Lecture 1.
Wightman axioms.

David Kazhdan

Introduction. In this series of lectures we will present Wightman axioms and show how this set of axioms can be used to derive deep and unexpected results on behavior of quantum field theories. These axioms constitute a consistent mathematical theory which can serve for a mathematician as a gateway to the strange world of quantum field theories. In spite of the failure to describe the structure of gauge theories Wightman axioms are very useful for developing of an intuition on quantum field theories.

1.0 Setup and notations. Let V be a pseudo-Riemannian manifold of signature $(1, d-1)$. In these lectures we will only consider the case $V = \mathbb{R}^{1,d-1}$ with constant quadratic form. We denote by $(\quad,\quad)\colon V \times V \to \mathbb{R}$ the corresponding quadratic form and will write v^2 instead of (v, v) for $v \in V$. We will sometimes use decomposition $V = \mathbb{R} \times \mathbb{R}^{d-1}$, where $\mathbb{R}$ (time) and $\mathbb{R}^{d-1}$ (space) are orthogonal subspaces equipped with constant positive (respectively negative) definite metrics. This decomposition is fixed throughout the lectures. We denote: $V_{space} = \{v \in V | v^2 < 0\}$; and $\overline{V}_{space}$ is the closure of V_{space}. The complement $V - \overline{V}_{space}$ consists of 2 connected components, V_+ and V_- (where $V_\pm \subset \mathbb{R}^\pm \times \mathbb{R}^{d-1}$).

For $m \in \mathbb{R}$ we put $\mathcal{O}_m^\pm = \{v \in \overline{V}_\pm | v^2 = m^2,\ v \neq 0\}$. Let dv be the standard measure on V and let $\mu_m = \mathrm{Res}\big|_{\mathcal{O}_m^+} \frac{dv}{v^2 - m^2}$ be the invariant volume form on $\mathcal{O}_m^+$.

Let $G = \mathrm{Spin}(V)$ be the double (universal for $d > 3$) cover of connected component of the group $\mathrm{SO}(1, d-1)$. The semidirect product $P = G \ltimes V$ is called the Poincare group. It acts on V by isometries. We denote by $\tau \in G$ the nontrivial element of the kernel of the map $G \to \mathrm{SO}(1, d-1)$.

We will use the quadratic form to identify V^* with V, but we will distinguish between vectors and covectors by reserving the letters v, w, etc. for vectors, and p, q, etc. for covectors.

Note that the cone V_+ is self-dual (i.e. $\{p \in V | (p, v) > 0\ \forall\, v \in V_+\} = V_+$). For a function f on V we will denote by $\mathcal{F}(f)$ its Fourier transform; so we have:

$$\mathcal{F}(f)(p) = \frac{1}{(2\pi)^{d/2}} \int\limits_V e^{-i(p,v)} f(v) dv.$$

Let $U\colon V \to \mathrm{Aut}(\mathcal{H})$ be a unitary representation of the vector group V in a Hilbert space $\mathcal{H}$.

Definition. Representation U is called *positive* if support of the corresponding spectral measure lies in $\overline{V}_+$.

If $U\colon P \to \mathrm{Aut}(\mathcal{H})$ is a unitary representation, then we say that U is positive if $U\big|_V$ is positive.

For a G-module H we will write $H = H^{even} \oplus H^{odd}$, where H^{even} (respectively H^{odd}) is the space of invariants (respectively antiinvariants) of τ. We will endow H with the structure of superspace given by this decomposition.

Let $R = R^{even} \oplus R^{odd}$ be a finite dimensional real representation of G. Since P acts transitively on V and stabilizer of 0 is G, we can associate to R a P-equivariant vector bundle $\mathcal{R}$ on V.

Let $\mathcal{S}^R = \mathcal{S}^R_{even} \oplus \mathcal{S}^R_{odd}$ be the space of Schwartz sections of this bundle. (Note that our bundle is trivialized equivariantly with respect to translations, and $\mathcal{S}^R$ is identified with the space of smooth rapidly decreasing R-valued functions on V.) We will often abbreviate $\mathcal{S} = \mathcal{S}^R$.

Finally we need some preparations on signs.

