Lecture 5.
Feynman graphs

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5.1Feynman graph expansion. The technique of Feynman graphs allows one to write down an asymptotic series for functional integral of a QFT in a neighborhood of a free QFT (see Witten's lecture 1). Nevertheless, the technique itself can be applied to a purely finite dimensional integral. In this lecture we will discuss this application only.

Let V be a finite dimensional real vector space, let V' be its dual. We will view elements of symmetric algebra $\mathrm{Sym}(V')$ as polynomial functions on V. We fix a nondegenerate positive definite quadratic form $b \in \mathrm{Sym}^2(V')$. Let $b^{-1} \in \mathrm{Sym}^2(V)$ be the corresponding quadratic form on V'.

Let $\mu_0 = e^{-b^{-1}/2}dv$ be the Gaussian measure on V'; we have $\mathcal{F}(\mu_0) = e^{-b/2}$.

Let P be a polynomial function on V'. We want to study the "perturbed" measure $\mu = e^{\varepsilon P}\mu_0$. This is a well-defined measure if $\varepsilon > 0$, and P is negative.

We are going to write down an asymptotic expansion for the Fourier transform $\mathcal{F}(\mu)$.

More precisely, we will obtain a formal Taylor series $\widetilde{\mu} \in \mathbb{C}[[V', \varepsilon]]$ in the following way.

Let $D(P)$ be the differential operator with constant coefficients on V corresponding to P. More precisely, D is a homomorphism of the algebra of polynomial functions on V' to the algebra of differential operators with constant coefficients on V, such that $D(x) = i\partial/\partial x$ for $x \in V$.

We write:

$$\mathcal{F}(e^{\varepsilon P}\mu_0) = e^{\varepsilon D(P)}(e^{-b/2}) = \sum_{n=0}^{\infty} \tfrac{1}{n!}\varepsilon^n D(P)^n(e^{-b/2}) \ .$$

This is obviously an element of $\mathbb{C}[[V', \varepsilon]]$.

This formal power series is connected with the original analytical problem via the notion of Borel summability. However in this lecture we will be interested in the formal expression only.

It is notationally convenient to perform "Wick rotation". So we consider the series

$$\widetilde{\mu}(v) = \mathcal{F}(e^{\varepsilon P}\mu_0)(iv) = \sum_{n=0}^{\infty} \tfrac{1}{n!}\varepsilon^n \widehat{P}^n(e^{b/2}) \ .$$

where $\widehat{P} = D(P(-ix))$, so that $\widehat{x} = \partial/\partial x$.

The series $\widetilde{\mu}$ is defined for arbitrary nondegenerate quadratic form b, and a polynomial function P over any field of characteristic 0. (The condition on b to be positive definite is of course irrelevant in this setting.)

There exists a combinatorial way to "compute" $\widetilde{\mu}$. To describe it we first fix some notations.

By a graph we will mean the data consisting of: two sets Γ_1 (edges) and Γ_0 (vertices) and a map $j\colon \Gamma_1 \to \Gamma_0 \times \Gamma_0$. We also assume an involution σ of Γ_1 is fixed such that $j(\gamma) = (x, y) \Leftrightarrow j(\sigma(\gamma)) = (y, x)$, and σ has no fixed points. We will only consider graphs with no isolated vertices (i.e. $\mathrm{pr}_1 \circ j$ is surjective). An edge of a graph is an element of Γ_1/σ. For a vertex $\gamma_0 \in \Gamma_0$ we define its "star" as $\Gamma(\gamma_0) = \{\gamma_1 \in \Gamma_1 | \ \mathrm{pr}_2 \circ j(\gamma_1) = \gamma_0\}$. The set of external vertices is $\Gamma_{\mathrm{ex}} = \{\gamma_0 \in \Gamma_0 | \#(\Gamma(\gamma_0)) = 1\}$, and the set of inner vertices is $\Gamma_{\mathrm{in}} = \Gamma_0 - \Gamma_{\mathrm{ex}}$. We put: $L(\gamma_0) = \mathrm{Sym}^{\#(\Gamma(\gamma_0))}(V)$; $\widetilde{L}(\Gamma) = \bigotimes_{\gamma_0 \in \Gamma_0} L(\gamma_0)$; $L(\Gamma) = \bigotimes_{\gamma \in \Gamma_{\mathrm{in}}} L(\gamma)$.

