Lecture 2
Eucledian formulation of Wightman QFT

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2.1 Analytic continuation of Wightman functions. In lecture 1 we were working with Minkowski space-time V. However it is often desirable to be able to work with Euclidean (Riemannian rather than pseudo-Riemannian) space-time as well. In this lecture we explain what data over the Euclidean space-time $E = \mathbb{R}^d$ corresponds to a QFT in Wightman axioms.

More precisely, let $V_{\mathbb{C}} = V \otimes \mathbb{C}$, $V_n^{\mathbb{C}} = V_n \otimes \mathbb{C}$, and consider the Euclidean affine space $E = i\mathbb{R} \times \mathbb{R}^{d-1} \subset V_{\mathbb{C}}$.

Going between QFT data on Minkowski space-time $V = \mathbb{R} \times (\mathbb{R})^{d-1}$ and the one on $E = i\mathbb{R} \times (\mathbb{R})^{d-1}$ is often called *Wick rotation.*

The plan is to obtain Euclidean counterparts of Wightman functions by analytically continuing $\mathcal{W}_n$ to an open domain in $V_n^{\mathbb{C}}$ and then restricting the obtained analytic functions to E.

We need some notations. Let $G_{\mathbb{C}}$ be complexification of G. We set: $\mathfrak{T} = V - iV_+ \subset V_{\mathbb{C}}$ (the *backward tube*); $\mathfrak{T}_n = (\mathfrak{T})^n \subset V_n^{\mathbb{C}}$ (the *extended tube*); $\widetilde{\mathfrak{T}}_n = G_{\mathbb{C}}(\widetilde{\mathfrak{T}}_n) \subset V_n^{\mathbb{C}}$ (the *permuted extended tube*) ; $J_n = \widetilde{\mathfrak{T}}_n \cap V_n$. Recall that we identified $V_n = \overline{V}_{n+1}$, so we have an action of the symmetric group Σ_{n+1} on V_n.

Proposition.

a) *Assume $d > 2$. Then for any $\sigma \in \Sigma_{n+1}$ the subset $J_n \cap \sigma(J_n) \subset V_n$ is open and nonempty.*

b) *If $d > 2$ then for any $\sigma \in \Sigma_{n+1}$ the subset $\widetilde{\mathfrak{T}}_n \cap \sigma(\widetilde{\mathfrak{T}}_n)$ is connected.*

c) *Put $E = i\mathbb{R} \times \mathbb{R}^{d-1} \subset V_{\mathbb{C}}$; $E_n = E \times \cdots \times E \subset V_n^{\mathbb{C}}$. Also let $E_n^0 \subset E_n$ be the complement to all diagonals, and: $\overline{E}_n = E_n/\Delta(E)$; $\overline{E}_n^0 = E_n^0/\Delta(E)$. Then $\bigcup_{\sigma \in \Sigma_{n+1}} \sigma(\widetilde{\mathfrak{T}}_n) \supset \overline{E_{n+1}^0}$.*

Proof. We start with the following result.

Lemma. *(Jost's theorem)* $J_n = \{(v_1, \dots, v_n) \in V_n | \forall (\lambda_1, \dots, \lambda_n) \in \mathbb{R}_+^n - \{0\}$: $(\lambda_1 v_1 + \dots + \lambda_n v_n) \in V_{space}\}$.

Proof of the Lemma. Assume that $(v_1, \dots, v_n) \in J_n$. Then for some $g \in G_{\mathbb{C}}$: $g(v_1), g(v_2), \dots, g(v_n) \in \mathfrak{T}$, hence $g(\lambda_1 v_1 + \dots + \lambda_n v_n) \in \mathfrak{T}$ for $(\vec{\lambda}) \in \mathbb{R}_+^n - \{0\}$. Let us write $\sum \lambda_k v_k = \xi - i\eta$, where $\eta \in V_+$. Then $(\sum \lambda_k v_k)^2 = (\xi)^2 - (\eta)^2 - 2i(\xi, \eta)$. The L.H.S. is real, hence $(\xi, \eta) = 0$, and $(\xi)^2 \leq 0$, i.e. $(\sum \lambda_k v_k)^2 < 0$.

