Lecture 3

Free QFT

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3.1 Some examples of free classical field theories. As was explained in Bernstein's lectures, for any classical field theory the space Sol of stationary points of Lagrangian carries a canonical symplectic form, or at least a closed 2-form.

The theory is called free if Sol is isomorphic to a linear space with a constant 2-form and linear action of the Poincaré group. Let us consider some examples.

Example 1. A scalar boson field in dimension $(1, d-1)$.
A classical field is a real-valued function on $\mathbb{R}^{1,d-1}$. The Langrangian is given by:

$$\mathcal{L}(\varphi) = |d\varphi|^2 - m^2\varphi^2.$$

The space of stationary points is: $\mathrm{Sol} = \{\varphi | \Box\varphi + m^2\varphi = 0\}$. It is linear, the symplectic form is constant. It is given by: $[\varphi_1, \varphi_2] = \int\limits_\Gamma *(\varphi_1 d\varphi_2 - \varphi_2 d\varphi_1)$ for $\varphi_1, \varphi_2 \in \mathrm{Sol}$, where Γ is a space-type hypersurface in V. By a space-like hypersurface we mean a complete smooth oriented hypersurface such that the normal vector at each point lies in V_+ (for example the hyperplane $\{t = \mathrm{const}\}$ is space-like). Since $*(\varphi_1 d\varphi_2 - \varphi_2 d\varphi_1)$ is closed and sufficiently rapidly decaying the integral does not depend on the choice of a hypersurface. (The condition of Schwartz decay in space directions is tacitly imposed on the classical solutions under consideration throughout the lecture).

Example 2. Spinors. Let V be 10-dimensional, or of any dimension $n = 8k+2$. Then the spinor representation S decomposes as $S = S^+ \oplus S^-$, and S^+, S^- are real. There exists a unique symmetric pairing π: $S^+ \times S^+ \to V$ (see §3.16 in Deligne's "Notes on spinors" on this site). We also have a pairing $(\quad,\quad)$: $S^+ \times S^- \to \mathbb{R}$ and the action map $s : S^\pm \otimes V \to S^\mp$. The space of classical fields is defined to be the space of maps: $\mathrm{Map}(V, \mathbb{R}^{0,1} \otimes S^+) = \Pi\overline{\mathrm{Sol}}$ where $\overline{\mathrm{Sol}} = \mathrm{Map}(V, S^+)$. We define Lagrangian by: $\mathcal{L}(\psi) = (\partial\psi, \psi)$. Here $d\psi \in \Omega^1(V, \mathbb{R}^{0,1} \otimes S^+) = \Gamma(V, \mathbb{R}^{0,1} \otimes S^+ \otimes V)$, and $\partial\psi \stackrel{\mathrm{def}}{=} s d\psi \in \Gamma(V, \mathbb{R}^{0,1} \otimes S^-)$. We have:

$\mathrm{Sol} = \{\psi|\partial\psi = 0\}$, it is a purely odd vector space. Giving a symplectic form on Sol is equivalent to giving a symmetric form on $\overline{\mathrm{Sol}}$. The symmetric form is constant and is given by: $(\psi_1, \psi_2) = \int\limits_\Gamma \pi(\psi_1, \psi_2)$ where Γ is a space-type hypersurface, and we use the isomorphism $V \simeq \Lambda^{d-1}(V^*)$ provided by the volume form on V to interpret $\pi(\psi_1, \psi_2)$ as a $(d-1)$-form on V.

Example 3. Free electro-dynamics in any dimension.

Let M be the space of 1-forms on V modulo the exact forms. The Lagrangian $\mathcal{L}$: $M \to \Omega^d(V)$ is as follows:

$$\mathcal{L}(\alpha) = \alpha \cdot (d * d)\alpha.$$

We have: $\mathrm{Sol} = \{\alpha | d * d\alpha = 0\}/\{\text{exact 1-forms}\}$. We will write down the symplectic form using Fourier transform. The Fourier transform of the exact sequence

$$0 \longrightarrow \Omega^0(V)/\mathbb{R} \xrightarrow{d} \Omega^1(V) \xrightarrow{*d*d} \Omega^1(V)$$

looks as follows:

$$0 \longrightarrow \mathcal{S}(V) \xrightarrow{\nu} \mathcal{S}(V) \otimes V \xrightarrow{\eta} \mathcal{S}(V) \otimes V$$

where $\mathcal{S}(V)$ is the Schwartz space, and:

$$\nu(f)(p) = f(p)p \quad \text{for } p \in V,\ f \in \mathcal{S}(V),$$
$$\eta(G)(p) = (G(p), p)p - (p, p)G(p).$$

Let $\mathcal{O}_0 = \{p | p^2 = 0,\ p \neq 0\}$. Let $\mathcal{N}$ denote a vector bundle over $\mathcal{O}_0$, such that $\mathcal{N}_p = p^\perp/\mathbb{R}p$.