Assume that for any $\gamma_0 \in \Gamma_0$ we are given an element $s(\gamma_0) \in L(\gamma_0)$. We then take $\ell_s = \bigotimes_{\gamma_0 \in \Gamma_0} s(\gamma_0) \in \widetilde{L}(\Gamma)$. For any map $\tilde{s}\colon \Gamma_0 \to V$ we define $s_{\tilde{s}}(\gamma_0) = (\tilde{s}(\gamma_0))^{\#(\Gamma(\gamma_0))} \in L(\gamma_0)$, and denote $\ell_{\tilde{s}} = \ell_{s_{\tilde{s}}}$.

Lemma 1. *There exists a unique linear function $\widetilde{\tau}_b^\Gamma$ on $\widetilde{L}(\Gamma)$ such that for any map $\tilde{s}\colon \Gamma_0 \to V$ we have:*

$$\widetilde{\tau}_b^\Gamma(\ell_{\tilde{s}}) = \tfrac{1}{\#\mathrm{Aut}(\Gamma)} \prod_{\gamma_1 \in \Gamma_1/\sigma} b\left(\tilde{s}(\mathrm{pr}_1 \circ j(\gamma_1)), \tilde{s}(\mathrm{pr}_2 \circ j(\gamma_1))\right) \ .$$

Proof is clear.

We can view $\widetilde{\tau}_b^\Gamma$ as a map $\tau_b^\Gamma\colon L(\Gamma) \to \mathrm{Sym}^{\#\Gamma_{\mathrm{ex}}}(V')$.

Example. Γ_n is a disjoint union of n copies of the graph: . (The nonoriented edge on the picture corresponds to two elements of Γ_1 permuted by σ.) Then $L(\Gamma) = \mathbb{C}$

canonically, because Γ_{in} is empty. We have:

$$\tau_b^{\Gamma_n}(1) = \frac{b^n}{2^n n!} \ .$$

Note the equality $\sum\limits_{n=0}^{\infty} \tau_b^{\Gamma_n}(1) = e^{b/2}$ where Γ_0 is the empty graph.

Let P, $\widetilde{\mu}$ be as above. We write $P = \sum\limits_{n\geq 2} P_n/n!$ where P_n is homogeneous of degree n (thus we assume for convenience that P contains no constant or linear term).

Theorem. *We have an equality of formal series:*

$$\widetilde{\mu} = \left(\sum_{\Gamma\in\mathfrak{g}} \varepsilon^{\#\Gamma_{\text{in}}} \tau_b^{\Gamma}(s_P) \right) e^{b/2}$$

Here $\mathfrak{g}$ is the set of isomorphism classes of graphs and $s_P(\gamma_0) = P_{\#\Gamma(\gamma_0)}$.

Proof. [Another proof appears in §1.3 of Witten's lecture 1].

The idea of the proof is as follows. We want to prove an equality between two expressions, which are power series in ε and in P_n's. We will show that both the left hand side and the right hand side satisfy the same system of linear differential equations in P_n's as variables, and then we will check that the initial conditions coincide.

So, let Γ be a graph and let $\lambda \subset \Gamma_{\text{ex}}$, be such that $|\lambda| = n$. We denote by Γ^λ the "cone over λ". This means that Γ^λ is a graph, such that $\Gamma_0^\lambda = \Gamma_0 \cup \{*\} - \lambda$ and $\Gamma_1^\lambda = \Gamma_1 - \{$all edges of Γ with an end in $\lambda\} \cup \{$all edges from $*$ to vertices connected with $\lambda\}$. (We add all edges connecting $*$ to a vertex in λ, and then erase the vertices belonging to λ.) Assume now that $n > 1$. Then $*$ is an internal vertex of Γ^λ.

For any $d \in S^n(V)$ and $s \in L(\Gamma)$ we define $s_\lambda^d = d \otimes s \in L(\Gamma^\lambda)$.