Let us prove the converse statement. So assume that the (convex) cone generated by $v_1, \dots, v_n$ does not intersect $\overline{V}_+ \cup \overline{V}_-$. Then this cone can be separated from $V_\pm$ by a hyperplane tangent to $\overline{V}_\pm$. This means that there exist $\alpha \in \overline{V}_+$ and $\beta \in \overline{V}_-$ such that: $(\alpha)^2 = (\beta)^2 = 0$, $(\alpha, v_k) < 0$, $(\beta, v_k) < 0$ for $k = 1, \dots, n$ and $(\alpha, w) > 0$ for $w \in V_+$; $(\beta, w) > 0$ for $w \in V_-$. Then $(\alpha, \beta) \neq 0$, and it is easy to show that there exists $g \in G_{\mathbb{C}}$ such that $g(\alpha) = i\alpha$, $g(\beta) = -i\beta$ and g is identity on the subspace orthogonal to $\langle \alpha, \beta \rangle$. This g satisfies the condition $g(v_k) \in \mathfrak{T}$ for $k = 1, \dots, n$.

Proof of the Proposition. a) It is obvious that this subset is open. So we must show it is nonempty, i.e. we are to find a point $\underline{v} = (v_1, \dots, v_n)$ such that $(v_1, \dots, v_n)$ and $(v_{\sigma(1)}, \dots, v_{\sigma(n)})$ satisfy the conditions of the last lemma. We can choose two linearly independent vectors $\ell_1, \ell_2 \in \mathbb{R}^{d-1} \subset V$ (since $d > 2$) and put $\underline{v} = (\ell_1, \dots, \ell_1) + \sigma^{-1}(\ell_2, \dots, \ell_2)$. Then it is immediate to check the conditions.

b) It is enough to show that $\sigma(\widetilde{\mathfrak{T}}_n) \cap \mathfrak{T}_n$ is connected for any σ.

We have a standard

Lemma. *Generic element of $G_{\mathbb{C}}$ can be written as $g = g' \cdot g_{\mathbb{R}}$, where $g_{\mathbb{R}} \in G$ and g' preserves an orthogonal decomposition of $V_{\mathbb{C}}$ into sum of subspace of dimension not greater than 2 defined over $\mathbb{R}$. Here by generic we mean an element belonging to an open (in fact real Zariski open) dense subset.*

Let us now finish the proof of b). Fix $\sigma \in \Sigma_{n+1}$ and consider $S = \{(g \in G_{\mathbb{C}}, \underline{v} \in V_n^{\mathbb{C}}) \mid g(\sigma(\underline{v})) \in \mathfrak{T}_n\}$. Take a point $(g, \underline{v}) \in S$.

Moving $(g, \underline{v})$ slightly inside S we can achieve that g is generic as in the last lemma. We can now take g', $g_{\mathbb{R}}$ as in the Lemma. Obviously $(g', g_{\mathbb{R}}^{-1}(\underline{v}))$ and $(g, \underline{v})$ lie in one connected component of S. Set $\underline{v}' = g_{\mathbb{R}}^{-1}(\underline{v})$.

Let $V_0 \subset V$ be the unique block in the decomposition corresponding to g' on which $(\ ,\ )$ is not negative, and let $\pi : V_n^{\mathbb{C}} \to (V_0^{\mathbb{C}})^n$ be the orthogonal projection. One sees immediately that for $\underline{v} \in \mathfrak{T}_n$ the segment connecting $\underline{v}$ with $\pi(\underline{v})$ lies in $\mathfrak{T}_n$. Also π commutes with g' and σ; it follows that this segment also lies in $\sigma^{-1}(g'^{-1}(\mathfrak{T}_n))$, hence $(g', \underline{v}')$ and $(g', \pi(\underline{v}'))$ are connected inside S.

If $\dim(V_0) = 1$ then $g'|_{V_0} = \pm Id$. So in that case we have $\pi(\underline{v}') \in \mathfrak{T}_n$, $\pm\sigma(\pi(\underline{v}')) \in \mathfrak{T}_n$, which is only possible if $\sigma = 1$ or $\sigma = w_0$ is the "long" permutation $(w_0(i) = n + 2 - i)$. But in these two cases $\sigma(\widetilde{\mathfrak{T}}_n) \supset \mathfrak{T}_n$, so there is nothing to prove anyway.