Lemma. a) *For any $G \in \operatorname{Ker}\eta$ we have $G(p) \in p^\perp$ for all $p \in \mathcal{O}_0$.*

b) *The map $\operatorname{Ker}\eta \to \Gamma(\mathcal{O}_0, \mathcal{N})$ given by $G \to G(p)/\mathbb{R}p$ is an isomorphism between* Sol *and $\{\gamma \in \Gamma(\mathcal{O}_0, \mathcal{N}) | \gamma(-p) = \overline{\gamma(p)}\}$. The symplectic form $[\ ,\ ]$ on* Sol *is constant and is given by $[\gamma_1, \gamma_2] = \int\limits_{\mathcal{O}_0^+} \operatorname{Im} \gamma_1 \overline{\gamma_2}$.*

Remark. It is a useful exercise to find an expression for the symplectic form $[\ ,\ ]$ in terms of Fourier transform in examples 1,2 also.

3.2 Clifford module. Quantization of a free classical field theory is based on the following linear algebra construction.

Let H be a real superspace, equipped with an even super-symplectic form $[\ ,\ ]: H \times H \to \mathbb{R}$. Let Cl^H be the corresponding Clifford (=Heisenberg) algebra.

A complex polarization of H is an automorphism $I: H \to H$, preserving $[\ ,\ ]$ and such that $I^2 = -\mathrm{id}$.

Define the maps $i_{\pm}: H \to H_{\mathbb{C}}$ by: $i_{\pm}(v) = v \pm i^{-1} Iv$. Let $H_{\pm} = \mathrm{Im}\ i_{\pm}$. Then $H_{\pm}$ is a complex isotropic subspace. We also have an antilinear isomorphism $^{-}: H_{+} \to H_{-}$. Define $(\ ,\ ): H_{+} \times H_{+} \to \mathbb{C}$ by $(h_1, h_2) = \frac{i}{2}[h_1, \bar{h}_2]$.

Claim. a) *$(\ ,\ )$ is a super-hermitian form on H_{+}.*

b) *$(\ ,\ )$ is positive iff: $[h, h] > 0$ for $h \in H^{\mathrm{odd}} - \{0\}$ and $[Ih, h] > 0$ for $h \in H^{\mathrm{even}} - \{0\}$.*

Proposition. *Let $(H, [\ ,\ ], I)$ be as before such that $(\ ,\ )$ is positive. Then*

a) *there exists a unique up to unique isomorphism triple $(j_I, \mathcal{D}, \Omega)$ where: j_I is a complex representation of Cl^H on $\mathcal{D}$ and $\Omega \in \mathcal{D} - \{0\}$ such that:*

$$j_I(\mathrm{Cl}^H)(\Omega) = \mathcal{D}$$

$$j_I(H_{-})(\Omega) = 0.$$

b) *There exists a unique positive Hermitian form $(\ ,\ )_{\mathcal{D}}$ on $\mathcal{D}$ such that $(\Omega, \Omega) = 1$ and $j_I(h)$ is an antiHermitian operator on $\mathcal{D}$ for $h \in H$.*

Remark. We have an embedding $\mathrm{Sym}(H_{+}) \hookrightarrow \mathrm{Cl}^H$, which induces an isomorphism: $j_{H_{+}}: \mathrm{Sym}(H_{+}) \tilde{\to} \mathcal{D}$, where $j_{H_{+}}(h) = j_I(h)(\Omega)$. The algebra Cl^H acts on $\mathcal{D} = \mathrm{Sym}(H_{+})$ by differential operators.

Remark. If H is infinite-dimensional, then the representation depends essentially on the complex polarization I. For finite dimensional H the L^2-completions of representations constructed via different polarizations are isomorphic.

Remark. The above construction for the finite dimensional case is discussed in Deligne's notes on quantization §3–7.

3.3 Examples of free QFT's.

Scalars. Let H be the space of real solutions of: $\Box\varphi + m^2\varphi = 0$ and $[\quad,\quad]$ be the form on H given by $[\varphi_1, \varphi_2] = \int_\Gamma *(\varphi_1 d\varphi_2 - \varphi_2 d\varphi_1)$ (Example 1 above).

To $f \in \mathcal{S}(V)$ there corresponds a classical observable, i.e. a function on H, given by $h \to \langle h, f\rangle_{L^2(V)}$. So in quantum theory an operator $\phi(f)$ should be defined for $f \in \mathcal{S}(V)$.