Lemma 2. *For any $d \in S^n(V)$, $\Gamma \in \mathfrak{g}$ and $s \in L(\Gamma)$ we have*

$$\frac{\hat{d}}{n!}(\tau_b(\Gamma)(s)) = \sum_{\lambda\subset\Gamma_{\text{ex}}, |\lambda|=n} \tau_b(\Gamma^\lambda)(s_\lambda^d)$$

(Here $\hat{d}$ is the differential operator with constant coefficients on V corresponding to d.)

Now we can prove the theorem. Let $l(P)$ denote the left hand side and let $r(P)$ be the right hand side of the equality. We want to write down a certain system of differential equations in P_n's as variables, which will be satisfied by both sides. It is easy to see from the definitions that $l(P)$ satisfies the system of equations

$$\frac{\partial l(P)}{\partial P_n}(Q) = (\varepsilon \widehat{Q}/n!)l(P)$$

where Q is an arbitrary polynomial of degree n on V' and $\widehat{Q}$ is the corresponding differential operator on V; in the left hand side Q is considered as a tangent vector to the space of polynomials of degree n.

But now Lemma 2 tells us precisely that the right hand side of this expression is equal to $\frac{\partial r(P)}{\partial P_n}(Q)$.

Therefore $l(P)$ and $r(P)$ satisfy the same system of differential equations. Let's now check the initial conditions. By the definition, $l\big|_{P=0} = e^{b/2}$. On the other hand, one also has $r\big|_{P=0} = e^{b/2}$ by the above Example. Hence we have $l(P) = r(P)$ for any P, so the theorem is proved.

Remark 1. The theorem and its proof along with all the statements below can be immediately generalized in the following way. We can take P to be an element of $\mathbb{C}[V][\![\varepsilon]\!]$ rather than a polynomial, i.e. $P = \sum_{m=0}^{\infty} \varepsilon^m P_{(m)}$ is a Taylor series in ε with polynomial functions on V' as coefficients.

Remark 2. This way of writing an asymptotic expansion for a measure uses Fourier transform, and hence the linear structure of V heavily. For example when one applies the Feynman graph expansion to the gauge theory, V being the space of connections modulo gauge transformation, then V has no natural linear structure. In this case one still can write an asymptotic expansion for functional integral, but an individual term assigned to a particular graph is not canonically defined.

We denote $Z = \sum_{\Gamma \in \mathfrak{g}} \tau_b^{\Gamma}(s_P)\varepsilon^{\#\Gamma_{\mathrm{in}}}$.

One of the difficulties in using this expansion is that there are "too many" terms; so we try to reduce it somehow. We denote $F = \sum_{\Gamma \in \mathfrak{g}_{\mathrm{con}}} \tau_b^{\Gamma}(s_P)\varepsilon^{\#\Gamma_{\mathrm{in}}}$, where $\mathfrak{g}_{\mathrm{con}}$ is the set of connected nonempty graphs.

Note that F (and all series that will be introduced below) lie in a subring $\mathbb{C}[V'][\![\varepsilon]\!] \subset \mathbb{C}[\![V', \varepsilon]\!]$.

Claim 1. $F = \log Z$.

Proof. Decomposing a graph into union of its irreducible components one can identify the two combinatorial expressions:

$$Z = \sum_{\Gamma \in \mathfrak{g}} \tau_b^\Gamma = \sum_{\Gamma_1,..,\Gamma_k \in \mathfrak{g}_{\mathrm{con}}; n_1,..,n_k} \frac{1}{n_1! n_2! .. n_k!} \tau_b^{\Gamma_1} \dots \tau_b^{\Gamma_k} = \exp(F)$$

where the summation is over all sets of distinct elements $\Gamma_i \in \mathfrak{g}_{\mathrm{con}}$.

We next want to study the "quasi-classical approximation" to our integral. So we introduce another variable $\hbar$ and consider the expression:

$$F_\hbar(v) = \log \mathcal{F}\left(e^{(b^{-1}/2+\varepsilon P)/\hbar} dv\right)(\hbar v)\ .$$

As an immediate corollary to the Theorem, we get:

Lemma 3. $F_\hbar = \hbar^{-1} \sum_{\Gamma \in \mathfrak{g}_{\mathrm{con}}} \hbar^{h^1(\Gamma)} \tau_b^\Gamma(s_P) \varepsilon^{\#\Gamma_{\mathrm{in}}}$ *where* $h^1(\Gamma) = \dim(H^1(\Gamma)) = \#\Gamma_1/\sigma - \#\Gamma_0$.