Suppose now that $\dim(V_0) = 2$. Set $G^0 = \mathrm{Spin}(V_0)$, $G^0_{\mathbb{C}} = \mathrm{Spin}(V_0 \otimes \mathbb{C})$, $\mathfrak{T}^0_n = V_0 + iV_0^+$. Then $G^0_{\mathbb{C}}/G^0$ is the circle S^1, and one can show that for $\sigma \neq 1$, w_0 the set $\{g \in G^0_{\mathbb{C}}/G^0 \mid g(\sigma(\mathfrak{T}^0_n)) \cap \mathfrak{T}^0_n \neq \emptyset\}$ coincides with $S^1 - \{1, -1\}$. For given g the set $g(\sigma(\mathfrak{T}^0_n)) \cap \mathfrak{T}^0_n$ is convex and hence connected; thus the set $S_{V_0} := \{(g \in G^0_{\mathbb{C}}, \underline{v} \in \mathfrak{T}^0_n \mid g(\sigma(\underline{v})) \in \mathfrak{T}_n\}$ has 2 connected components. It is easily seen that these 2 components are permuted by conjugation by an element of G, hence they lie in the same connected component of S. But we have shown that any point in S is connected to a point in $(Stab_{V_0} \times V_0) \cap S$, and hence to a point in $S_{V_0} \subset S$. (Here we use that the action of g' on V_0 has determinant 1, because otherwise it has eigenvalues $1, -1$, which does not happen for generic g.)

Since G transitively permutes 2-dimensional subspaces of V of signature $(1, 1)$ we see that S is connected, which implies the statement.

c) For $v \in V_{\mathbb{C}}$ let us write $v = (v^0, \vec{v})$, $v_0 \in \mathbb{C}$, $\vec{v} \in \mathbb{C}^{d-1}$. It is clear that for $(v_1, \ldots, v_{n+1}) \in E^0_{n+1}$ there exists $g \in \mathrm{SO}(n+1)$, such that $(gv_i)^0 \neq (gv_j)^0$ for $i \neq j$. Then we can permute $(v_1, \ldots, v_{n+1})$ so as to get: $(v_i)^0 > (v_j)^0$ for $i > j$. But this implies that the corresponding vector in $\overline{V}_{n+1}$ lies in $\widetilde{\mathfrak{T}}_n$.

Now we are ready to prove

Theorem. *Let $d > 2$. Then W_n extends to an analytic function on $\bigcup_{\sigma \in \Sigma_{n+1}} \sigma(\widetilde{\mathfrak{T}}_n)$. These* *functions are Σ_{n+1} and $G_{\mathbb{C}}$-equivariant. (The Σ_{n+1}-equivariance is understood as usually in the "super" sense.)*

Proof. Since $\mathfrak{F}(W_n)$ is supported in V_+ it follows that W_n extends to an analytic function on $\mathfrak{T}$ (which will be denoted by the same letter). Since W_n is G-equivariant, this analytic function is G-equivariant.

To ensure that W_n extends to a single-valued analytic function on $\widetilde{\mathfrak{T}}$, which then is automatically $G_{\mathbb{C}}$-equivariant, it is enough to check that $W_n(x) = W_n(g(x))$ for $g \in G_{\mathbb{C}}$, $x \in \mathfrak{T}_n$ and $g(x) \in \mathfrak{T}_n$.

If $g^{-1}\mathfrak{T}_n \cap \mathfrak{T}_n \neq \emptyset$ then also $g^{-1}\mathfrak{T} \cap \mathfrak{T} \neq \emptyset$. Let us take $v \in g^{-1}\mathfrak{T} \cap \mathfrak{T}$. It is easy to see that for $v \in \mathfrak{T}$ the set $\{h \in G_{\mathbb{C}} | h(v) \in \mathfrak{T}\}$ is connected. Hence there exists a neighborhood $U \subset \mathfrak{T}_n \cap g^{-1}\mathfrak{T}_n$ of the point $(v, \ldots, v)$ such that g^{-1} lies in the connected component of 1 of $\{h \in G_{\mathbb{C}} | h(U) \subset \mathfrak{T}\}$. From G-invariance of W_n it follows that for fixed $x \in V_n^{\mathbb{C}}$ the function $W_n(g(x))$ defined on some open subset of $G_{\mathbb{C}}$ is locally constant. Hence $W_n(x) = W_n(g(x))$ for $x \in U$. Since U is open in the connected set $g^{-1}\mathfrak{T}_n \cap \mathfrak{T}_n$ we can conclude that $W_n(x) = W_n(g(x))$ for all $x \in g^{-1}\mathfrak{T}_n \cap \mathfrak{T}_n$, so analyticity of W_n in $\widetilde{\mathfrak{T}}$ is proved.