We say that a complex superspace $H = H^{even} \oplus H^{odd}$ is a superHermitian space if it is endowed with an even $\mathbb{R}$-bilinear pairing $(\quad,\quad) : H \times H \to \mathbb{C}$ such that $(\lambda x, y) = (x, \overline{\lambda} y) = \lambda(x, y)$ and $(x, y) = (-1)^{|x|\cdot|y|}\overline{(y, x)}$. Note that since $(\quad,\quad)$ is even we have $(x, y) = 0$ for $x \in H^{even}$, $y \in H^{odd}$.

H is called superHilbert space if $(H^{even}, (\quad,\quad))$ and $(H^{odd}, -i(\quad,\quad))$ are Hilbert spaces. For $T \in \mathrm{End}(H)$ the (super)adjoint operator T^* is defined by: $(Tx, y) = (-1)^{|x|\cdot|T|}$ (x, T^*y). An operator is superHermitian symmetric if $T^* = T$. We will often omit the word super.

Now we are ready to formulate:

1.1 Wightman axioms.
Data:

1) A finite dimensional real representation ρ: $G \to \mathrm{Aut}\, R$.
2) A positive unitary representation U: $P \to \mathrm{Aut}(\mathcal{H})$ in a (super)Hilbert space $\mathcal{H}$.
3) A dense P-invariant subspace $\mathcal{D} \subset \mathcal{H}$, and a P-invariant vector $\Omega \in \mathcal{D}$, $\|\Omega\| = 1$.
4) A linear map ϕ: $\mathcal{S}^R \to \mathrm{End}(\mathcal{D})$, called *the field map* such that:

α) ϕ is P-equivariant where P acts on $\mathrm{End}(\mathcal{D})$ by conjugation.

β) $\phi(f)$ is superHermitian-symmetric.

γ) For any $\psi_1, \psi_2 \in \mathcal{D}$ the map $f \to \langle \psi_1, \phi(f)\psi_2 \rangle$ from $\mathcal{S}$ to $\mathbb{C}$ is continuous ($\mathcal{S} = \mathcal{S}^R$ is equipped with the Schwartz topology).

δ) The space $\mathcal{D}$ is generated by vectors $\phi(f_1)\phi(f_2), \ldots, \phi(f_n)\Omega$, for $f_1, \ldots, f_n \in \mathcal{S}$.

ε) (space-locality) If $f_1, f_2 \in \mathcal{S}$ are such that for any $v_1 \in \mathrm{supp}(f_1)$, $v_2 \in \mathrm{supp}(f_2)$ we have $(v_1 - v_2) \in V_{space}$ (such f_1, f_2 are called *space-like separated*), then $[\phi(f_1), \phi(f_2)] = 0$ where $[\quad,\quad]$ stands for supercommutator.

Remark. It is often assumed that the space of V-invariants in $\mathcal{H}$ is 1-dimensional (uniqueness of vacuum).

Such set of data satisfying the above properties is called QFT in Wightman axioms (or simply Wightman QFT).

1.2 Wightman's functions. Let us denote: $V_n = V \times V \times \cdots \times V$ (n times); $\mathcal{R}_n = \mathcal{R}^{\boxtimes n}$ (this is a bundle on V_n). The space of Schwartz sections of $\mathcal{R}_n$ will be denoted by $\mathcal{S}_n$.

For $n \geq 1$ the Schwartz theorem and the property (γ) imply the existence of unique tempered $\mathbb{C}$-valued distribution $\mathcal{W}_n \in \mathcal{S}'_n$ such that

$$\mathcal{W}_n(f_1 \times f_2 \times \cdots \times f_n) = \langle \Omega, \phi(f_1)\phi(f_2)\ldots\phi(f_n)\Omega\rangle$$

for $f_1, \ldots, f_n \in \mathcal{S}$.

These $\mathcal{W}_n$ are called *Wightman functions*, or correlation functions of the theory. (We will use the word "function" for a section of an appropriate vector bundle when it causes no confusion).

1.3 Reconstruction of QFT from Wightman functions. We are going to show that a QFT is uniquely determined by its Wightman functions. So we assume that a representation ρ of G and a collection of distributions $\mathcal{W}_n \in \mathcal{S}_n$ are given, and that $\mathcal{W}_n$ are Wightman functions of some Wightman QFT $(\rho; \mathcal{H}, \mathcal{D}, \Omega; \phi)$. We want to reconstruct $\mathcal{H}, \mathcal{D}, \Omega$ and ϕ.