Let us construct a positive complex polarization I on $(H, [\quad,\quad])$. We describe I in terms of Fourier transform. The Fourier transform gives an isomorphism $H = \{f \in \mathcal{S}_{\mathbb{C}}(\mathcal{O}_m) | f(-p) = \overline{f(p)}\}$, where $\mathcal{O}_m = \{p \in V - \{0\} | p^2 = m^2\}$. We have: $\mathcal{O}_m = \mathcal{O}_m^+ \sqcup \mathcal{O}_m^-$ (union of 2 connected components). Restriction to $\mathcal{O}_m^+$ defines an isomorphism $H \tilde{\to} \mathcal{S}_{\mathbb{C}}(\mathcal{O}_m^+)$. Since $\mathcal{S}_{\mathbb{C}}(\mathcal{O}_m^+)$ has natural complex structure and $\mathcal{S}_{\mathbb{C}}(\mathcal{O}_m^+) \subset \mathcal{S}_{\mathbb{C}}(\mathcal{O}_m) = H_{\mathbb{C}}$ is an isotropic subspace, we get a positive complex polarization of H. Consider the "canonical" representation j_I: $\mathrm{Cl}^H \to \mathrm{End}\,\mathcal{D}$ described above. Let $\mathcal{H} = \overline{\mathcal{D}}$ be the Hilbert space completion with respect to the invariant Hermitian form (see Proposition). Since the action of P on H commutes with I, we get an action of P on $\mathcal{H}$.

We define the field operators ϕ: $\mathcal{S}(V) \to \mathrm{End}(\mathcal{D})$ as the composition

$$\mathcal{S}(V) \ni f \to \mathcal{F}(f) \to \mathcal{F}(f)|_{\mathcal{O}_m} \xrightarrow{j_I} \mathrm{End}\mathcal{D}$$

where we consider the restriction $\mathcal{F}(f)|_{\mathcal{O}_m}$ as an element of H.

Let us finally analyze the support of the spectral measure of $\mathcal{H}$ considered as a V-module. We obviously have: $\mathrm{Spec}\,\rho\big|_V = \mathcal{O}_m^+$, where ρ denotes the action of P on H_+. Recall that j_{H_+}: $\mathrm{Sym}\, H_+ \tilde{\to} \mathcal{D}$ is an isomorphism.

Hence $\mathrm{Spec}\, U\big|_V$ is the closure of the set $\{0\} \cup \mathcal{O}_m^+ \cup \mathcal{O}_m^+ + \mathcal{O}_m^+ \cup \ldots$ where we use the notation $A + B = \{a + b \,|\, a \in A, b \in B\}$, and U stands for the representation of P on $\mathcal{H}$. It is easy to see that $\mathrm{Spec}\, U\big|_V = \{0\} \sqcup \mathcal{O}_m^+ \sqcup \{p \in V_+ | p^2 \geq 2m\}$.

Spinors. We keep the notations of Example 2 above. Let $H = \Pi(\{\psi | \partial\psi = 0\})$. (Recall that Π denotes change of parity of a superspace).

A classical observable is associated with $f \in \mathcal{S}^{\mathcal{S}^-}(V)$, and is given by $h \to \int_V (f(v), h(v))\, dv$.

In Fourier coordinates the classical field equation becomes: $s(p, \mathcal{F}(\psi)(p)) = 0$. Since $\mathcal{D}^2 = \Box$ we see that for any $\psi \in H$: $\mathrm{supp}(\mathcal{F}(\psi)) \subset \mathcal{O}_0$.

Let $\mathcal{A}$ be a subbundle of the trivial bundle over $\mathcal{O}_0$ with fiber S^+, such that $\mathcal{A}_p \subset S^+ = \operatorname{Ker} s(p)$.

Then $H \cong \{f \in \mathcal{S}_{\mathbb{C}}(\mathcal{O}_0, \mathcal{A}) | f(-p) = \overline{f(p)}\}$. The symplectic form on Sol is the same as symmetric pairing on H, and it is given by: $(f_1, f_2) = \int\limits_{\mathcal{O}_0^+} \frac{\pi(f_1(p), f_2(-p))}{p} \mu_0$, where we use linear dependence of the vectors $\pi(f_1(p), f_2(-p))$, p, and denote by $\frac{\pi(f_1(p), f_2(-p))}{p}$ the coefficient of proportionality. (This formula is equivalent to the one given in Example 2).

Using the decomposition $\mathcal{O}_0 = \mathcal{O}_0^+ \sqcup \mathcal{O}_0^-$, we obtain a polarization and the canonical representation $j_I\colon \mathrm{Cl}^H \to \operatorname{End} \mathcal{D}$. If U is, as usually, the action of P on $\mathcal{D}$, then $\operatorname{Spec} U\big|_V = \overline{V}_+$.

The field operators are also defined similarly to the above. Namely, for $f \in \mathcal{S}(V, S^-)$ the section $\varkappa(f)$ such that $\varkappa(f)(p) = s(p, \mathcal{F}(f)\big|_{\mathcal{O}_0}(p))$ lies in H because $s(p)^2 = 0$ for $p \in \mathcal{O}_0$. We define $\phi(f) = j_I(\varkappa(f)) \in \operatorname{End} \mathcal{D}$.

Photones (Free quantum electro-dynamics = free abelian gauge theory). Here Sol $\cong \{f \in \mathcal{S}(\mathcal{O}_0, \mathcal{N}) | f(-p) = \overline{f(p)}\}$. Polarization comes again from the decomposition: $\mathcal{O}_0 = \mathcal{O}^+ \sqcup \mathcal{O}^-$. Thus we obtain the canonical representation $j_I\colon \mathrm{Cl}^H \to \operatorname{End} \mathcal{D}$.

