Lecture 4.
Scattering theory.

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4.1 Introduction.

This important branch of functional analysis is not so popular as it should be; one of the reasons is that the phenomena studied in scattering theory has no finite-dimensional analog.

The situation is roughly as follows.

Let A_0 be a self-adjoint operator in a Hilbert space H (the function space). Let λ be a point in the *continuous* spectrum of A_0 and let f_λ^0 be the corresponding "eigenfunction" (quotation marks are put to remind that f_λ^0 lies not in the Hilbert space H, but rather in an appropriate larger function space, i.e. f_λ is a linear functional on a dense subspace in H.)

We consider a "small" perturbation $A = A_0 + B$ of our self-adjoint operator A_0, where B is a self-adjoint operator such that $(\sqrt{1+A_0^2})^{-1}B$ is compact, and we look for a λ-eigenfunction f_λ of the operator A.

Remark. This problem makes sense only if λ lies in the continuous spectrum of A_0; in particular it never makes sense in finite dimensions. Indeed, the spectrum of an operator moves as we perturb the operator; so an isolated point of the spectrum in general does not lie in the spectrum of perturbed operator.

We look for an eigenfunction of the form $f_\lambda = f_\lambda^0 + g$. We have:

$$Af_\lambda = \lambda f_\lambda \Leftrightarrow (A-\lambda)g = Bf_\lambda^0 \quad \text{so formally}$$
$$g = (A-\lambda)^{-1}B(f_\lambda^0) \ .$$

Of course the last equality makes no sense since $(A-\lambda)$ is not invertible. Nevertheless in some cases one can show existence of the limits $g_\pm = \lim_{\varepsilon\to\pm 0}(A-\lambda\pm i\varepsilon)^{-1}B(f_\lambda^0)$. Then we put $f_\pm^\lambda = f_\lambda^0 + g_\pm$.

Let us consider a concrete example of the above situation.

In this example one can actually show existence of the limits $\lim_{\varepsilon\to\pm 0}(A-\lambda\pm i\varepsilon)^{-1}B(f^0_\lambda)$. We will present another way to obtain the two eigenfunctions $f^\lambda_\pm$. We leave it as an exercise to the reader to check that the two procedures lead to the same answer.

We take $H = L^2(E)$, where E is a d-dimensional Euclidean space. Let $A_0 = -\Delta + m^2$, and let B be multiplication by a function $b \in C^\infty_c(E)$.

Claim. *For $f \in C^\infty_c(E)$ we have:*

$$\|Be^{itA_0}f\| \sim |t|^{-d/2} \qquad \textit{for} \qquad t \to \pm\infty$$

Corollary. *For any $f \in C^\infty_c(E)$ the L^2-limits $\lim_{t\to\pm\infty} e^{itA}e^{itA_0}f$ exist provided $d > 2$.*

Proof of Corollary. Denote $f^t = e^{itA}e^{-itA_0}f$. Then we have:

$$\begin{aligned}\left\|\frac{\partial f^t}{\partial t}\Big|_{t_0}\right\| &= \|iAe^{it_0A}e^{-it_0A_0}f - ie^{it_0A}A_0e^{-it_0A_0}f\| \\ &= \|e^{it_0A}Be^{-it_0A_0}f\| = \|Be^{-t_0A_0}f\| \ .\end{aligned}$$

So by the previous claim $f_+ = \int\limits_0^\infty \left(\frac{\partial f^t}{\partial t}\right) dt$ converges if $d > 2$. The proof for f_- is analogous.

Obviously: $\|f_\pm\|_{L^2} = \|f\|_{L^2}$, hence there exist well-defined operators $W_\pm$: $H \to H$, such that $W_\pm(f) = f_\pm$. Moreover, we see that:

$$\|(AW_\pm - W_\pm A_0)(f)\| = \lim_{t\to\pm\infty} \|e^{itA}Be^{-itA_0}(f)\| = 0 \ ,$$

hence $AW_\pm = W_\pm A_0$.

If the image of W_+ is equal to the image of W_-, we define the *S-matrix* by: $S = W_+^*W^-$. Thus $S \in \mathrm{End}(H)$, and S commutes with A_0.

4.2 System of n particles (potential scattering). This is a generalization of the last problem. We take $H = L^2(E^n)$, $A = -\sum\limits_{i=1}^n \Delta_i + \sum B_{ij}$, where B_{ij} is multiplication by a function $b(v_i - v_j)$, for $b \in C^\infty_c(E)$, and Δ_i is the Laplace operator along the i^{th} copy of E.