Let us put $\widetilde{\mathcal{S}} = \bigoplus\limits_n \mathcal{S}_n$, $\mathcal{W} = \prod\limits_n \mathcal{W}_n \in \widetilde{\mathcal{S}}'$ (where we use the standard notation L' for the topological dual of a topological vector space L). The space $\widetilde{\mathcal{S}}$ carries an algebra structure: $f_1 \cdot f_2 = f_1 \boxtimes f_2$. For $f \in \mathcal{S}_n$ we define $f^* \in \mathcal{S}_n$ by: $f^*(v_1, \ldots, v_n) = \pm f(v_n, \ldots, v_1)$ (the sign is determined by the super-rule as usually). Consider a bilinear form $\{\ ,\ \}$ on $\widetilde{\mathcal{S}}$ given by:

$$\{f_1, f_2\} = \mathcal{W}(f_1^* \cdot f_2).$$

Claim. $\{f_1, f_2\} = \langle \phi(f_1)\Omega, \phi(f_2)\Omega\rangle$.

Proof. Clear.

We can recover $\mathcal{H}$ as the Hilbert space completion of $(\widetilde{\mathcal{S}} \otimes \mathbb{C}, \{\ ,\ \}')$, where $\{\ ,\ \}'$ is the unique Hermitian scalar product on $\widetilde{\mathcal{S}} \otimes \mathbb{C}$, such that $\{\ ,\ \}'|_{\widetilde{\mathcal{S}}} = \{\ ,\ \}$. Then we can take $\mathcal{D}$ to be the image of the obvious map $\widetilde{\mathcal{S}} \otimes \mathbb{C} \to \mathcal{H}$ and define Ω as the image of $1 \in \widetilde{\mathcal{S}}$ under this map. Since P acts on $\widetilde{\mathcal{S}}$ preserving $\{\ ,\ \}$ this action defines an action of P on $\mathcal{H}$. Finally, we get an algebra map $\phi\colon \widetilde{\mathcal{S}} \to \mathrm{End}(\mathcal{D})$.

Before we formulate the next Claim let us fix some notations. Let $\Delta\colon V \to V_n$ be the diagonal embedding, and $\overline{V}_n = V_n/\Delta$ be the quotient space. Since $\mathcal{R}_n$ is P-equivariant it defines a bundle $\overline{\mathcal{R}}_n$ on $\overline{V}_n$. Let $\overline{\mathcal{S}}_n$ be the space of Schwartz sections of $\overline{\mathcal{R}}_n$. For $f \in \mathcal{S}_n$ let $\overline{f} \in \overline{\mathcal{S}}_n$ be its averaging over $\Delta(V)$, and let pr^* be the dual map on distributions.

We identify $\overline{V}_n \tilde{\to} V_{n-1}$ by:

$$(v_1, v_2, \ldots, v_n) \to (v_2 - v_1, \ldots, v_n - v_{n-1}).$$

Claim. *$\mathcal{W}_n$ satisfy the following properties:*

1) *$\mathcal{W}_n$ is P-invariant for all n.*
 In particular, $\mathcal{W}_n$ is $\Delta(V)$-invariant; hence $\mathcal{W}_n = pr^*(W_n)$ for unique $W_n \in \overline{\mathcal{S}}'_n$.
2) $\operatorname{supp}\mathcal{F}(W_n) \subset (\overline{V}_+)^{n-1}$.
3) $\mathcal{W}_n(f^*) = \overline{\mathcal{W}_n(f)}$.
4) $\mathcal{W}(v_1, \ldots, v_i, v_{i+1}, \ldots, v_n) = \pm\mathcal{W}(v_1, \ldots, v_{i+1}, v_i, \ldots, v_n)$ *if* $v_i - v_{i+1} \in V_{space}$. *(Here the sign comes from the $\mathbb{Z}/2$ grading on the space of test function $\mathcal{S}_n = \mathcal{S}_n^+ \oplus \mathcal{S}_n^-$.)*
5) $\mathcal{W}(f^* \cdot f) \geq 0$.

Proof. 1) and 3)–4) are clear. Let us prove 2).