Now by Proposition a) for any $\sigma \in \Sigma_{n+1}$ the set $\sigma^{-1}(\widetilde{\mathfrak{T}}) \cap \widetilde{\mathfrak{T}}$ contains an open non-empty subset of V_n. Moreover, from the Jost's theorem it follows that

$$\sigma^{-1}(\widetilde{\mathfrak{T}}) \cap \widetilde{\mathfrak{T}} \cap V_n \subset (V_{space})^n.$$

But $W_n\big|_{(V_{space})^n}$ is Σ_{n+1} (super)equivariant. Hence W_n and $\sigma^*(W_n)$ agree on a neighborhood of the set of real points in $\sigma^{-1}(\widetilde{\mathfrak{T}}) \cap \widetilde{\mathfrak{T}}$, and hence on the whole of $\sigma^{-1}(\widetilde{\mathfrak{T}}) \cap \widetilde{\mathfrak{T}}$ by Proposition b). So the collection of functions $\sigma^*(W_n)$ for all $\sigma \in \Sigma_n$ give a well-defined analytic function on $\bigcup_{\sigma \in \Sigma_{n+1}} \sigma(\widetilde{\mathfrak{T}}_n)$, which is obviously Σ_{n+1}-(super)equivariant. The theorem is proven.

We denote the restriction of these functions to $\overline{E_n^0}$ by $\overline{\mathfrak{S}}_n$. We also put $\mathfrak{S}_n = \pi^*(\overline{\mathfrak{S}}_n)$ where $\pi : E_n^0 \to \overline{E_n^0}$ is the projection.

Definition. The functions $\mathfrak{S}_n$ are called *Schwinger functions.*

From the theorem it follows that $\mathfrak{S}_n$ is a real-analytic $\mathrm{SO}(n) \ltimes E$ and Σ_n equivariant function.

Remark. The theorem (and its proof) relies only on weak Wightman's axioms. Hence Schwinger functions can be constructed from a set of $\mathcal{W}_n \in \widetilde{\mathcal{S}}'_n$ which satisfy weak Wightman axioms.

Remark. The assumption that $d > 2$ is not essential in the Theorem. A method which allows to treat also the case $d = 2$ and furhter extend the analyticity domain (the so called "edge of the wedge Theorem") is explained e.g. in [Streater-Wightman], §3.3.

Remark. It was explained that QFT has two formulations: Minkowski and Euclidean. The experience says that it is more convenient to work technically with Euclidean picture, though the basic notions are naturally formulated in Minkowski one.

Remark. The analytic Schwinger functions on $\Sigma_{n+1}(\widetilde{\mathfrak{T}}_n)$ seem to be the correct object of consideration. So it is natural to ask whether they can be further extended to some larger domain.

The domain $\Sigma_{n+1}(\widetilde{\mathfrak{T}}_n)$ is not holomorphically convex.

It may be worthwhile to describe its convex hull and look at the extension of Schwinger functions to it.

2.2 Euclidean formulation of Wightman QFT. Recall that decomposition $E = (i\mathbb{R}) \times \mathbb{R}^{d-1}$ is fixed. Let θ denote reflection at the hyperplane $\{0\} \times \mathbb{R}^{d-1}$; put: $E^+ = (i\mathbb{R}_{>0}) \times \mathbb{R}^{d-1}$; $E^+_n = E^+ \times E^+ \times \cdots \times E^+ \subset E_n$. Let $\mathcal{E}_n$ denote the space of Schwartz sections of $\mathcal{R}^{\mathbb{C}}_n|_{E_n}$, and $\mathcal{E}^0_n \subset \mathcal{E}_n$ be the subspace of sections with support in E^0_n.