Let π be a partition of $\{1,\ldots,n\}$ in a union of k disjoint sets $\pi_1,\ldots,\pi_k$ ("clusters").

Consider the decomposition $E^{\pi_\ell} \simeq E \times \overline{E}_{\pi_\ell}$, where $\overline{E}_{\pi_\ell} = \{(v_{i_1}, \ldots, v_{i_{|\pi_\ell|}}) \in E^{\pi_\ell} \mid \sum v_i = 0\}$, and the isomorphism is given by $(v_{i_1}, \ldots, v_{i_{|\pi_\ell|}}) \to (w_\ell = \frac{1}{|\pi_\ell|} \sum v_i;\ (v_{i_1} - w_\ell, v_{i_2} - w_\ell, \ldots v_{i_{|\pi_\ell|}} - w_\ell))$.

We have obvious decompositions $E = \prod_{\ell=1}^{k} E^{\pi_\ell}$, and the corresponding decomposition of the function space:

$$H = L^2(E^n) = \bigotimes_{\ell=1}^{k} L^2(E^{\pi_\ell}) = \bigotimes_{\ell=1}^{k} \left(L^2(E) \otimes L^2(\overline{E}_{\pi_\ell})\right)$$

The operator A_0 obviously decomposes as: $A_0 = \sum_{\ell=1}^{k} A_\ell$, where A_ℓ acts along E^{π_ℓ}; and furthermore $A_\ell = A'_\ell + A''_\ell$, where A'_ℓ is minus the Laplace operator acting along E, and A''_ℓ acts along $\overline{E}_{\pi_\ell}$. Let $H_\ell = \oplus H^\lambda_\ell \subset L^2(\overline{E}_{\pi_\ell})$ be the sum of eigenspaces of A''_ℓ where λ runs over the discrete spectrum of A''_ℓ. Put $H_0(\pi) = \bigotimes_{\ell=1}^{k} (L^2(E) \otimes H_\ell) \subset L^2(E^n) = H$. Let $J_\pi \colon H_0(\pi) \hookrightarrow H$ denote the natural isometric inclusion; thus $J_\pi(H_0(\pi))$ is an A_0-invariant subspace in H. We denote $J = \oplus J_\pi \colon \bigoplus_\pi H_0(\pi) \to H$.

Theorem 1. a) *There exist strong limits* $W_\pm = \lim_{t \to \pm\infty} e^{itA} J e^{-itA_0}$.

b) $W_\pm$ *is an isometric imbedding and we have:* $A \circ W_\pm = W_\pm \circ A_0$.

c) $W_\pm$ *is an isomorphism.*

We omit the proof. Just remark that the proof of a) and b) is routine, while c) is a difficult theorem, the full proof of which was obtained only recently.

4.3 Haag-Ruelle theory. We now apply the method of scattering theory to QFT.

Claims and methods of scattering theory discussed below apply only to Minkowski, not to the Euclidean picture.

We will describe the following construction.

Let $(U, \mathcal{H} \supset \mathcal{D} \ni \Omega_i)$ be a scalar QFT. Suppose that:

a) $\mathcal{H}^V = \mathbb{C}\Omega$ (uniqueness of vacuum).

b) There exists a P-invariant Hilbert subspace $H \subset \mathcal{H}$ with a P-equivariant isomorphism $H \simeq L^2(\mathcal{O}^+_m)$, $m > 0$.

c) Mass spectrum of $\mathcal{H}/H \oplus \mathbb{C}\Omega$ does not contain 0 and m.

d) The span of $\langle\phi(f)\Omega\rangle$, $f\in\mathcal{S}$ is not contained in $H^\perp$.

Let $(U_0, \mathcal{H}_0 \supset \mathcal{D}_0, \phi_0)$ be the free field theory of mass m. Then one can construct two P-equivariant isometric imbeddings $W_\pm : \mathcal{H}^0 \hookrightarrow \mathcal{H}$.

Remark. We cannot hope that $W_\pm$ is an isomorphism, because $\Box\big|_{\mathcal{H}}$ can have discrete spectrum outside of $\{0,\ m^2\}$ while for free theory it does not. An ideal theorem would assert that a free theory constructed on discrete part of the spectral decomposition of $\Box$ on $\mathcal{H}$ maps isomorphically to $\mathcal{H}$ (compare with potential scattering described in the previous section). Unfortunately there is no known way to formulate a precise theorem of this sort.