Recall that in classical situation (Example 3) the space Sol is a subspace in $\Omega^1(V)/d\Omega^0(V)$. So to any 2-form $\omega \in \Omega^2(V)$ there corresponds a classical gauge-invariant observable:

$$\Omega^1/d\Omega^0 \ni \alpha \to \int\limits_V d\alpha \wedge *\omega$$

Consider the map $\varkappa\colon \Omega^2(V) = \mathcal{S}^{\Lambda^2 V} \to H = S_{\mathbb{C}}(\mathcal{O}_0, \mathcal{N})$ given by $\varkappa(\omega)(p) = i_p \mathcal{F}(\omega)(p) \in p^\perp$. Now we define the field map: $\phi(\omega) = j_I(\varkappa(\omega))$.

3.4 Free QFT of arbitrary spin. A QFT is called free of mass m if the field operator $\phi(f)$ depends only on $\mathcal{F}(f)|_{\mathcal{O}_m}$. (In other words, the classical field equation $\Box + m^2 = 0$ is satisfied on quantum level).

All examples considered in the previous section are of this kind.

In this section we give a construction of a free QFT of mass m which generalizes these examples.

To describe it we need some preliminaries.

Recall that $G = \mathrm{Spin}(d-1,1)$, and $\tau \in G$ is the spin-element.

We fix the mass of theory $m \geq 0$. Let St denote the stabilizer of a point $p \in \mathcal{O}^+_m$.

If $m > 0$ then $St = \mathrm{Spin}(p^\perp) \cong \mathrm{Spin}(d-1)$. If $m = 0$ then St has nontrivial unipotent radical N and $St/N = \mathrm{Spin}(p^\perp/\mathbb{R}p) \cong \mathrm{Spin}(d-2)$. Let G' denote a maximal semisimple subgroup of St. (So $G' = St$ if $m > 0$, and $G' = St/N$ if $m = 0$).

We fix a real representation R of G, and ρ of G'. We will view R, ρ as representations of real algebraic groups defined over $\mathbb{R}$.

ρ can be also viewed as representation of the group St. We fix a nonzero morphism $i : R|_{St} \to \rho$.

We do not have to assume that R, ρ are irreducible, but it is convenient to require that τ acts on R and ρ by constant, i.e. R and ρ are sums of irreducible representations of equal spin. This constant will be denoted by $(-1)^{\mathrm{spin}(R)}$, $(-1)^{\mathrm{spin}(\rho)}$ respectively. Obviously $(-1)^{\mathrm{spin}(R)} = (-1)^{\mathrm{spin}(\rho)}$.

Let $\widetilde{St}$ denote the subgroup $\{g \in G | g(p) = \pm p\}$; let $\widetilde{G'}$ denote a maximal semisimple subgroup of $\widetilde{St}$. (So $\widetilde{G'} = \widetilde{St}$ if $m > 0$, and $\widetilde{G'} = \widetilde{St}/N$ if $m = 0$). Clearly G' is a subgroup of $\widetilde{G'}$ of index 2.

More precisely, if $m > 0$ then $\widetilde{G'} \cong Pin(n)$. If $m = 0$ then $\widetilde{G'} \cong (\mathrm{Spin}(d-2) \times \mathbb{Z}_4)/(\tau, -1)$.

$\widetilde{G'}$ carries the structure of algebraic group over $\mathbb{R}$. Note that the nontrivial connected component of $\widetilde{G'}$ has *no real points.*

We assume that complexification of ρ extends to a representation of $\widetilde{G'}$ in such a way that the map i is $\widetilde{G'}$-equivariant. (The representation extends automatically if $m = 0$ or if d is even).

Let K be the normalizer in $\widetilde{G'}(\mathbb{C})$ of the subgroup of real points $\widetilde{G'}(\mathbb{R}) = G'(\mathbb{R})$. Then K is a compact Lie subgroup in the Lie group $\widetilde{G'}(\mathbb{C})$; moreover, K has 2 connected components and the connected component of 1 is $G'(\mathbb{R})$.

We have: $K = \{g \in \widetilde{G'}(\mathbb{C}) | \pi(g) \in \mathrm{SO}(d-1,1)(\mathbb{R})\}$ where $\pi : \mathrm{Spin} \to \mathrm{SO}$ is the projection.

Note that K is *not* the set of real points of $\widetilde{G'}$. More precisely, for $g \in K - G'(\mathbb{R})$ we have $\overline{g} = g \cdot \tau$.

Since K is compact, the space of $\rho_{\mathbb{C}}$ carries a K-invariant Hermitian form $\langle \quad , \quad \rangle$. It also yields a quadratic form $(\quad , \quad)$, where $(x, y) = \langle x, \overline{y} \rangle$. The quadratic form restricted to the subspace of real vectors (the space of ρ) is real and positive definite.