Claim.

a) *Schwinger functions satisfy the following properties:*

" *i)*" $\mathfrak{S}_n$ *is a real-analytic* $\widehat{P}$*-invariant and* Σ_n *(super) invariant real valued function on* E^0_n *where* $\widehat{P} = \widehat{G} \ltimes E$, $\widehat{G} = \mathrm{Spin}(E)$;

If moreover $\mathfrak{S}_n$ *come from a strong (as opposed to weak) QFT then they satisfy:*

" *ii)*" $\mathfrak{S}_n$ *is of moderate growth; i.e.* $\mathfrak{S}_n$ *defines a tempered distribution on* $\mathcal{E}^0_n$*.*

" *iii)*" *Reflection positivity.*

Put: $\mathcal{E}^+_n = \{F \in \mathcal{E}^0_n | \mathrm{supp}\, F \subset E^+_n\}$*. Also let* $\mathcal{E}^+ = \oplus \mathcal{E}^+_n$*,* $\widetilde{\mathcal{E}} = \oplus \mathcal{E}^0_n$*, and* $\widetilde{\mathfrak{S}} = \Pi \mathfrak{S}_n \in (\widetilde{\mathcal{E}})'$*. Then for* $F \in \mathcal{E}^+$ *we have:* $\widetilde{\mathfrak{S}}(\theta(F) \boxtimes F) \geq 0$*.*

" *iv)*" *Let* $\overline{\mathcal{E}}_n$ *be the space of Schwartz sections of* $\overline{\mathcal{R}^{\mathbb{C}}_n}|_{\overline{E}_n}$*, and let* $\overline{\mathcal{E}^+_n} \subset \overline{\mathcal{E}}_n$ *be the subspace of sections with support in* $E^+_{n-1} \subset \overline{E}_n \cong E_{n-1}$*. For* $r \in \mathbb{R}^+$ *and a differential operator* D *consider the seminorm on* $\overline{\mathcal{E}^+_n}$ *defined by:*

$$|f|_{r,D} = \sup_{x \in (\mathbb{R}_{>0} \times \mathbb{R}^{d-1})^n} |(1+|x|^2)^{r/2} D\mathcal{F}(f)(x)|$$

(here for $x \in V$*,* $x = (x_0, \vec{x})$ *we put* $|x|^2 = x_0^2 + |\vec{x}^2|$*). Then* $\overline{\mathfrak{S}}_n$ *defines a distribution on* $\overline{\mathcal{E}^+_n}$ *which is continuous with respect to the seminorms* $| \quad |_{r,D}$*.*

b) *The construction of §2.1 provides a bijection between Wightman QFT's and collections of functions* $\mathfrak{S}_n$ *satisfying the conditions i)–iv) above.*

Proof. See [Osterwalder-Schrader(1975)].

It should be mentioned that this theorem is not really useful, because the property iv) of the list is impossible to check in practice.

A more useful list is obtained by replacing the property iv) by the following axiom:

Axiom iv$'$). (Linear growth condition) There exist $\alpha, \beta, r \in \mathbb{R}_{>0}$ such that

$$|\mathfrak{S}_n(f)| < \alpha (n!)^\beta |f|_{nr}$$

for all $f \in \mathcal{E}^0_n$ where the seminorm $|f|_{nr}$ is given by $|f|_{nr} = \sup_{x \in E^0_n, D} |(1-x^2)^{r/2} Df(x)|$ and D runs over the set of differential monomials of degree less then nr.

This property is more tractable, but it is more restrictive than iv).

Some remarks are in order. For Schwinger functions constructed from a (strong) Wightman QFT the reflection positivity property is equivalent to the positivity property (5) of Wightman axioms. However, to formulate it we do not have to assume anything about the behavior of $\mathfrak{S}_n$ on diagonals. (Note that if $F_1, F_2 \in \mathcal{E}^+$ then $\theta(F_1) \boxtimes F_2$ automatically vanishes on diagonals, so $(F_1)\boxtimes F_2\in\widetilde{\mathcal{E}})$. So it can be formulated for Schwinger functions of weak QFT. It is not clear how to restate this property directly in Minkowski picture (which is more intuitive from the physical point of view) for weak Wightman QFT.