Remark. The construction can be carried out under weaker assumptions. Namely one can omit condition d), and replace condition c) by requirement that the theory has positive mass, and m^2 is an isolated point of the discrete spectrum of $\Box$.

However the assumption that 0 is an isolated point of $\mathrm{Spec}(U\big|_V)$ is essential. Application of scattering to a massless QFT confronts much more serious difficulties than in the massive case.

Remark. The construction does not work for small d. Consider the case $d = 1$. Then QFT reduces to quantum mechanics. The spectrum of Hamiltonian of the free theory $\mathrm{Spec}\, H_0 = m\mathbb{Z}_+$ is totally discrete, thus scattering theory is not applicable. One can prove a slightly weaker statement in the case $d = 2$. Our argument will work for $d \geq 4$.

We now turn to construction of imbeddings $W_\pm$. We try to imitate the construction of the previous section (potential scattering).

Let $\Lambda \subset V^+$ be a G-invariant neighborhood of $\mathcal{O}^+_m$, such that $\Lambda \cap \mathrm{Spec}\, U\big|_{H^\perp} = \emptyset$. For any function f on V such that $\mathrm{supp}\,\mathcal{F}(f) \subset \Lambda$ and $t\in\mathbb{R}$ let $f^t \in \mathcal{S}(V)$ be the function such that:

$$\mathcal{F}(f^t)(p_0,\vec{p}) = e^{it\left(p_0 - \sqrt{m^2-\vec{p}^2}\right)}\mathcal{F}(f)(p_0,\vec{p})\,.$$

Note that $f^t = f$ provided $\mathrm{supp}(\mathcal{F}(f)) \subset \mathcal{O}^+_m$.

In the rest of the lecture we prove the following

Theorem 2. a) *If $d \geq 4$ then for any $f_1,\ldots,f_n \in \mathcal{S}(V)$ such that* $\mathrm{supp}(\mathcal{F}(f_i)) \subset \Lambda$ *the limits* $\lim_{t\to\pm\infty} \phi(f_1^t)\ldots\phi(f_n^t)\Omega$ *exist.*

b) *There exists a unique isometric imbedding* $W_\pm : \mathcal{H}_0 \hookrightarrow \mathcal{H}$, *such that for any* $f_1, \dots, f_n \in \mathcal{S}(V)$ *with* $\operatorname{supp}(\mathcal{F}(f)) \subset \lambda$ *we have:*

$$W_\pm\left(\phi_0(f_1)\dots\phi_0(f_n)\Omega\right) = \lim_{t\to\pm\infty} \phi(f_1^t)\dots\phi(f_n^t)\Omega$$

The imbeddings $W_\pm$ *are* P*-equivariant.*

Remark. For a free field theory the vector $\phi_0(f_1^t)\dots\phi_0(f_n^t)\Omega$ does not depend on t. Indeed, the field operator $\phi_0(f)$ for a free field theory depends on $\phi(f)\big|_{\mathcal{O}_m^+}$ only, and $\mathcal{F}(f^t)\big|_{\mathcal{O}_m^+}$ does not depend on t.

Remark. Let L be the space of functions satisfying the conditions of Lemma 1. Fourier transform identifies L with the space of Schwartz functions on $\mathcal{O}_m^+$. Let χ be a Schwartz function of one variable such that $\mathcal{F}(\chi)$ is supported in a small neighborhood of 0. For $f \in L$ let us put $f_\chi(x) = \chi(x_0)\cdot f(x)$ where $x = (x_0, \vec{x}\,)$. Then $f_\chi \in \mathcal{S}(V)$, and the Fourier transform of f_χ is concentrated in a neighborhood of $\mathcal{O}_m^+$ being the convolution of the function $\mathcal{F}(f)$ concentrated on $\mathcal{O}_m^+$ and a function concentrated in a small neighborhood of 0 on the time-axis.

This way we can fix an initial imbedding j of (a dense subspace of) $\mathcal{H}_0$ to $\mathcal{H}$ such that $j(\phi_0(f_1)\cdots\phi_0(f_k)\Omega) = \phi(f_{1\chi})\cdots\phi(f_{k\chi})\Omega$ for any functions $f_1, .., f_k \in L$. Then one checks directly that:

$$\phi(f_{1\chi}^t)\cdots\phi(f_{k\chi}^t)\Omega = U\left((t,\vec{0}\,)\right) j(U_0\left((-t,\vec{0}\,)\right)\phi_0(f_1)\cdots\phi_0(f_k)\Omega).$$

Thus the situation is quite analogous to the potential scattering considered in the previous section.