Now let us return to QFT.

To ρ there corresponds a G-equivariant real vector bundle $\mathcal{A}_\rho$ on $\mathcal{O}_m^+$; its complexification will be denoted by $\mathcal{A}_\rho^{\mathbb{C}}$. Let H be the space of Schwartz sections of $\mathcal{A}_\rho^{\mathbb{C}}$ viewed as a real vector space.

As usual we get the structure of a super-space on H by declaring the parity of H to be $(-1)^{\mathrm{spin}(\rho)}$.

The bundle $\mathcal{A}_\rho^{\mathbb{C}}$ carries a G-equivariant Euclidean and Hermitian metrics, obtained from metrics $(\quad , \quad)$, $\langle \quad , \quad \rangle$ on the space of $\rho_{\mathbb{C}}$ in the standard way. We will abuse notations by denoting the metrics on the vector bundle by $(\quad , \quad)$, $\langle \quad , \quad \rangle$ also. Of course $(\quad , \quad)|_{\mathcal{A}_\rho}$ is real and positive.

A super-symplectic form on H is given by: $[h_1, h_2] = \int\limits_{\mathcal{O}_m^+} Im \left\langle h_1, \overline{h_2} \right\rangle$ if H is even, and $[h_1, h_2] = \int\limits_{\mathcal{O}_m^+} Re \left\langle h_1, \overline{h_2} \right\rangle$ if H is odd. The obvious complex structure on H yields the positive polarization to be denoted I. Thus we get the canonical representation $\mathcal{D} \ni \Omega$ and its Hilbert space completion $\mathcal{H}$.

To provide the field map $\phi : \mathcal{S}^R \to \mathrm{End}(\mathcal{D})$ we first construct a map $\varkappa : \mathcal{S}^R \to H$ and then take $\phi = j_I \circ \varkappa$.

To the morphism $i : R|_{St} \to \rho$ there corresponds a map $\iota : R \to \Gamma(\mathcal{A}_\rho)$, where $\Gamma(\mathcal{A}_\rho)$ is the space of sections.

Now we can define $\varkappa$. For $f \otimes r \in \mathcal{S}^R = \mathcal{S} \otimes R$ we put $\varkappa(f \otimes r) = \mathcal{F}(f)|_{\mathcal{O}_m^+} \cdot \iota(r)$ where $\mathcal{F}$ stands for Fourier transform as usual.

The data of QFT is constructed.

Remark. In fact the maps $\varkappa$, ϕ are defined on a larger function space then the space of Schwartz section $\mathcal{S}^R$. Indeed, let F be a Schwartz function on $\mathbb{R}^{d-1}$. Let f be a generalized function on $V = \mathbb{R} \times \mathbb{R}^{d-1}$ of the form: $f = \delta_{t=t_0} \times F$ where t is the coordinate on $\mathbb{R}$. Then it is easy to see that $\mathcal{F}(f)|_{\mathcal{O}_m^+}$ is a Schwartz function on the hyperboloid $\mathcal{O}_m^+$. Thus $\varkappa$ and ϕ are defined on the generalized sections of $\mathcal{R}$ of the form $f \cdot r$ for f as above and $r \in \mathbb{R}$.

Let us now check the axiom ϵ) of Wightman QFT (the space-locality property).

We need the following

Key lemma. *For any r_1, $r_2 \in R$ the real-valued function on $\mathcal{O}_m^+$ defined by $P_{r_1,r_2}(p) = (\iota(r_1), \iota(r_2))|_p$ extends to a polynomial function on $\mathcal{O}_m$. This polynomial function is even if R and ρ have trivial spin, and odd otherwise.*

Proof of the Lemma. We can view $\mathcal{A}_\rho$ as an algebraic vector bundle on the real algebraic variety σ_m, and $\iota(r)$ for $r \in R$ – as an algebraic section of this vector bundle.

Since the metric on $\mathcal{A}_\rho$ is algebraic as well, the first part of the claim is clear.

For any $g \in G(\mathbb{C})$ such that $g(p) = -p$ we have by the definition

$$(\iota(r_1), \iota(r_2))_{-p} = (\iota(g(r_1)), \iota(g(r_2)))_p = (i(g(r_1)), i(g(r_2)))$$

In particular the last equality holds for any $g \in K - G'(\mathbb{R})$.

As was mentioned above for $g \in K - G'(\mathbb{R})$ we have $g^{-1}\overline{g} = \tau$. So for such g:

$$\begin{aligned}(i(g(r_1)), i(g(r_2))) &= \langle i(g(r_1)), \overline{i(g(r_2))}\rangle = \langle (i(r_1)), \overline{g}(i(r_2))\rangle = \langle i(r_1), g^{-1}\overline{g}(i(r_2))\rangle \\ &= \langle i(r_1), \tau(i(r_2))\rangle = (-1)^{\mathrm{spin}(\rho)}\langle i(r_1), i(r_2)\rangle = (-1)^{\mathrm{spin}(\rho)}(\iota(r_1), \iota(r_2))_p\end{aligned}$$

So for r_1, $r_2 \in R$ we proved that $(\iota(r_1), \iota(r_2))_{-p} = (-1)^{\mathrm{spin}(\rho)} (\iota(r_1), \iota(r_2))_p$. Since the map $(r_1, r_2) \to P_{r_1,r_2}$ is G-equivariant it follows that the same holds for any $p' \in \sigma_m^+$, and the Lemma is proven.