Proof. Apply the same formal procedure to the quadratic form $\tilde{b} = \hbar b$ and polynomial $\widetilde{P} = \hbar^{-1} P$, and note that for a connected graph Γ we have $\tau_{\hbar b}^\Gamma(s_{P/\hbar})(\hbar v) = \hbar^{h^1(\Gamma)-1} \tau_b^\Gamma(s_P)(v)$.

5.2 Quasi-classical (low-loop) approximations. Recall that the "classical" approximation to Fourier transform is the Legendre transform. An appropriate version of the definition of Legendre transform is as follows.

If f is a function on V', then we can view its differential df as a map $df \colon V' \to V$.

If df is an isomorphism then the Legendre transform of f is defined by

$$L(f)(v) = \left\langle v, (df)^{-1}(v) \right\rangle - f\left((df)^{-1}(v)\right)$$

i.e. $L(f)(v)$ is the critical value of the function $v - f$ where v is considered as a linear function on V'.

Let now $G \in \mathbb{C}[\![V, \varepsilon]\!]$ be a formal power series. Then its differential dG (in the V'-direction) is an element of $\mathrm{Hom}(V', V) \otimes \mathbb{C}[\![V, \varepsilon]\!]$.

Assume that $G = g_0 + g_1 + \varepsilon G_1$ where g_0 is a nondegenerate quadratic form on V', and g_1 contains no terms of degree less then 3 in V. Then dG is invertible, i.e. there exists $H = (dG)^{-1} \in \mathrm{Hom}(V, V') \otimes \mathbb{C}[\![V, \varepsilon]\!]$ with no constant term such that $dG \circ H = id$ and $H \circ dG = id$. In this situation the Legendre transform $L(G) \in \mathbb{C}[[V', \varepsilon]]$ can be defined by the same formula.

To describe the next term of the asymptotics we define for any function f on V' as above a new function $H(f)$ on V as follows. Let $Hess_f(p) = \det\left(\frac{\partial^2 f}{\partial v_i \partial v_j}\right)|_p$ be the Hessian of f (we assume that a constant volume form on V is fixed so that the determinant of the quadratic form is taken with respect to that volume form).

We put: $H(f)(v) = Hess\left((df)^{-1}(v)\right)$.

For $G = g_0 + g_1 + \varepsilon G_1 \in \mathbb{C}[[V', \varepsilon]]$ as above we can define $Hess(G)$ and $H(G) \in \mathbb{C}[[V', \varepsilon]]$ by the same formula.

Claim 2. a) *Let $F_0 = \sum_{\Gamma \in T} \tau_b^\Gamma(s_P)\varepsilon^{\#\Gamma_{\mathrm{in}}}$, where T is the set of (nonempty) trees. Then F_0 is the Legendre transform of $\frac{b^{-1}}{2} - \varepsilon P$.*

b) *Let $F_1 = \sum_{\Gamma \in \mathfrak{g}_1} \tau_b^\Gamma(s_P)\varepsilon^{\#\Gamma_{\mathrm{in}}}$ where $\mathfrak{g}_1$ is the set of one-loop connected graphs (i.e. graphs with $h^0(\Gamma) = h^1(\Gamma) = 1$. Then $F_1 = \log\left(H\left(\frac{b^{-1}}{2} - \varepsilon P\right)^{-1}\right)$.* (We assume that the background volume form is such that $\det(\frac{b^{-1}}{2}) = 1$. Then $H(\frac{b^{-1}}{2} - \varepsilon P) \in 1 + \varepsilon\mathbb{C}[\![V', \varepsilon]\!] + V'\mathbb{C}[\![V', \varepsilon]\!]$, and $\log(H(\frac{b^{-1}}{2} - \varepsilon P))$ is a well-defined element of $\mathbb{C}[\![V', \varepsilon]\!]$).