Proof of the Theorem. We start with two auxiliary estimates needed in the proof. The first is a standard fact from analysis.

Lemma 1 (locality for free fields)**.** *Let* $f \in C^\infty(V)$ *be a solution of the Klein-Gordon equation, i.e.* $\operatorname{supp}\mathcal{F}(f) \subset \mathcal{O}_m^+$. *Assume that* $f(0,\vec{x}\,)$ *and* $\frac{\partial f}{\partial x_0}(x_0,\vec{x})\big|_{x_0=0}$ *are Schwartz functions on* $\mathbb{R}^{d-1}$. *Then for some constant* C *we have:*

a) $\sup\limits_{\vec{x}\in\mathbb{R}^{d-1}} |f(x_0,\vec{x}\,)| \le C\,(1+|x_0|)^{-(d-1)/2}$

b) $\int\limits_{\mathbb{R}^{d-1}} |f(x_0,\vec{x})|d\vec{x} \le C(1+|x_0|)^{(d-1)/2}$

We omit the proof; it is an exercise in stationary phase method (see lecture 1, §1.6, Claim 1b) for a similar argument).

The second ingredient is the following estimate on truncated Wightman functions (see lecture 3, §3.5 for the definition).

Proposition 1. *There exist a differential operator with constant coefficients D, a C^0 function of tempered growth F, and a polynomial function P on V_n such that $\mathcal{W}_n^{\mathrm{tr}} = D(F)$ and the following estimate holds.*

Let $[1..n] = I \cup J$ be a partition of $[1..n]$ into union of 2 disjoint subsets; let $a_1, \dots, a_n \in \mathbb{R}^{d-1} \subset V$ be such that $a_i \neq a_j$ for $i \in I$, $j \in J$. Then for all $t_1, ..., t_n \in \mathbb{R} \subset V$ we have:

$$|F(t_1+a_1,\dots,t_n+a_n)| < |P(t_1+a_1,\dots,t_n+a_n)|e^{-m\min_{i\in I,j\in J}(\sqrt{-(a_i-a_j)^2}-|t_i-t_j|)}.$$

Remark. The particular case $n=2$ of the Proposition is implied by lecture 1, §1.6, Claim 1b). Also part a) of the same Claim guarantees that at least $\mathcal{W}_n^{\mathrm{tr}}(v_1,..,v_i,v_{i+1}+\lambda a,..,v_n+\lambda a) \to 0$ for any $v_1,\dots,v_n$ provided $a^2<0$, which obviously follows from the Proposition.

The statement we will actually need is the following immediate

Corollary. *Let i_t denote the embedding of $(\mathbb{R}^{d-1})^n$ to V_n, $\prod \vec{v}_i \to \prod (t,\vec{v}_i)$, and let $\pi : (\mathbb{R}^{d-1})^n \to (\mathbb{R}^{d-1})^{n-1} = (\mathbb{R}^{d-1})^n/\Delta$ be the projection to quotient modulo diagonal. For a Schwartz function h on V_n the restriction $i_t^*(h * \mathcal{W}_n^{\mathrm{tr}})$ is a smooth function invariant under diagonal translations, hence $i_t^*(h*\mathcal{W}_n^{\mathrm{tr}}) = \pi^*(F_t^h)$ for some function F_h^t on $(\mathbb{R}^{d-1})^{n-1}$. Then $F_h^t \in L^1\left((\mathbb{R}^{d-1})^{n-1}\right)$ for any t, and $\left|F_t^h\right|_{L^1}$ is bounded by a constant independent of t.*

Proof of the Proposition. We start with an auxiliary result. Recall from the first lecture that the Fourier transform of Wightman functions has the property: $\mathcal{F}(\mathcal{W}_n(p_1,\dots,p_n)) = 0$ if $p_1+\dots+p_k \notin \overline{V}_+$ for some k. One can easily show:

Lemma 2. *Let $V_+(m)$ be $\{p \in V_+ | p^2 > m\}$. Then we have $\mathcal{F}(\mathcal{W}_n^{\mathrm{tr}}(p_1,\dots,p_n)) = 0$ provided $p_1 + \dots + p_k \notin \overline{V_+(m)}$ for some k where m is the mass of the theory. Also $\mathcal{F}(\mathcal{W}_n^{\mathrm{tr}}(p_1,\dots,p_n))$ is concentrated on the hyperplane $\sum\limits_{i=1}^{n} p_i = 0$.*

We are ready to prove the Proposition.