We have to check that $W_n(f) = 0$ if for some i $\operatorname{supp}(\mathcal{F}(f))$ lies in the domain $\{(v_1, .., v_n) \in V_n | v_i \in V - \overline{V}_+\}$. For such f we can find $g \in \mathcal{S}_{n+1}$ with $\overline{g} = f$ satisfying: $\operatorname{supp}\mathcal{F}(g) \subset \{(v_1, \ldots, v_{n+1}) \in V_{n+1} | v_1 + \cdots + v_{i+1} \in V - \overline{V}_+\}$. So we have to show that for such g: $\mathcal{W}_n(g) = 0$. We can assume $g = g_1 \otimes g_2$ for $g_1 \in \mathcal{S}_{i+1}, g_2 \in \mathcal{S}_{n-i}$, and $\operatorname{supp}\mathcal{F}(g_1) \subset \{(v_1, \ldots, v_{i+1}) \in V_{i+1} | v_1 + \cdots + v_{i+1} \subset V - \overline{V}_+\}$. We have $\mathcal{W}_n(g) = \langle\phi(g_1)^*\Omega, \phi(g_2)\Omega\rangle$. But the spectral support of $\phi(g_1^*)\Omega$ is contained in the set $\{v_1 + \cdots + v_{i+1} | (v_1, \ldots, v_{i+1}) \in \operatorname{supp}\mathcal{F}(g_1)\}$. On the other hand, it is contained in $\overline{V}_+$ by the positivity condition. Hence, it is empty, and $\phi(g_1^*)\Omega = 0 \Rightarrow \mathcal{W}_n(g) = 0$.

Remark. By a standard theorem from analysis property 2) is equivalent to the following:
 2′) W_n is a boundary value of an analytic function on $V_{n-1} + i(V_+)^{n-1}$.
 It is not hard to see the following

Theorem. *The above constructions provide a bijection between the following sets of data:*

a) *ρ; $\mathcal{H} \supset \mathcal{D} \ni \Omega$; ϕ satisfying Wightman's axioms;*
b) *representation ρ of G and a collection of distributions $\{\mathcal{W}_n\}$ satisfying properties 1) – 5) above.*

Remark. Sometimes one considers theories satisfying weaker axioms. Namely, let $\mathcal{S}_n^0 \subset \mathcal{S}_n$ be the subspace of sections which vanish on diagonals together with all partial derivatives. A QFT satisfying weak Wightman's Axioms is (a representation ρ together with) a collection of $\mathcal{W}_n \in (\mathcal{S}_n^0)'$ satisfying 1), 2′), 3), 4) above.

Note that an essential property 5) can not be formulated if $\mathcal{W}_n$ are defined on $\mathcal{S}_n^0$ only because $f^* \cdot f$ does not lie in $\mathcal{S}^0$.

1.4 Spin-statistics Theorem. This is a statement that our convention on the parity of the basic superspace R is the only right one. More precisely, we have:

Theorem. *Consider the Wightman axioms with the sign in the axiom of space-locality changed. Any theory satisfying this modified set of axioms is trivial. (The field map $\phi = 0$.)*

Proof. Let us take a 2-dimensional subspace $U \subset V$ of signature (1,1). Let σ' be the element of $\mathrm{SO}(d-1, 1)$ which acts by -1 on U and by 1 on the orthogonal complement;

let σ be an element in the complexification of $\mathrm{Spin}(d-1,1)$ which projects to σ'. Then $\sigma^2 = \tau$ is the spin element. Also note that the eigenspaces of σ in the space of $R \otimes \mathbb{C}$ are defined over $\mathbb{R}$. If τ acts trivially in R then this is obvious because σ is real; otherwise τ acts by -1 and we use the identity $\bar{\sigma} = \tau\sigma$ to see that the $\pm i$ eigenspace of τ is invariant under the conjugation.

In the next lecture (§2.2) we will show that W_n extends to an analytic $G_{\mathbb{C}}$-equivariant function on a certain open domain in the complexification $\overline{V}_n \otimes \mathbb{C}$. In particular W_2 is analytic in the domain $\widetilde{\mathcal{T}} := \{v \mid v^2 \notin \mathbb{R}^{\geq 0}\}$. This analytic function is also denoted W_2.

