptotic freedom for ϕ^4 theory is one of the reason people believe this theory does not exist.

Remark. In spite of this, physicists often use Lgarangians which are not asymptotically free and even nonrenormalizable for describing quantum field theories. In this case, it is usually meant that the the theory is considered with a permanent UV cutoff, which is not sent to infinity; thus the theory only makes sense as a "low energy effective theory". See Witten's lecture II-1 for more details.

2.2. The Gross-Neveu model.

Now we are ready to consider a concrete field theory, on whose example we will be able to illustrate the method of renormalization group. This is a 2-dimensional, purely fermionic theory, which is called the $(\bar{\psi}\psi)^2$-model, or the Gross-Neveu model.

This model exhibits the following nice properties:

1. It is described by a local Lagrangian and is both renormalizable and asymptotically free.

2. It exists nonperturbatively; this is a nontrivial fact which can be proved by applying a nonperturbative version of the renormalization group method.

3. It has dynamical symmetry breaking and dynamical mass generation (these things go together as the symmetry which is broken is the one prohibiting mass).

4. This theory is in 2 dimensions, but we can formally continue its large N expansion to $2+\varepsilon$ (using dimensional regularization techniques), and the obtained amplitudes will be regular at $\varepsilon = 1$. Thus we obtain a 3-dimensional "theory" which is non-renormalizable but exists.

5. This theory is soluble (integrable). This means, its S-matrix can be computed exactly for a any fixed N and the Green's functions can be computed exactly in the large N limit (where N is the number of particles). Computation of the S-matrix leads to the theory of quantum groups (Yangians) – see Witten's lecture II-3, where a similar model is studied.

6. It is closely related to 2-dimensional nonlinear sigma-model to a homogeneous space with positive curvature (see Witten's lecture II-6).

Let us now define the Gross-Neveu model. It is a theory of N massless Dirac fermions in two dimensions with a quartic interaction. Namely, let V be a complex N-dimensional vector space with a positive definite Hermitian form $(,) : \bar{V} \times V \to \mathbb{C}$, and let ψ be a Dirac spinor on $\mathbb{R}^{1,1}$ with coefficients in V. Consider the Lagrangian in Minkowski space

$$(4.7) \qquad L(\psi) = \int d^2x(i(\bar{\psi}, D\psi) + \frac{g^2}{2}(\bar{\psi}, \psi)^2).$$

Let us determine the symmetries of this Lagrangian. Let S_+, S_- be the spin bundles over $\mathbb{R}^{1,1}$ (these are real, 1-dimensional representations of $Spin(1,1)$ with spins $1/2$ and $-1/2$. Let $S = S_+ \oplus S_-$. By definition, ψ is a section of $\Gamma(S) \otimes V$, so $\psi = (\psi_+, \psi_-)$. In terms of the chiral components (ψ_+, ψ_-), Lagrangian (4.7) can be written as

$$(4.8) \qquad L(\psi) = \int d^2x(i(\bar{\psi}_+, D\psi_+) + i(\bar{\psi}_-, D\psi_-) + g^2((\bar{\psi}_+, \psi_-) + (\bar{\psi}_-, \psi_+)).$$

We see that (4.8) has an obvious global $U(N)$ symmetry (unitary transformations of V), as well as a chiral symmetry $\psi_\pm \to \pm\psi_\pm$. This chiral symmetry is the one that prevents a mass term $m(\bar{\psi}, \psi)$. We will see that in quantum theory, the chiral symmetry will be broken and the mass will be dynamically generated.

Recall that in 2 dimensions spinor fields have scaling dimension $1/2$. This implies that the theory defined by (4.7) is purely renormalizable (i.e. classically conformally invariant), with a single renormalizable coupling g.

Remark. One can show that the $U(N)$ symmetry survives in quantum theory, and so no new couplings will appear in the process of renormalization.

We will see below that the Gross-Neveu model is asymptotically free for $g^2 > 0$ and IR free for $g^2 < 0$. So we could hope to have a nice theory only for $g^2 > 0$. Note that unlike ϕ^4-theory, the positivity of the coefficient at the quartic term in Minkowski signature is not a problem, since our theory is fermionic and positivity of g^2 does not create a divergence in the functional integral.

To study the Gross-Neveu model, it will be useful to introduce an auxiliary real scalar field σ (with values in $\mathbb{R}$), and consider the Lagrangian

$$L_\sigma(\psi,\sigma) = \int d^2x(i(\bar{\psi}, D\psi) + g\sigma(\bar{\psi},\psi) - \frac{1}{2}\sigma^2). \tag{4.9}$$

It is clear that this Lagrangian is equivalent to (4.7): indeed, (4.7) is obtained from (4.9) by integrating out σ (i.e. setting σ to equal its stationary point, $\sigma = g(\bar{\psi},\psi)$.