Let us denote: $\mathcal{W}'(v_1,\dots,v_n) = \mathcal{W}^{\mathrm{tr}}(v_{i_1},\dots,v_{i_k}; v_{j_1}, v_{j_2},\dots,v_{j_{n-k}})$ and $\mathcal{W}''(v_1,\dots,v_n) = \mathcal{W}^{\mathrm{tr}}(v_{j_1},\dots,v_{j_{n-k}}; v_{i_1},\dots,v_{i_k})$ where $i_1 < i_2 < \dots < i_k$ are the elements of I, and $j_1 < \dots < j_{n-k}$ are the elements of J. From the Lemma it follows that

$$\mathcal{F}(\mathcal{W}')(p_1,\dots,p_n) = 0 \quad \text{if} \quad p_{i_1} + \dots + p_{i_k} \notin \overline{V_+(m)};$$
$$\mathcal{F}(\mathcal{W}'')(p_1,\dots,p_n) = 0 \quad \text{if} \quad -(p_{i_1} + \dots + p_{i_k}) \notin \overline{V_+(m)}$$

Let $\widetilde{\chi}$ be a smooth function of one variable, such that $\widetilde{\chi}(x) = 1$ for $x > m$, $\widetilde{\chi}(x) = -1$ for $x < -m$, and its Fourier transform has asymptotics $\widetilde{\varphi} = \mathcal{F}(\widetilde{\chi}(t)) \sim e^{-m|t|}$ as $t \to \pm\infty$. [It is a standard fact from calculus that such a function exists].

Let us put $\chi(p_1,\dots,p_n) = \widetilde{\chi}(p_{i_1}^{(0)} + \dots + p_{i_k}^{(0)})$, where $p_i = (p_i^{(0)}, \vec{p}_i)$ for $p_i^{(0)} \in \mathbb{R}$, $\vec{p}_i \in \mathbb{R}^{d-1}$ is the usual decomposition. Then we see that $\chi\mathcal{F}(\mathcal{W}') = \mathcal{F}(\mathcal{W}')$, $\chi\mathcal{F}(\mathcal{W}'') = -\mathcal{F}(\mathcal{W}'')$. Let φ be Fourier transform of χ; so φ is concentrated on a 1-dimensional subspace. Then we have $\varphi * \mathcal{W}' = \mathcal{W}'$, $\varphi * \mathcal{W}'' = -\mathcal{W}''$, where $*$ stands for convolution.

Since $\mathcal{W}_n$ is a tempered distribution it can be represented as $\mathcal{W}_n = D(F)$ where F is a continuous function of polynomial growth and D is a differential operator with constant coefficients. We can assume that D is invariant under the action of symmetric group Σ_{n+1}. Then the functions $F' = F(v_{i_1},..,v_{i_k}; v_{j_1},..,v_{j_{n-k}})$ and $F'' = F(v_{j_1},..,v_{j_{n-k}}; v_{i_1},..,v_{i_k})$ satisfy: $D(F') = \mathcal{W}'$ and $D(F'') = \mathcal{W}''$. Then obviously: $\varphi * F' = F'$ and $\varphi * F'' = -F''$

The space-locality property implies that: $F'(v_1,\dots,v_n) = F''(v_1,\dots,v_n) = F(v_1,\dots,v_n)$ provided $(v_i - v_j)^2 < 0$ for $i \in I$, $j \in J$.

So for $|(a_i - a_j)^2| > (t_i - t_j)^2$ we have:

$$F(t_1 + a_1,\dots,t_n + a_n) = \tfrac{1}{2}(F'(t_1 + a_1,\dots,t_n + a_n) + F''(t_1 + a_1,\dots,t_n + a_n))$$
$$= \tfrac{1}{2}\varphi * (F' - F'') = \tfrac{1}{2}\int\limits_{-\infty}^{\infty} (F' - F'')((t_1 + a_1)^t, (t_2 + a_2)^t,\dots,(t_n + a_n)^t)\widetilde{\varphi}(t)dt$$

where we used the notation: $(t_i + a_i)^t = t_i + a_i - t$ if $i \in I$, and $(t_j + a_i)^t = t_j + a_j$ for $j \in J$. The expression in the integral vanishes if $t < \min(\sqrt{-(a_i - a_j)^2} - |t_i - t_j|)$. So since F is of polynomial growth we get the desired upper bound for the integral.