Proof of space-locality. It is enough to check that if $\varphi_1, \varphi_2 \in \mathcal{S}^R$ are space-like separated then the scalar product $[\varkappa(\varphi_1), \varkappa(\varphi_2)]$ vanishes.

Let us take F_1, $F_2 \in \mathcal{S}(\mathbb{R}^{d-1})$ such that $\mathrm{supp}(F_1) \cap \mathrm{supp}(F_2) = \emptyset$. Let $f_j = \delta_{t=0} \times F_j$ for $j = 1, 2$. For r_1, $r_2 \in R$ the expression $f_j \cdot r_j$ is a generalized section of $\mathcal{R}$. As was mentioned above $\varkappa$ is actually defined on such sections.

Since $\varkappa$ and $[\quad,\quad]$ are P-equivariant it sufficies to check that $[\varkappa(f_1 \cdot r_1), \varkappa(f_2 \cdot r_2)] = 0$ for f_j, r_j as above.

Put $g_j = \mathcal{F}(f_j)$. Then the functions g_j satisfy $g_j(p + (t, \vec{0})) = g_j(p)$, so we can write $g_j(t, q) = g_j(q)$ where $t \in \mathbb{R}$, $q \in \mathbb{R}^{d-1}$. We also have $g_j(-p) = \overline{g_j(p)}$ since f_j is a real function. So:

$$[\varkappa(f_1 \cdot r_1), \varkappa(f_2 \cdot r_2)] = \text{const} \cdot \int\limits_{\sigma_m^+} \left(g_1 \overline{g_2} - (-1)^{\text{spin}(\rho)} g_2 \overline{g_1}\right) (\iota(r_1), \iota(r_2))\, d\mu$$

$$= \text{const} \cdot \int\limits_{\mathbb{R}^{d-1}} \left(g_1(q)\overline{g_2(q)} - (-1)^{\text{spin}(\rho)} g_2(q)\overline{g_1(q)}\right) P_{r_1,r_2}(|q|, q) \cdot |q|^{-1} dq$$

(notations of the Key Lemma). It can be rewritten as:

$$\int\limits_{\mathbb{R}^{d-1}} \left(g_1(q)\overline{g_2(q)} - (-1)^{\text{spin}(\rho)} g_1(-q)\overline{g_2(-q)}\right) P_{r_1,r_2}(|q|, q) \cdot |q|^{-1} dq$$

$$= \int\limits_{\mathbb{R}^{d-1}} (g_1(q)\overline{g_2(q)}) \left(P_{r_1,r_2}(|q|, q) - (-1)^{\text{spin}(\rho)} P_{r_1,r_2}(|q|, -q)\right) \cdot |q|^{-1} dq$$

By the Key Lemma

$$P_{r_1,r_2}(|q|, q) - (-1)^{\text{spin}(\rho)} P_{r_1,r_2}(|q|, -q) = P_{r_1,r_2}(|q|, q) - P_{r_1,r_2}(-|q|, q)$$

and $P_{r_1,r_2}(|q|, q)$ is a polynomial function.

But then it follows that the function $Q(q) = |q|^{-1}(P_{r_1,r_2}(|q|, q) - P_{r_1,r_2}(-|q|, q))$ is again polynomial.

Fourier transform of multiplication by a polynomial function is a differential operator, in particular is local. Thus we have:

$$[\varkappa(f_1 \cdot r_1), \varkappa(f_2 \cdot r_2)] = \int\limits_{\mathbb{R}^{d-1}} (g_1(q)\overline{g_2(q)}) Q(q) dq = (Q(q) \cdot g_1, g_2)_{L^2(\mathbb{R}^{d-1})}$$

$$= (\hat{Q}(F_1), F_2)_{L^2(\mathbb{R}^{d-1})} = 0$$

where $\hat{Q}$ is the differential operator with constant coefficients corresponding to Q and the last equality is obvious because F_1, F_2 have separated supports.

Space-locality is proved.

3.5 Wightman functions of a free field theory; truncated Wightman functions. The next statement characterizes free theories in terms of Wightman functions.