2.3. The large N limit.

The Gross-Neveu model becomes especially simple in the limit $N \to \infty$. More precisely, in this limit the theory is exactly solvable. This means, we can compute the asymptotic expansion of the Green's functions in this theory in terms of the variable $1/N$. This is a kind of perturbation expansion which is useful also for many other theories. The advantage of this expansion versus perturbation expansions with respect to other parameters (like g) is that $1/N$ is not changed by the renormalization group flow.

We will solve the Gross-Neveu model in the large N limit in two ways: 1) by explicitly summing the perturbation series; and 2) by using presentation (4.9).

Remark. It is not claimed that there is a well defined quantum field theory at $N = \infty$; we just have a family of theories depending on N whose Green's functions have a nice asymptotic expansion in N as $N \to \infty$.

We start with summing the perturbation series. We note that the perturbation series for bosonic theories (with a cutoff) usually diverges, and one has to apply Borel summation to get a nonperturbative result. However, in purely fermionic theories (like the Gross-Neveu model) the perturbation theory may converge, defining an analytic (not only C^∞) function of the perturbation parameter. To understand why it is so, it is best to look at two integrals

$$I_1(h) = \int_{\mathbb{R}^n} e^{-\frac{v^2+(v^2)^2}{h}} dv, I_2(h) = \int_{\mathbb{R}^{0,2n}} e^{-\frac{(\psi,A\psi)+(\psi,A\psi)^2}{h}} d\psi$$

where $\mathbb{R}^{0,2n}$ is an odd vector space, and A a skewsymmetric nondegenerate matrix. Clearly, the first integral gives a divergent perturbation series, while the second gives a polynomial. For infinite dimensional integrals, convergence is not obvious even for purely fermionic theories, but one can show that for the Gross-Neveu model the perturbation series indeed converges. (see the paper of Gawedzki and Kupiainen, CMP, v.102(1),p.1-30).

Consider Lagrangian (4.7) for large N. It is clear that the first (kinetic) term is of order N and the second (quartic) term is of order N^2. Therefore, if set $g_N = \overline{g}N^{-1/2}$, where $\overline{g}$ is fixed, and consider the sequence of theories with N-dimensional V and coupling constant g_N, then both terms in the Lagrangian scale in the same way for large N (they are both of order N). Thus, the large N limit for our theory is similar to its quasiclassical limit (i.e. $1/N$ plays the role of the Planck constant).

Let us consider the Feynman rules for our theory. For this we will choose an orthonormal basis e_i in V and write $\psi = \sum \psi_j e_j$, where ψ_j are complex valued spinors. The Feynman rules are as follows: the fermion propagator (in momentum space of Minkowski signature) is $\frac{ip}{p^2}$ (as usual), and the 4-vertex corresponding to the term $(\bar{\psi}_j, \psi_{j'})(\bar{\psi}_k, \psi_{k'})$ carries the factor $ig^2\delta_{jj'}\delta_{kk'}$. This makes in natural to draw the 4-vertex as two separate (oriented) arcs:

$$\begin{matrix} i & j \\ \Big) & \Big(\\ \bar{i} & \bar{j} \end{matrix}$$

(bars over i, j remember the orientation). In particular, every Feynman diagram is a union of lines and circles, like

$$\Big) OOO \Big(\tag{4.10}$$

For large N, each circle gives a factor of N (because of summation over the index that lives on this circle) and every "tangency" of two circles (a 4-vertex) gives a factor of g^2. Thus the order of the contribution from such diagram is N^K, where N is the number of circles minus the number of tangencies.

Now consider the perturbation expansion of the 1-particle irreducible 2-point function Γ_2 for ψ. This means we should consider diagrams with one line and any number of circles. For convenience let us close up the line (this will only give us an extra factor of N). Then we get diagrams with circles only. For any such diagram D, consider the graph $\Gamma(D)$ in which vertices are the circles of D and edges are tangencies of circles. We see that $\Gamma_2 = \sum_D A_D$, where A_D is the amplitude of D, and for large N one has $A_D \sim N^{-n}$, where n is the number of cycles in $\Gamma(D)$. (note that $\Gamma(D)$ is always connected).

Thus, the leading term of the large N expansion of our theory is given by "tree" diagrams, i.e. diagrams for which $\Gamma(D)$ (not D itself!) is a tree.