Let now $r \in R$ be a σ eigenvector; let v be a point in $U \otimes \mathbb{C} \cap \widetilde{\mathcal{T}}$. We claim that $\langle r \otimes r, W_2(v)\rangle = 0$. Indeed we have $\langle r \otimes r, W_2(v)\rangle = \langle \sigma(r) \otimes \sigma(r), W_2(-v)\rangle = \mathrm{spin}(R) \cdot \langle r \otimes r, W_2(-v)\rangle$ where $\mathrm{spin}(R)$ is the τ eigenvalue on R; here the first equality follows from σ-equivariance, and the second is obvious from $\sigma^2 = \tau$.

On the other hand by our assumption $\langle r_1 \otimes r_2, W_2(v)\rangle = -\mathrm{spin}(R) \cdot \langle r_2 \otimes r_1, W_2(-v)\rangle$ for $v \in V^{space}$. Since $V^{space} \subset \widetilde{\mathcal{T}}$, we get $\langle r \otimes r, W_2(v)\rangle = 0$ for $v \in V^{space}$, and hence for all $v \in \widetilde{\mathcal{T}}$.

For any real valued Schwartz function f on V we have $\langle \phi(f \cdot r)\Omega, \phi(f \cdot r)\Omega\rangle = \int \langle f \cdot r \times f \cdot r, \mathcal{W}_2\rangle = \int \langle \mathcal{F}(f) \cdot r \times \mathcal{F}(f) \cdot r, \mathcal{F}(\mathcal{W}_2)\rangle = \int \mathcal{F}(f) \cdot \overline{\mathcal{F}(f)}\, \langle r \otimes r, \mathcal{F}(W_2)\rangle$. This shows that the distribution $\langle r \otimes r, \mathcal{F}(W_2)\rangle$ is nonnegative.

Now let us take $v \in V^+$. Then we get

$$0 = \langle r \otimes r, W_2(iv)\rangle = \int \exp(-v,p)\, \langle r \otimes r, \mathcal{F}(W_2)\rangle\,.$$

Since $\exp(-v,p)$ is positive this implies that $\langle r \otimes r, \mathcal{F}(W_2)\rangle = 0$, hence $(\phi(f \cdot r)\Omega)^2 = 0$, so the theory is trivial.

We finish by some elementary properties of Wightman functions, and their relation to properties of the field map.

1.5 Mass spectrum of a theory. Let us denote $\Delta_+(p, m^2) = \mathcal{F}(\delta_{\mathcal{O}_m^+})(p)$.

Claim. *For any scalar field theory* $\phi\colon \mathcal{S}(V) \to \mathrm{End}(\mathcal{D})$ *there exists a positive measure* $d\rho$ *on* $[0,\infty)$ *such that:*

$$W_2(p) = \int \Delta_+(p,m) d\rho(m)\ .$$

Proof. We have to show that $\mathcal{F}(W_2) = \int d\rho(m)\delta_{\mathcal{O}_m^+}$ for a positive measure $d\rho$. Since $\mathcal{O}_m^+$ is an orbit of the group G it is obvious that any G-invariant distribution supported in $\sqcup \mathcal{O}_m^+ = \overline{V_+}$ can be written down in such form for some distribution ρ. In order to see that ρ is positive we have to show that $\mathcal{F}(W_2)$ is positive.

For any $f \in \mathcal{S}$ we have $\mathcal{W}_2(f \boxtimes f) = \langle \phi(f)\Omega, \phi(f)\Omega\rangle \geq 0$. Hence for $g = \mathcal{F}(f)$ we have $\mathcal{F}(\mathcal{W}_2)\,(g \boxtimes g) \geq 0$.

We know that $\mathcal{F}(\mathcal{W}_2)$ is concentrated on the anti-diagonal and

$$\mathcal{F}(\mathcal{W}_2)(p,-p) = \mathcal{F}(W_2)(p)$$

Now note that $f \in \mathcal{S}$ iff $g = \mathcal{F}(f) \in \mathcal{S}^{\mathbb{C}}$ and $g(-p) = \overline{g(p)}$. Thus for any such g

$$\mathcal{F}(W_2)(g \cdot \overline{g}) = \mathcal{F}(\mathcal{W}_2)(g \boxtimes g) \geq 0$$

which means that $\mathcal{F}(W_2)$ is positive.