The crucial step of the proof is the following

Lemma 3.

$$|\mathcal{W}_n^{\mathrm{tr}}(g_1^t, \ldots, g_n^t)| \leq C(1 + |t|)^{(1-d)(n-2)/2}$$

for some C and all t, where $g_j^t = f_j^t$ or $g_j^t = \frac{\partial f_j^t}{\partial t}$ or complex conjugate for each j.

Proof of the Lemma. Let us work in Fourier coordinates; let pr be the projection $pr : V_n \to (\mathbb{R}^{d-1})^n$.

Of course $\mathcal{W}_n^{\mathrm{tr}}(g_1^t, \ldots, g_n^t) = \int \mathcal{F}(\mathcal{W}_n^{\mathrm{tr}})(-p)\mathcal{F}(g_1^t \times \cdots \times g_n^t) = \int pr_*(\mathcal{F}(W_n^{\mathrm{tr}})(-p) \cdot \mathcal{F}(g_1^t \times \cdots \times g_n^t))$. Let us look more closely at the last integral.

For this purpose it is convenient to write $\mathcal{F}(f_i)$ (respectively $\mathcal{F}(\overline{f}_i)$) in the form $\tilde{\psi}_i \cdot \tilde{\eta}_i$ where $\tilde{\psi}_i$ is a Schwartz function with support in Λ (respectively in $-\Lambda$), and $\tilde{\eta}_i$ is the pull-back of a Schwartz function under the projection on $\mathbb{R}^{d-1}$.

Let η_i be given by $\mathcal{F}(\eta_i) = \delta_{\mathcal{O}_m^+} \cdot \tilde{\eta}_i$ (respectively $\mathcal{F}(\eta_i) = \delta_{\mathcal{O}_m^-} \cdot \tilde{\eta}_i$).

Using the projection formula we can rewrite:

$$pr_* \left(\mathcal{F}(\mathcal{W}_n)(-p) \cdot \mathcal{F}(\prod g_i^t) \right) = pr_*[T_t \left(\mathcal{F}(\mathcal{W}_n)(-p) \cdot \prod \tilde{h}_i \right)] \cdot pr_*[T_{-t} \left(\mathcal{F}(\prod \eta_i) \right)]$$

Here $T_t(f)(p) = \exp(it \cdot p_0) f(p)$ and $\tilde{h}_i = \tilde{\psi}_i$ if $g_i^t = f_i^t$ or complex conjugate, and $\tilde{h}_i = \tilde{\psi}_i \cdot \pm i(p_0 - \sqrt{m^2 - \vec{p}^2})$ if $g_i^t = \frac{\partial f_i^t}{\partial t}$ or complex conjugate.

Let h denote $\mathcal{F}(\prod \tilde{h}_i)$. Then the integral we need to estimate equals to

$$\int\limits_{(\mathbb{R}^{d-1})^n} pr_* \Big[T_t \left(\mathcal{F}(\mathcal{W}_n)(-p) \cdot \mathcal{F}\left(\prod \tilde{h}_i \right) \right) \Big] \cdot pr_* \Big[T_t \left(\overline{\mathcal{F}\left(\prod \eta_i \right)} \right) \Big] = \int\limits_{(\mathbb{R}^{d-1})^n} i_t^*(F * h) \cdot i_{-t}^* \left(\prod \eta_i \right)$$

By Corollary to the Proposition $i_t^*(\mathcal{W}_n * h) = \pi^*(F_t^h)$, for some function $F_t^h \in L^1((\mathbb{R}^{d-1})^{n-1})$, and $|F_h^t|_{L^1} < C$ where C does not depend on t.

Thus we have:

$$\int\limits_{(\mathbb{R}^{d-1})^n} |pr^* i_t^*(\mathcal{W}_n * h) \cdot pr_1 i_{-t}^*(\eta_1)| \le C \cdot \int\limits_{\mathbb{R}^{d-1}} |j_t^*(\eta_1)| \le C \cdot (1+|t|)^{(d-1)/2}$$

by Lemma 1b), where j_t is the imbedding $\mathbb{R}^{d-1} \hookrightarrow V$, $\vec{v} \to (t, \vec{v})$), and $pr_i : (\mathbb{R}^{d-1})^n \to \mathbb{R}^{d-1}$ is the projection to the i-th component. Hence

$$\int\limits_{(\mathbb{R}^{d-1})^n} i_t^*(\mathcal{W}_n * h) \cdot i_{-t}^* \left(\prod \eta_i\right) \le C(1+|t|)^{(d-1)/2}$$

$$\cdot \max_{(\mathbb{R}^{d-1})^n} \prod_{k=2}^{n} pr_k^* j_{-t}^*(\eta_j) \le C(1+|t|)^{-(n-2)(d-1)/2}$$

where the last inequality follows from Lemma 1a). Lemma 3 is proved.