2.3 Schwinger functions and measures on the map-spaces. In fact, the following picture should stand behind the formalism of Schwinger functions. Assume we have a (quasi) measure ν on $\mathcal{E}'$ (where $\mathcal{E}$ is the space of Schwartz sections of $\mathcal{R}^{\mathbb{C}}\big|_E$), such that if $F_1,\dots,F_n\in\mathcal{E}$ and $\mathrm{supp}(F_i)\cap\mathrm{supp}(F_j)=\emptyset$ for $i\neq j$ then the function $\alpha\to\alpha(F_1)\dots\alpha(F_n)$ is measurable. Then we can define a distribution in $(\mathcal{E}^0_n)'$ for any n by requirement that its value on $F_1\boxtimes\dots\boxtimes F_n$ equals $\int\limits_{\mathcal{E}'}\alpha(F_1)\dots\alpha(F_n)$. This distribution should coincide with $\mathfrak{S}_n$; thus $\mathfrak{S}_n$ describes n-th moment of ν. If the measure ν exists, it is uniquely determined by its moments $\mathfrak{S}_n$. However it does not necessarily exist for given $\mathfrak{S}_n$ coming from a Wightman QFT.

So working with $\mathfrak{S}_n$ is a way to talk about the properties of a measure when the measure itself is not defined.

The formulation of QFT involving a measure on $\mathcal{E}'$ has another serious drawback. In general one is interested in field theories where the fields are not sections of a vector bundle, but rather maps from a manifold to another manifold (no linear structure on the target space). Then we can not even say on what space the measure ν is defined. Indeed, in the linear case the measure lives on the space of generalized sections of a vector bundle. But there is no notion of a generalized map from a manifold to another manifold. The formalism of Schwinger functions also does not work in this setting: to define moments of a measure one needs linear structure on the underlying space.

The difficulty can be overcome if $d=1$. In this case for a linear QFT the measure is actually defined and is concentrated on the subspace of continuous sections. So one gets a reasonable theory considering the space of continuous maps from $V=\mathbb{R}$ to M and introducing a measure on this space. This is illustrated in the next two examples.

Example 1. (Wightman QFT for $d=1$) In this case QFT is equivalent to usual quantum mechanics. Precisely, let us put: $\mathcal{H}=L^2(\mathbb{R})$, let $A\in\mathbb{C}[u,\partial_u]$ be a Hamiltonian. Let A be of the form $A = -\frac{d^2}{du^2} + r(u)$, where $r(u) \to +\infty$ for $u \to \pm\infty$. Then A has the smallest eigenvalue, and the corresponding eigenspace is one dimensional. Assume that the smallest eigenvalue of A is 0. Let Ω be the corresponding eigenfunction. Also put: $\mathcal{D} = \mathcal{S}(R)$, then $\Omega \in \mathcal{D} \subset \mathcal{H}$. Finally, let $\widehat{u} \in \mathrm{End}(\mathcal{D})$ be the operator of multiplication by the function $f(u) = u$. We define the action of $P = \mathbb{R}$ on $\mathcal{H}$ by $t \to e^{iAt}$ and put:

$$\phi(f) = \int f(t) e^{iAt} \widehat{u} e^{-iAt} dt.$$

Let us describe the corresponding measure on $\mathcal{S}'(\mathbb{R})$. We start with Gaussian measure ν_0 with covariance $(-(\frac{d}{dt})^2 + m^2)^{-1}$, where $m^2 > 0$ is such that $r(u) - m^2 u^2 > 0$ for $|u| \gg 0$. (So for $f \in \mathcal{S}$ we have $\mathcal{F}(\nu_0)(f) = e^{((-(d/dt)^2 + m^2)^{-1} f, f)}$.) We look for ν of the form $\nu = a\nu_0$ where a is a function. It is enough to define a on a set of measure 1. We define a by the formula:

$$a(\alpha) = e^{-\int\limits_{-\infty}^{\infty} (r(\alpha(t)) - m^2 \alpha(t)^2) dt}$$

Using Sobolev theorem one can show that ν_0 is essentially concentrated on the set of such continuous functions α that the above integral converges. Hence the measure $\nu = a\nu_0$ is correctly defined.

The fact that the measure ν and the field operators defined above give the same set of Schwinger functions follows from the so-called Feynman-Kac formula, see Gawedzki's lecture 1, §1.3.

Example 2. (Abstract QFT for $d = 1$) Let M be an arbitrary compact Riemannian manifold, $\mathcal{H} = L^2(M)$, Δ is the Laplacian, $A = -\Delta + r(m)$. In this case one still can describe a measure ν on the space of continuous maps $\mathrm{Map}(\mathbb{R}, M)$ (one can get rid of the compactness assumption by introducing requirements on asymptotic behavior of r; Example 1 is a particular case of this situation).