Now, it is easy to see that the amplitude of the diagram

$$\Big) O$$

vanishes (it contains the factor $\int_{|p|\le\Lambda} \frac{ip}{p^2} d^2p$ which is zero by symmetry arguments). Therefore, there is no nontrivial trees which give a contibution of order N^0 to the 2-point function (indeed, such a tree would have to have only one 1-valent vertex, which is impossible for a nontrivial tree); thus the propagator does not get a correction in perturbation theory in the top order in N.

Let us now consider the 4-point function $\langle \psi_j \bar{\psi}_k \bar{\psi}_l \psi_m \rangle$. Arguing similarly to the 2-point case, we see that all leading contributions (in N) come from diagrams D for which $\Gamma(D)$ is a tree with only two 1-valent vertices. Any such tree must be a chain. Thus all diagrams of importance in the leading order are the following diagrams, which we call D_n and D'_n (where n is the number of bubbles):

$$\begin{array}{c} j \qquad \bar{k} \\ \Big) ooo \Big(\\ \bar{l} \qquad m \end{array}$$

and

$$\begin{array}{c} j \qquad \bar{l} \\ \Big) ooo \Big(\\ \bar{k} \qquad m \end{array}$$

Thus, computing the leading term of the 4-point function reduces to summing a geometric series. Namely, we have

$$\Gamma_4^{jklm}(p_1, p_2, p_3, p_4)|_{N=\infty} = \delta_{jk}\delta_{lm}F(s) + \delta_{jm}\delta_{kl}F(t), \tag{4.11}$$

where

$$s = (p_1 + p_2)^2, t = (p_3 + p_4)^2, \tag{4.12}$$

i, k, l, m are the labels of the components of ψ_i for which the 4-point function is computed, and p_1, p_2, p_3, p_4 are the momenta corresponding to labels i, j, k, l. In this formula, the first summand comes from diagrams D_n and the second from D'_n. Thus,

$$F(s) = \sum_{n\geq 0} A_{D_n}. \tag{4.13}$$

The amplitude of D_n is easily computed. Since the Feynman integrals are IR convergent, it suffices to introduce an ultraviolet cutoff Λ. Then

$$A_{D_n} = (ig^2)^{n+1}A^n, \tag{4.14}$$

where

$$A = \frac{iN}{\pi}\ln(-\Lambda^2/s) \tag{4.15}$$

(remember that a fermionic loop contributes a minus sign). Thus, after introducing a counterterm $Cg^4\ln(\Lambda/\mu)(\bar{\psi}, \psi)^2$ in the Lagrangian, using some scale μ, we will get by summing the geometric series:

$$F(s) = \frac{ig^2}{1 + \frac{g^2N}{\pi}\ln(-s/\mu^2)}. \tag{4.16}$$

Formula (4.16) implies that the coupling g has the meaning of the strength of the interaction at the scale μ: indeed, if we take p_i such that $(p_i + p_j)^2 = -\mu^2$ then $\Gamma_4^{jklm} = ig^2(\delta_{jk}\delta_{lm} + \delta_{jm}\delta_{kl})$.

Another consequence of formula (4.16) is that our theory is asymptotically free for large N. Indeed, as $s \to -\infty$, function (4.16) behaves as $1/(\ln(-s))$, i.e. goes to 0. Moreover, we can compute the β-function exactly: from the renormalization group equation and (4.16) we get:

$$\beta(g) = -\frac{g^3 N}{\pi} + O(1), \ N \to \infty. \tag{4.17}$$

This is the exact β-function to leading order in $1/N$.

Thus, we should expect that our theory is well defined in the UV limit.

Let us now consider what happens to this theory in the infrared limit. Then we notice that formula (4.16) has an unfortunate property that the function $F(s)$ has a pole at $s = -\mu^2 e^{-\pi/g^2 N}$. Thus, the Γ-function has a pole outside of 0 for "Euclidean" momenta. This cannot happen in a well defined quantum field theory: it would mean that the theory contains a "tachyon" – particle of imaginary mass.

So what is wrong with our theory? There could be two things: either the theory is "sick" (does not exist) or we have been doing perturbation theory around a "wrong" (unstable) vacuum.

In fact, the situation (which can be expected if we believe the theory exists) is as follows. This theory classically has no mass gap, so the quasiclassical approximation is not enough to determine what it does quantum mechanically. What in fact happens is that the theory has two different realizations, which are permuted by the chiral symmetry (see Witten's lecture II-1). In particular, the chiral symmetry is broken quantum mechanically. These two vacuum states are massive (i.e. mass is dynamically generated), and the expectation value of $(\bar{\psi}, \psi)$ at them is nonzero: we have two opposite values for two different vacua. However, these expectation values are exponentially small (in terms of the coupling), so they cannot be distinguished in perturbation theory.