Proof of the Claim. From Lemma 3 we see that $F_\hbar(v) = F_0(v)/\hbar + F_1(v) + O(\hbar)$. Thus the claim follows from the stationary phase approximation applied to the integral

$$\mathcal{F}(f)(iv) = \frac{1}{(2\pi)^{d/2}} \int_{V'} e^{(v,p) + (\varepsilon P - b^{-1}/2)(p)} dp$$

One can also directly show the equality of the two combinatorial expressions:

$$\begin{aligned} F_0\left(d\left(\frac{b^{-1}}{2} - \varepsilon P\right)|_p\right) &= \frac{b^{-1}}{2}(p) - \varepsilon \sum_n \frac{P_n}{(n-1)!}(p) + \varepsilon \sum_n \frac{P_n}{n!}(p) \\ &= \langle p, d\Big(\frac{b^{-1}}{2} - \varepsilon P\Big)|_p\rangle - (b^{-1}/2 - \varepsilon P)\Big(d\big(\frac{b^{-1}}{2} - \varepsilon P\big)\Big) \end{aligned} \tag{5.1}$$

and

$$F_1\Big(d(\frac{b^{-1}}{2}-\varepsilon P)|_p\Big)=-\log Hess\Big(\frac{b^{-1}}{2}-\varepsilon P\Big) \tag{5.2}$$

Here (5.1) is equivalent to $F_0=L\big(\frac{b^{-1}}{2}-\varepsilon P\big)$ and (2) is equivalent to $F_1=\log\big(H\big(\frac{b^{-1}}{2}-\varepsilon P\big)^{-1}\big)$.

Let us sketch the combinatorial proof of (5.1) and (5.2).

Let us identify V and V' by means of b and use an equality $\varepsilon^{\#\Gamma_{\rm in}}\tau_b^\Gamma(s_P)\circ d\big(\frac{b^{-1}}{2}-\varepsilon P\big)=\sum(-1)^{c_i}\varepsilon^{\#\Gamma^i_{\rm in}}\tau_b^{\Gamma^i}(s_P)$ where the graph Γ^i is obtained from the graph Γ by adding c_i vertices of arbitrary valencies, each one of which is connected with an external vertex of Γ in such a way that any external vertex of Γ is connected with no more then one new vertex.

Now it is not hard to check that in the expansion of $F_0\circ\big(d\big(\frac{b^{-1}}{2}-\varepsilon P\big)\big)$ all terms with ε^i for $i\geq 2$ cancel, and identify the sum of remaining terms with the RHS of (5.1).

Likewise in the expansion of $F_1\circ\big(d\big(\frac{b^{-1}}{2}-\varepsilon P\big)\big)$ only the terms with one-particle irreducible one-loop graphs ("circles") do not cancel; the sum of these terms is identified with $\sum_{n=1}^\infty\varepsilon^i\mathrm{tr}(dP)^n/n=-\log\det(Id-dP)$ which coincides with the RHS of (5.2).

5.3 Effective potential. Let us call a nonempty graph 1-*particle irreducible* (or just *1-irreducible*) if it is connected and remains connected after removal of any internal edge. To any connected graph there corresponds a unique tree with a 1-particle irreducible graph assigned to each vertex together with an identification of the set of edges coming to a vertex of a tree with the set of external vertices of the corresponding graph. Thus "computation" of F_0 can be reduced to summation over trees and over 1-particle irreducible graphs separately. Denote $F_{1-\rm irr}=\sum_{\Gamma\in\mathfrak{g}_{1-\rm irr}}\varepsilon^{\#\Gamma_{\rm in}}\tau_b^\Gamma(s_P)$ where $\mathfrak{g}_{1-\rm irr}$ is the set of 1-particle irreducible graphs. (Notice that the graph $\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$ is 1-particle irreducible by the definition).

Claim 3. $F_{1-\rm irr}=b-b^*(L(F))$ *where* b *is viewed as a map from* V *to* V' *and* L *is the Legendre transform.*

Proof. By Remark 1 after the proof of the theorem we can apply claim 2 to $P'=\varepsilon^{-1}(b^{-1})^*(F_{1-\rm irr}-b/2)\in\mathbb{C}[V']⟦\varepsilon⟧$.

From the above combinatorial observation we see that $F = F_P = (F_0)_{P'}$. Hence claim 2 implies that $F = L(b^{-1}/2 - \varepsilon P') = L\left((b^{-1})^*(b - F_{1-\mathrm{irr}})\right)$. Since Legendre transform is involutive we get the claim.

$F_{1-\mathrm{irr}}$ is called the effective potential of the theory.