The measure can be characterized as follows. Let $t_1, \dots, t_k$ be points in $\mathbb{R}$, and $U_1, \dots, U_k \subset M$ be open subsets. Put $X^{t_1,\dots,t_k}_{U_1,\dots,U_k} = \{f \in \mathrm{Hom}(\mathbb{R}, M) | f(t_i) \in U_i \text{ for } i = 1, \dots, k\}$. Let $K_t(m_1, m_2)$, $m_1, m_2 \in M$, $t \in \mathbb{R}_{>0}$ be the fundamental solution of the differential equation

$\frac{df}{dt} = Af$. Then ν is the unique measure, such that:

$$\nu(X_{U_1,\ldots,U_k}^{t_1,\ldots,t_k}) = \int\limits_{U_1\times\ldots\times U_k} \prod_{i=1}^{k-1} K_{t_{i+1}-t_i}(m_i, m_{i+1}) dm_1 \ldots dm_k.$$

But there is no easy way to translate it to a Wightman-type picture.

We finish by a technical variation on the definition of Wightman functions which will be used later.

2.4 Time-ordering. Let $\theta_0 \in \mathrm{SO}(1, d-1)$ be reflection at the hyperplane $\mathbb{R}^{d-1} \subset V$ (change of time direction). Then θ_0 can be lifted to an element $\theta \in G_{\mathbb{C}}$, but not to an element of G. Hence the analytic continuations of $\mathcal{W}_n$ we constructed in section 1 are θ-invariant, though Wightman functions themselves are not.

We will now present a variant of the definition of $\mathcal{W}_n$ which does not have this drawback.

Namely, the time-ordered Wightman function $\mathcal{W}_n^T$ is given by: $\mathcal{W}_n^T(v) = \lim\limits_{\varepsilon\to+0} \mathfrak{S}_n(v + i\varepsilon v)$. For $pr(v) \in J_{n-1}$ we have: $\mathcal{W}_n^T(v) = \mathcal{W}_n(v)$, where pr is the projection $V_n \to \overline{V}_n$. The function $\mathcal{W}_n^T$ is θ and Σ_n-invariant (unlike W_n).

Like a usual Wightman function, $\mathcal{W}_n^T$ is lifted from a unique function W_n^T on $\overline{V}_n$.

Let us describe W_2^T for the free scalar QFT of mass m; we will denote it by $\Delta_F(\ ,m^2)$, where F stands for Feynman. It can be expressed through functions $\Delta_+(\ ,m^2) = \mathcal{F}(\delta_{\sigma_m^+})$, and $\Delta_-(\ ,m^2) = \mathcal{F}(\delta_{\sigma_m^-})$.

Claim. *We have:*

a) $\Delta_F(v,m^2) = \Delta_\pm(v,m^2)$ *for* $v \in \overline{V}_\pm$*;* $\Delta_F(v,m^2) = \Delta_+(v,m^2) = \Delta_-(v,m^2)$ *for* $v \in V$*;*

b) $(-\Box + m^2)(\Delta_F(\ ,m^2)) = 2\pi i\,\delta_0$*, i.e.* W_2^T *is the fundamental solution of* $(-\Box + m^2)$*;*

c) $\mathcal{F}(\Delta_F(\ ,m^2)) = \frac{1}{2\pi i(p^2+m^2+i0)} \stackrel{\text{def}}{=} \lim\limits_{\varepsilon\to+0} \frac{1}{2\pi i(p^2+m^2+i\varepsilon)}$ *(this is a well-defined distribution).*

[Just to remind how such things look, we recall that $\frac{1}{x+i0}$ and $\frac{1}{x-i0}$ are well-defined distributions on $\mathbb{R}$ and we have: $\frac{1}{2\pi i}\left(\frac{1}{x-i0} - \frac{1}{x+i0}\right) = \delta_0$ and $\frac{1}{2}\left(\frac{1}{x+i0} + \frac{1}{x-i0}\right) = \left(\frac{1}{x}\right)_{\text{p.v.}}$, where $\left\langle\left(\frac{1}{x}\right)_{\text{p.v.}}, f\right\rangle \stackrel{\text{def}}{=} \lim\limits_{\varepsilon\to+0} \int\limits_{-\infty}^{-\varepsilon} \frac{1}{x} f(x)dx + \int\limits_{\varepsilon}^{\infty} \frac{1}{x} f(x)dx$.]