We are now ready to prove the Theorem. Denote $v^t = \phi(f_1^t)\dots\phi(f_n^t)\Omega$. To show that the limit $\lim\limits_{t\to\infty} v^t$ exists it suffices to check that the integral $\int\limits_0^\infty \left|\frac{\partial v^t}{\partial t}\right| dt$ converges. We have:

$$\left(\frac{\partial v^t}{\partial t}\right)^2 = \left(\sum_i \phi(f_1^t)\dots\phi(\dot{f}_i^t)\dots\phi(f_n)\Omega, \sum_j \phi(f_1^t)\dots\phi(\dot{f}_j^t)\dots\phi(f_n)\Omega\right)$$

$$= \sum_{i,j} \mathcal{W}_{2n}\left(\overline{f}_n^t, \overline{f}_{n-1}^t, \dots, \bar{\dot{f}}_i^t, \dots, \overline{f}_1^t, f_1^t, \dot{f}_j^t, \dots, f_n\right)$$

Let us decompose $\mathcal{W}_{2n}$ as sum of products of truncated Wightman functions, and consider the corresponding decomposition of each summand in the last sum.

Each factor containing $\mathcal{W}_1^{\mathrm{tr}}$ obviously vanishes. By the previous lemma each factor containing $\mathcal{W}_i^{\mathrm{tr}}$ is bounded by $\mathrm{const}\cdot(1+|t|)^{(i-2)(1-d)/2}$; in particular each factor is bounded.

Let us look at a factor of the form $\mathcal{W}_k^{\mathrm{tr}}(g_1^t, ..., g_k^t)$ where at least one of g_i^t is a time derivative of f_ℓ^t (or complex conjugate). If $k = 2$ then such a factor vanishes. Indeed, if say $g_1^t = \overline{\dot{f}_i^t}$ and $g_2^t = \dot{f}_j^t$ then $\mathcal{W}_2^{\mathrm{tr}}(g_1^t, g_2^t) = \left\langle \phi(f_i^t)\Omega, \phi(\dot{f}_j^t)\Omega \right\rangle = 0$ since $\phi(f_j^t)\Omega$ does not depend on t. (Here we use that $\operatorname{supp} \mathcal{F}(f_j) \cap \operatorname{supp} U\big|_V = \mathcal{O}_m^+$.) The other cases are absolutely parallel.

We now see that each non-zero summand contains a factor of the form $\mathcal{W}_k^{\mathrm{tr}}(g_1^t, \dots, g_k^t)$ with $k \ge 3$. Furthermore, it either contains at least 2 factors $\mathcal{W}_k^{\mathrm{tr}}$ with

$k \geq 3$ or at least one factor $\mathcal{W}_k$ with $k \geq 4$, because the term has the form $\mathcal{W}^{\mathrm{tr}}_{k_1} \dots \mathcal{W}^{\mathrm{tr}}_{k_\ell}$ with $k_1 + \dots + k_\ell = 2n$ being even, and each $k_i \geq 2$. In any case it is bounded by $\mathrm{const} \cdot (1+|t|)^{(1-d)/2 \cdot \sum (k_i - 2)} \leq \mathrm{const} \cdot (1+|t|)^{1-d}$. Thus $|\dot{v}^t| \leq \mathrm{const} \cdot (1+|t|)^{(1-d)/2}$ and $\int\limits_{-\infty}^{\infty} \dot{v}^t \, dt$ converges absolutely if $d \geq 4$.

Proof of b). Let us check that if $v^t = \phi(f_1^t) \dots \phi(f_k^t)\Omega$, $w^t = \phi(g_1^t) \dots \phi(g_\ell^t)\Omega$ for f_i, g_j as above then $\lim\limits_{t \to \pm\infty} (v^t, w^t) = (\phi_0(f_1) \dots \phi_0(f_k)\Omega, \phi_0(g_1) \dots \phi_0(g_\ell)\Omega)$. Let us again decompose $(v^t, w^t) = \mathcal{W}_{\ell+k}(\overline{f}^t_k, \overline{f}^t_{k-1}, \dots, \overline{f}^t_1, g_1^t, \dots, g_\ell^t)$ into sum of products of truncated Wightman functions. From the above arguments we see that every term containing $\mathcal{W}^{\mathrm{tr}}_1$ vanishes, and every term containing $\mathcal{W}^{\mathrm{tr}}_k$ for $k \geq 3$ tends to 0 as $t \to \pm\infty$. Besides, $\mathcal{W}^{\mathrm{tr}}_2(\overline{f}, g) = (\phi(f)\Omega, \phi(g)\Omega) = (\phi_0(f)\Omega, \phi_0(g)\Omega)$. Thus we have

