LECTURE 5: RENORMALIZATION GROUPS (CONTINUED)

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In this lecture we will return to the Wilsonian point of view on the method of renormalization group, which was discussed in the first lecture. Namely, we will discuss the exact renormalization group equation, which describes the renormalization group flow in the space of all Lagrangians. This equation together with asymptotic freedom provides a tool for constructing nonperturbative field theories.

We follow J.Polchinski's paper "Renormalization and effective Lagrangians", Nuclear Physics B231, 1984, p. 269-295.

For simplicity we will consider the theory of a scalar bosonic field in 4 dimensions, as in the first lecture. We consider the theory with a cutoff propagator $\frac{P(p^2/\Lambda^2)}{p^2+m^2}$, where P is a smooth positive nonincreasing function, such that $P(a)=1$ for $a\le 1$ and $P(a)\to 0$ sufficiently fast as $a\to\infty$.

The free action for a massive scalar associated to this cutoff can be written in momentum space as

$$S_0(\phi,\Lambda_0)=\frac{1}{2}\int\hat\phi(p)\hat\phi(-p)\frac{p^2+m^2}{P(p^2/\Lambda_0^2)}\frac{d^4p}{(2\pi)^4} \tag{5.1}$$

(hat denotes the Fourier transform). This action is very big if high Fourier modes in ϕ are present, so it "suppresses" high Fourier modes in the path integral. Also, when $\Lambda_0\to\infty$, this action obviously tends to the standard action for a massive scalar.

Introduce a perturbation of the action S_0 by "interaction" terms:

$$S_I^0(\phi)=\int(\frac{1}{2}g_1^0\phi^2+\frac{1}{2}g_2^0(\nabla\phi)^2+\frac{g_3^0}{4!}\phi^4)d^4x, \tag{5.2}$$

where g_i^0 are formal variables, and consider the total action $S(\phi,\Lambda_0)=S_0(\phi,\Lambda_0)+S_I^0(\phi)$. Consider the theory whose Green's functions are defined by the path integral

$$Z(J)=\int D\phi\ e^{-S(\phi,\Lambda_0)+\int J\phi d^4x}. \tag{5.3}$$

Because we are considering a cutoff theory, all Feynman diagrams are well defined, so we get a perturbation series.

Now consider some smaller scale $\Lambda_R<\Lambda_0$. We are interested in physics at the scale Λ_R. This means, we assume that $m<<\Lambda_R$ and we are interested in values of $Z(J)$ for such J that $\hat J(p)=0$ for $p^2>\Lambda_R^2$.

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The physics at the scale Λ_R will be described by an effective Lagrangian, which is obtained from the original Lagrangian by "integrating out" degrees of freedom between Λ_R and Λ_0.

To study the effective Legrangian, we introduce an arbitrary interaction term $S_I(\phi,\Lambda)$ and consider the integral

$$Z(J,S,\Lambda)=\int D\phi\ e^{-S_0(\phi,\Lambda)+S_I(\phi,\Lambda)+\int J\phi d^4x} \tag{5.4}$$

Let us look for such $S_I(\phi,\Lambda)$ that $Z(J,S,\Lambda)$ is independent on Λ, and write down a differential equation for such $S_I(\phi,\Lambda)$.

The independence of Z of Λ yields

$$\frac{d}{d\Lambda}\int D\phi e^{-S_0(\phi,\Lambda)+S_I(\phi,\Lambda)+\int J\phi d^4x}=0. \tag{5.5}$$

Differentiating, we get

$$\begin{aligned}&\int D\phi\Big(\frac{1}{2}\int\hat{\phi}(p)\hat{\phi}(-p)(p^2+m^2)\frac{d}{d\Lambda}P(p^2/\Lambda^2)\frac{d^4p}{(2\pi)^4}-\frac{dS_I(\phi,\Lambda)}{d\Lambda}\Big)\times\\ &\qquad e^{-S_0(\phi,\Lambda)-S_I(\phi,\Lambda)+\int\hat{J}(p)\hat{\phi}(-p)\frac{d^4p}{(2\pi)^4}}=0.\end{aligned} \tag{5.6}$$

¿From this one can get

$$\begin{aligned}\frac{\partial S_I}{\partial\Lambda}=-\frac{1}{2}\int d^4p(2\pi)^4(p^2+m^2)^{-1}\frac{\partial P(p^2/\Lambda^2)}{\partial\Lambda}\times\\ \Big(\frac{\delta S_I}{\delta\phi(-p)}\frac{\delta S_I}{\delta\phi(p)}+\frac{\delta^2 S_I}{\delta\phi(-p)\delta\phi(p)}\Big).\end{aligned} \tag{5.7}$$

Indeed, substituting (5.7) into (5.6), we get the integral of a complete derivative, i.e. zero; here we heavily use the fact that $\hat{J}(p)$ has disjoint support with $\frac{\partial P(p^2/\Lambda^2}{\partial\Lambda}$. And it can be shown that (5.7) follows from (5.6).

Note that equation (5.7) corresponds to a linear equation for e^{S_I}:

$$\frac{\partial e^{S_I}}{\partial\Lambda}=-\frac{1}{2}\int d^4p(2\pi)^4(p^2+m^2)^{-1}\frac{\partial P(p^2/\Lambda^2)}{\partial\Lambda}\frac{\delta^2 e^{S_I}}{\delta\phi(-p)\delta\phi(p)}. \tag{5.8}$$

Equation (5.8) can be regarded as an infinite-dimensional heat equation. It is called the exact renormalization group equation. If $S_I(\phi,\Lambda)$ is the solution of this equation such that $S_I(\phi,\Lambda_0)=S_I^0(\phi)$, then $S_I(\phi,\Lambda_R)$ is the effective action at the scale Λ_R. This equation allows to prove the renormalizability of ϕ^4 theory without use of graph techniques, in particular of Weinberg's theorem (see Polchinski's paper).