Claim. *Let $\mathcal{W}_n$ be the Wightman functions of the free scalar* QFT *of mass m. Then we have $\mathcal{W}_2(v_1, v_2) = \mathcal{F}(\delta_{\mathcal{O}_m^+})\ (v_1 - v_2)$; $\mathcal{W}_n = 0$ if n is odd, and*

$$\mathcal{W}_{2k}(v_1, \ldots, v_{2k}) = \sum \mathcal{W}_2(v_{i_1}, v_{j_1})\mathcal{W}_2(v_{i_2}, v_{j_2}) \ldots \mathcal{W}_2(v_{i_k}, v_{j_k}),$$

where the sum is taken over all partitions of $\{1, \ldots, n\}$ in pairs $(i_1, j_1), \ldots, (i_k, j_k)$, and $i_r < j_r$.

Proof is straightforward.

This statement can be easily generalized to other free QFT's.

It also can be reformulated using the so-called *truncated Wightman functions.*

Since these objects will be used in the next lecture, let us give their definition. We set:

$$\mathcal{W}_n^{\mathrm{tr}} \stackrel{\mathrm{def}}{=} \sum_{\pi=(\pi_1,\ldots,\pi_k)} (-1)^{k-1}(k-1)! \prod_{\ell=1}^{k} \mathcal{W}_{|\pi_\ell|}(v_{i_1}, v_{i_2}, \ldots, v_{i_{|\pi_\ell|}})$$

Here the sum is over all decompositions of $\{1, .., n\}$ into the union of disjoint nonempty subsets $\pi_1, .., \pi_k$, (the order of $\pi_1, .., \pi_k$ does not matter), and $i_1 < i_2 \ldots < i_{|\pi_\ell|}$ are the elements of π_ℓ.

An equivalent equality is

$$\mathcal{W}_n = \sum_{\pi=(\pi_1,\ldots,\pi_k)} \prod_{\ell=1}^{k} \mathcal{W}_{|\pi_\ell|}^{\mathrm{tr}}(v_{i_1}, \ldots, v_{i_{|\pi_\ell|}})$$

Corollary. *For a free QFT we have $W_n^{\mathrm{tr}} = 0$ for $n > 2$.*

3.6 Gaussian measures. One can also easily describe free field theory in Euclidean formulation.

For a free QFT one can make sense of the measure ν on the space of tempered distributions discussed in lecture 2 (§2.3); the corresponding measure is Gaussian. Here is the formal definition for the case of the scalar theory.

Claim. *Let* $\mathfrak{S}\ (e_1, \dots, e_n)$ *be the Schwinger functions of the free Euclidean scalar* QFT *of mass* m. *Then* $\mathfrak{S}\ (e_1, \dots, e_n)$ *is the moment of a Gaussian measure* ν_m *on the space of tempered distributions* $\mathcal{S}'(E)$ *with covariance* $(-\Box_E + m^2)^{-1}$. *[So for* $f \in \mathcal{S}$ *we have* $\mathcal{F}(\nu_m)(f) = e^{((-\Box_E+m^2)^{-1}f,f)}$*].*

Remark. Gaussian measure on a finite dimensional real vector space H corresponding to a quadratic form $\langle\ \ ,\ \ \rangle$ is characterized by the equalities

$$\nu(1) = 1; \quad \nu\left(\frac{\partial P}{\partial h}\right) = -h^{\vee}\nu(P)$$

for any $h \in H$, and $P \in \mathrm{Sym}(H')$ a polynomial function on H. (Here $\frac{\partial}{\partial h}$ is a constant vector field, and $h^{\vee} = \langle h,\ \ \rangle$ is a linear functional on H.)

In our setting of kernel spaces ($H = \mathcal{S}'$, $H' = \mathcal{S}$ and $\langle f, f\rangle = \big((-\Box + m^2)f, f\big)$) this is still so, if one requires that the equality holds for all h for which it makes sense, namely for all $h \in \mathcal{S} \subset \mathcal{S}'$.

Reconstruction of the Hilbert space of a free QFT in terms of the Gaussian measure is based on the next Lemma.

Let I be a positive polarization of H, $[\ \ ,\ \ ]$ and let $\langle h_1, h_2\rangle = [Ih_1, h_2]$. Then we have an imbedding $\mathcal{D} = \mathrm{Sym}(H_+) \hookrightarrow A_H$ defined by the decomposition $H_{\mathbb{C}} = H_+ \oplus H_-$.

Lemma. *For* $F_1, F_2 \in \mathcal{D}$ *we have* $(F_1, F_2)_{\mathcal{D}} = \nu(F_1\overline{F}_2)$

We finish the lecture with discussion of

3.7 Normal ordering. In general in QFT one is interested in behavior of correlation functions near diagonals. Naively speaking, we would like to consider along with operator-valued generalized function $\phi(v)$ also an operator-valued generalized function $\phi^n(v)$. As usual such operator-valued functions should be encoded in correlation functions on V_k whose "value at $(v_1, \dots, v_k)$" equals to $(\Omega, \phi^{n_1}(v_1)\cdot\ldots\cdot\phi^{n_k}(v_k)\Omega)$. If the naive definition would work, these correlators would obviously be the restrictions of the original Wightman function to the corresponding diagonal. Of course this is not possible, since Wightman functions are singular along diagonals. Techniques to overcome this difficulty are discussed in Witten's lecture 3; here we only present a simple device, which leads to an answer in the free theory case (see also §§3.2, 3.5 in Witten's lecture 3).