Thus, in order to find the quantum vacuum states, we have to use nonperturbative methods. We will use formula (4.9), which contains an auxiliary bosonic field σ.

Consider the Euclidean path integral with Lagrangian (4.9) (remember that we have to change the signs when going from Minkowski to Euclidean space, and remove the i in front of the Dirac operator):

$$\int D\psi D\sigma e^{-\int d^2x((\bar{\psi}, D\psi) - g\sigma(\bar{\psi},\psi) + \frac{1}{2}\sigma^2)}. \tag{4.18}$$

In this integral, ψ has $4N$ real components, and integration is carried out over all of them. Remembering the asymptotics of the terms of the Lagrangian in terms of N, we see that it is convenient to replace σ with $\sigma N^{1/2}$ and rewrite this path integral as

$$\int D\psi D\sigma e^{-N\int d^2x(\frac{1}{N}(\bar{\psi}, D\psi) - \frac{g}{\sqrt{N}}\sigma(\bar{\psi},\psi) + \frac{1}{2}\sigma^2)} \tag{4.19}$$

Now, since the path integral (4.19) is Gaussian in ψ, we can integrate ψ out. We get (up to an overall factor)

$$\begin{aligned} &\int D\sigma \det{}^N(D - g\sqrt{N}\sigma) e^{-N\int d^2x\sigma^2/2} = \\ &\int D\sigma e^{N \mathrm{Tr}\ln(iD - g\sqrt{N}\sigma) - N\int d^2x\sigma^2/2}. \end{aligned} \tag{4.20}$$

The large N limit of last integral can be treated by the stationary phase method. In particular, the expectation value of σ for large N is such that it makes the functional

$$I(\sigma) = \mathrm{Tr}\ln(D - g\sqrt{N}\sigma) - \int d^2x\sigma^2/2 \tag{4.21}$$

stationary.

We look for constant functions σ. Then the stationarity equation for (4.21) is

$$\sigma = -\mathrm{Tr}\frac{g\sqrt{N}}{D - g\sqrt{N}\sigma} = -\int \frac{d^2p}{(2\pi)^2}\frac{g\sqrt{N}}{iP - g\sqrt{N}\sigma}, \tag{4.22}$$

where P is the operator of Clifford multiplication by p. Replacing σ with $-i\rho$, we get

$$\rho = Tr\int \frac{d^2p}{(2\pi)^2}\frac{g\sqrt{N}}{P + g\sqrt{N}\rho}, \tag{4.23}$$

It can be shown that this equation has two opposite nonzero solutions as well as the zero solution. To find the nonzero solutions, the right hand side, which is divergent, needs to be renormalized in the UV, using some scale μ. The renormalized equation has the form

$$\rho = \lim_{\Lambda\to\infty}\Big(\int_{|p|=\Lambda}\frac{d^2p}{(2\pi)^2}\frac{g\sqrt{N}}{P + g\sqrt{N}\rho} - \frac{g\sqrt{N}}{2\pi}\ln(\Lambda/\mu)\Big). \tag{4.24}$$

Simplifying (4.24), we get

$$\begin{aligned}\rho = 2\lim_{\Lambda\to\infty}\frac{g^2N\rho}{4\pi}\ln[\frac{\mu^2}{g^2N\rho^2}(1 + g^2N\rho^2/\Lambda^2)] = \\ 2\frac{g^2N\rho}{4\pi}\ln[\frac{\mu^2}{g^2N\rho^2},\end{aligned} \tag{4.25}$$

(the factor 2 in the front is due to the fact that P is a 2 by 2 matrix) which implies that

$$\rho = \frac{\mu}{g\sqrt{N}}e^{-\pi/g^2N}. \tag{4.26}$$

Thus, the expected physical mass of the fermions is

$$m_p = g\sqrt{N}\rho = \mu e^{-\pi/g^2N}. \tag{4.27}$$

Another way to find the explicit expression for the nonzero solutions of equation (4.23) is to compute the effective potential for ρ. The computation is analogous to the above computation of the 4-point function, and yields (up to scaling

$$V_{eff}(m) = \frac{m^2}{2} + \frac{g^2Nm^2}{4\pi}(\ln\frac{m^2}{\mu^2} - 1). \tag{4.28}$$

The critical points of this function are 0 and $\pm m_p$, where m_p is given by (4.27). However, the point $m = 0$ is unphysical (=does not define a quantum vacuum): at this point $V''_{eff}(m) = -\infty$, so the point is a local maximum of the effective potential.

Thus, we have shown that our theory exhibits dynamical mass generation. This is a special case of so called "dimensional transmutation" – a phenomenon in which a bare dimensionless coupling (g) transforms dynamically into a dimensionful measurable quantity (m).