$$\lim_{t \to \pm\infty} (\phi(f_1^t) \dots \phi(f_k^t)\Omega, \phi(g_1^t) \dots \phi(g_\ell^t)\Omega) = \lim_{t \to \pm\infty} (\phi_0(f_1^t) \dots \phi_0(f_k^t)\Omega,$$
$$\phi_0(g_1^t) \dots \phi_0(g_\ell^t)\Omega) = (\phi_0(f_1) \dots \phi_0(f_k)\Omega, \phi_0(g_1) \dots \phi_0(g_\ell)\Omega)$$

We can now finish the proof. For free field theory the space $\mathcal{D}$ is generated by the vectors $\phi_0(f_1) \dots \phi_0(f_n)\Omega$ for f_i as above, because $\phi_0(f)x$ depends only on $\mathcal{F}(f)\big|_{\mathcal{O}_m^\pm}$ for $f \in \mathcal{S}$, $x \in \mathcal{H}$.

For $x \in \mathcal{H}$ let us choose functions f_i such that $x = \sum \phi_0(f_{i_1}) \dots \phi_0(f_{i_s})\Omega$, and consider the limit $\lim\limits_{t \to \pm\infty} \sum \phi(f^t_{i_1}) \dots \phi(f^t_{i_s})\Omega$. We proved already that for any $y \in \mathcal{H}$, and functions g_j such that $y = \sum \phi_0(g_{j_1}) \dots \phi_0(g_{j_r})\Omega$ we have

$$\left(\lim_{t \to \pm\infty} \sum \phi(f^t_{i_1}) \dots \phi(f^t_{i_s})\Omega, \lim_{t \to \pm\infty} \sum \phi(g^t_{j_1}) \dots \phi(g^t_{j_r})\Omega \right) = (x, y).$$

Hence the limits do not depend on the choice of f_i, g_j, and give correctly defined isometric imbedding. The Theorem is proved.

4.4 Scattering matrix.

Proposition 2. *We have:* $\mathrm{Im}(W_+) = \mathrm{Im}(W_-)$.

Thus we can define the S-matrix $S = W_+^* W_- \in \mathrm{End}(\mathcal{H}_0)$. It is interesting that S can be described in terms of the Wightman functions of the theory.

The precise formulation is as follows. Let $\tilde{f}$ denote the partial Fourier transform of f in space variables. Vector $\vec{v} \in \mathbb{R}^{d-1}$ is called a velocity of f if we have $\vec{v} = \vec{p}/\sqrt{\vec{p}^2 + m^2}$ for $\vec{p}$ such that $(p_0, \vec{p}) \in \operatorname{supp}(\tilde{f})$ for some real number p_0.

Theorem. (Lehmann-Symanzik-Zimmerman)

a) *The residue* $\mathcal{F}(\mathcal{W}_n^T)\big|_{(\mathcal{O}_m^+)^n}$ *is a well-defined distribution on* V_n *supported on* $(\mathcal{O}_m^+)^n$, *where* $\mathcal{W}_n^T$ *is the time-ordered Wightman function (see lecture 2, §2.4 for the definition).*

b) *Assume* $f_i \in \mathcal{S}$ *and the set of velocities of* f_i *and* f_j *does not intersect for* $i \neq j$. *Then:*

$$\sum_{I \subset (1,\ldots,n)} (-1)^{|I|} \left\langle W_- \left(\prod_{i \in I} \phi_0(f_i)\Omega \right), W_+ \left(\prod_{j \notin I} \phi_0(f_j)\Omega \right) \right\rangle =$$
$$= \left\langle \operatorname{Res}(\mathcal{F}(\mathcal{W}_n^T)), \mathcal{F}(f_1) \times \ldots \times \mathcal{F}(f_n) \right\rangle .$$

It is an elementary combinatorial problem to deduce an expression for matrix coefficients of the S-matrix: $\langle W_- \prod \phi_0(f_1)\Omega, W_+ \prod \phi_0(f_j)\Omega \rangle$ in terms of $\mathcal{F}(\mathcal{W}^T)$ from the last formula.

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