Let $(\mathcal{H}, \phi, \mathcal{D}, \Omega)$ be the free scalar QFT. (The assumption that the theory is scalar is not essential, and made for notational convenience).

For any function $f \in \mathcal{S}$ let us write $f = f_+ + f_-$ where $f_\pm \in \mathcal{S}_\mathbb{C}$ and $\mathcal{F}(f_\pm)|_{\mathcal{O}_m^\mp} = 0$.

Then define normally ordered power of the field operator by

$$: \phi(f)^n : \stackrel{\text{def}}{=} \sum \binom{n}{i} \phi(f_+)^i \phi(f_-)^{n-i}.$$

Claim.

a) *For any $v \in V$, $x, y \in \mathcal{D}$ and $f \in \mathcal{S}$ the limit $\lim\limits_{f \to \delta_v} (: \phi(f)^n : (x), y)$ exists; we will write $(: \phi^n(v) : (x), y) \stackrel{\text{def}}{=} \lim\limits_{f \to \delta_v} (: \phi(f)^n : x, y)$. Thus the correlator $(\phi(v_1), .., : \phi^n(v_i) :, .., \phi(v_n))$ is a well-defined distribution.*

b) *If $d = 2$, then for $f \in \mathcal{S}$ there exists a well-defined endomorphism $: \phi^n(f) : \in \mathrm{End}(\mathcal{H})$ such that $(: \phi^n(f) : (x), y) = \int f(v) (: \phi^n(v) : (x), y)\, dv$.*

In fact Claim b) is valid for much wider class of 2-dimensional theories than free ones; see §8.6 of [Glimm-Jaffe].

One can define normal ordering more algebraically as follows.

Recall that we have $H = \{\varphi | \Box \varphi + m^2 \varphi = 0\}$, $[\varphi_1, \varphi_2] = \int *(\varphi_1 d\varphi_2 - \varphi_2 d\varphi_1)$. Consider a Lagrangian subspace $L \subset H$, $L = \left\{\varphi \colon \frac{\partial}{\partial t} \varphi\big|_{t=0} = 0\right\}$. Since L is isotropic we have an embedding: $\mathrm{Sym}(L) \hookrightarrow \mathrm{Cl}^H$. Then the map $F \to j_I(F)\Omega$ induces an isomorphism $j_L \colon \mathrm{Sym}(L) \tilde{\to} \mathcal{D}$ (because $H_\mathbb{C} = L \oplus H_-$). Also, the projection along H_- induces an isomorphism $u \colon H_+ \tilde{\to} L$.

Now consider the endomorphism of $\mathrm{Sym}\, L$ given by the composition of the arrows:

$$\begin{tikzcd}
\mathrm{Sym}\, L \arrow[rr, "u^*"] & & \mathrm{Sym}\, H_+ \arrow[dl, "j_{H_+}"] \\
& \mathcal{D} \arrow[ul, "j_L^{-1}"] &
\end{tikzcd}$$

We denote this endomorphism by $F \to :F:$.

Lemma.

a) *For* $\ell \in L$ *we have* $:\phi^n(\ell) := j_I(:\ell^n:)$. *(The LHS is defined by Remark in §3.4).*

b) $F - {:}F{:} \in \bigoplus_{k=0}^{n-1} \operatorname{Sym}^k(L)$ *for* $F \in \operatorname{Sym}^n L$.

c) $(j_L(:F:), j_L(G))_{\mathfrak{D}} = 0$ *for* $F \in \operatorname{Sym}^n L$, $G \in \operatorname{Sym}^k(L)$ $k < n$.

Lemma. *For any* $F_1, F_2 \in \operatorname{Sym}(L)$ *we have:*

$$(j_L(F_1), j_L(F_2))_{\mathfrak{D}} = \nu_{\langle\ ,\ \rangle\big|_L}(F_1 \cdot \overline{F}_2)$$

Let $\varkappa \in \operatorname{End}(\operatorname{Sym} L)$ be the second order differential operator corresponding to the form $\langle\quad,\quad\rangle$. (So we have: $\varkappa(a^k) = k(k-1)\,\langle a, a\rangle\, a^{k-2}$ for $a \in L$.)

Lemma. ${:}F{:} = \exp\left(-\frac{\varkappa}{2}\right) F$.

Corollary. $:\exp(\ell^\vee) := \exp\left(-\frac{(\ell,\ell)}{2}\right)\exp(\ell^\vee)$, *for* $\ell \in L$.

References

B. Simon, *The* $P(\phi)_2$ *Euclidean (quantum) field theory; §II.5*, Princeton University Press, 1974.

For a discussion of the Gaussian measure approach to free QFT see Chapter 3 of *loc. cit.* or

J. Glimm, A. Jaffe, *Quantum Physics. (A functional integral point of view), §6.2*, Springer-Verlag, 1987.