Definition. The *mass spectrum* of a QFT is support of the measure $d\rho$. The *mass* of the theory is the infinum of its mass spectrum with 0 removed.

1.6 Asymptotics of Wightman functions. The next statement is a baby example of relation between analytical behavior of Wightman functions and properties of the field map. Estimates of this kind will be used essentially in the lecture on scattering theory (see lecture 4, §4.3, Proposition 1 and Claim 1).

Claim.

a) Uniqueness of vacuum in a QFT *is equivalent to the equality:*

$$\lim_{\lambda \to \infty} \mathcal{W}_n(v_1, \ldots, v_i, v_{i+1} + \lambda a, \ldots, v_n + \lambda a) = \mathcal{W}_i(v_1, \ldots, v_i)\mathcal{W}_{n-i}(v_{i+1}, \ldots, v_n)$$

being true for all i, n and a such that $a^2 < 0$.

b) Assume that mass of the theory is $m_0 > 0$. Then $W_2(v) \sim e^{-m_0\sqrt{-v^2}}$ as $v^2 \to -\infty$.

Proof. a) For any $f_1, \ldots, f_n \in \mathcal{S}$ we have

$$\begin{aligned}\mathcal{W}_n\left(f_1, \ldots, f_i, T_{\lambda a}(f_{i+1}), \ldots, T_{\lambda a}(f_n)\right) &= \langle \phi(f_i)\ldots\phi(f_1)\Omega, \phi(T_{\lambda a}(f_{i+1}))\ldots\phi(T_{\lambda a}(f_n))\Omega\rangle \\ &= \langle \phi(f_i)\ldots\phi(f_1)\Omega\,,\, U(T_{\lambda a})(\phi(f_{i+1})\ldots\phi(f_n)\Omega)\rangle\end{aligned}$$

where $T_{\lambda a}$ stands for translation by λa. From the positivity condition it follows that the weak limit $\lim\limits_{\lambda \to \infty} U(T_{\lambda a})$ exists as a bilinear form (i.e. all matrix coefficients converge), and equals to the projection on $\mathcal{H}^V$.

Hence

$$\lim_{\lambda \to \infty} \langle \phi(f_i)\ldots\phi(f_1)\Omega, U(T_{\lambda a})\left(\phi(f_{i+1})\ldots\phi(f_n)\Omega\right)\rangle = \langle \mathrm{pr}(\phi(f_i)\ldots\phi(f_1)\Omega), \mathrm{pr}(\phi(f_{i+1})\ldots\phi(f_n)\Omega)\rangle$$

where pr is the projection on $\mathcal{H}^V$. The statement follows.

b) We have

$$W_2(v) = \int\limits_V \mathcal{F}(W_2)(p)e^{i(v,p)}\,dp = \int\limits_0^\infty d\rho(m) \int\limits_{\mathcal{O}_m^+} e^{(iv,p)}\,d\mu_m(p)$$

where μ_m is the invariant measure on $\mathcal{O}_m^+$. For $v \in \mathbb{R}^{d-1} - \{0\}$ it can be rewritten as:

$$\text{const} + \int\limits_{m_0}^\infty d\rho(m) \int\limits_{\mathbb{R}^{d-1}} e^{i(v,q)}\,dq/\sqrt{m^2 - q^2}$$

(Remember that $q^2 = (q, q)$ is negative for $q \in \mathbb{R}^{d-1} - \{0\}$).

The integrand is analytic in the region $Re(m_0^2 - q^2) > 0$; it also decreases as $(\vec{p})^2 \to -\infty$ for fixed p_0, where $p = (\vec{p}, p_0)$ is decomposition into space and time components. Hence we can shift the contour of integration from $\mathbb{R}^{d-1}$ to $\Gamma \subset \mathbb{C}^{d-1}$ where Γ is $\mathbb{R}^{d-1} + im_0 v/\sqrt{-v^2}$ squeezed slightly towards $\mathbb{R}^{d-1}$ in a small neighborhood of the singular point $im_0 v/\sqrt{-v^2}$ to avoid the singularity. Then the estimate becomes obvious.

We will usually assume that uniqueness of vacuum holds for the theory under consideration.