Notes on spinors

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The present notes owe a lot to Bourbaki's treatment in Alg., Ch. 9. I have learned of the analogy between spinorial and oscillator representations (see 3.5, 3.8, 3.10) from a lecture R. Howe gave in 1978. The §5 has been inspired by a conversation with Witten.

1. Super Brauer group.

Our aim is to explain and investigate the super analogues of the notions of division algebra, semi-simple algebra, central simple algebra and Brauer group.

The super Brauer group has been introduced by C. T. C. Wall: Graded Brauer Groups, J. Reine Angew. Math. 213 (1963-64), p. 187–199. In loc. cit., Theorem 3 p. 194, a description of its structure more precise than our 1.21 is given.

1.1. The sign rule forces the following definitions. The *opposite* A^0 of a super ring A has the same underlying additive group, and its multiplication law is given by

$$(x, y) \longmapsto (-1)^{p(x)p(y)} y\, x$$

for x, y homogeneous, with parity $p(x), p(y) \in \mathbb{Z}/2$. A left super A-module is turned into a right super A^0-module by

$$m\, a = (-1)^{p(m)p(a)} a\, m.$$

If M an N are two left super modules, $\underline{\mathrm{Hom}}_A(M, N)$ is mod 2 graded, with even (resp. odd) part the additive group of even (resp odd) additive morphisms $f\colon M \to N$ such that

$f(am) = (-1)^{p(f)p(a)}a, f(m)$. The even part of $\underline{\mathrm{Hom}}_A(M, N)$ is the group $\mathrm{Hom}_A(M, N)$ of morphisms from M to N in the category of left super modules .

If f: $A \to B$ is a morphism of super rings, the *commutant* of A in B is the super ring of $b \in B$ for which the super commutator $ab - (-1)^{p(a)p(b)}ba$ vanishes for $a \in f(A)$. The *commutant* A' of a left super A-module M is the commutant $\underline{\mathrm{End}}_A(M) = \underline{\mathrm{Hom}}_A(M, M)$ of A in $\underline{\mathrm{End}}_{\mathbb{Z}}(M)$. Example: for M the left module A, $A' = \underline{\mathrm{End}}_A(A)$ is identified with A^0 by $f \mapsto f(1)$. If we view M as a right A'^0-module, the ring of right multiplications by elements of A'^0 is the commutant in the ordinary, ungraded, sense. The bicommutant, i.e. the commutant of M viewed as an A'-module, coincide with its ungraded analogue.

The *center* of A is the commutant of A in A.

If Z is super commutative and A, B are super algebras over Z, i.e. are provided with a: $Z \to A$ and b: $Z \to B$, the tensor product algebra $A \otimes_Z B$ is $A \otimes_Z B$ with the product law $(a \otimes b).\ (a' \otimes b') = (-1)^{p(b)p(a')}(aa') \otimes (bb')$: the image of A and B in $A \otimes_Z B$ super commute.

A divisor ring (= field, possibly non commutative) is a ring D such that the left D-module D is simple. Adding "super" everywhere, one obtains the notion of super division ring. Equivalent definition: $D \neq 0$ and every homogeneous element $x \neq 0$ is invertible. The usual proof gives:

Proposition 1.2. *The commutant of a simple super module is a super division ring.*

Proof. If $a' \in A'$ is homogeneous, its kernel and image are graded submodules...

A left super module M is *semi-simple* if it is a sum of simple super modules.

Proposition 1.3. *Assume that* 2 *is invertible in* A. *Then, a left super-module* M *is semi-simple if and only if the underlying ungraded module* $|M|$ *is.*

Proof. Only if: it suffices to see that if M is simple, the underlying ungraded module $|M|$ is semi-simple. Let M_1 be the sum of the simple submodules of $|M|$. The parity map α: $x \mapsto (-1)^{p(x)}x$ is an automorphism of (A, M). By transport of structures, M_1 is stable by α. It is hence a graded submodule (we use here that 2 is invertible). The simplicity of M forces $M = M_1$ and $|M| = M_1$ is semi-simple.

If: It suffices to see that any super submodule N of M is a direct factor. By assumption, $|N|$ is a direct factor of $|M|$, meaning that there is a retraction $f\colon |M| \to |N|$. The map $(f + \alpha(f))/2$ is a graded retraction; it makes N a direct factor of M.

From now on in this §, k is a field of characteristic $\neq 2$. We say <u>*algebra*</u> *and* <u>*module*</u> *for super algebra and super left module. Non super objects will be called ungraded. Algebras and modules are assumed finite dimensional over the ground field (usually k).*

1.4. An algebra A is *semi-simple* if the category of its modules is semi-simple. Equivalent conditions: the A module A is semi-simple; A admits a faithful semi-simple module M.

Proof. We write $\prod$ for the parity change functor $X \mapsto X \otimes k$, for k viewed as an odd k-vector space, with basis noted 1^-. The equivalence of the conditions follows from: any module is a quotient of some $A^n \oplus (\prod A)^m$; if M is faithful, A is a submodule of some $M^n \oplus (\prod M)^m$.

Corollaries of 1.3:

Corollary 1.5. *An algebra is semi-simple if and only if the underlying ungraded algebra is.*

Corollary 1.6. *If M is a semi-simple module, A maps onto its bicommutant A''.*

By 1.3 and the identity 1.1 of the bicommutant with its ungraded analog, we are reduced to the ungraded case (Bourbaki Alg. Ch. 8, §4 n^o 2, Th. 1)

Corollary 1.7. *If k' is a separable extension of k, A is semi-simple if and only if the k'-algebra $A \otimes_k k'$ is.*

By 1.5, we are reduced to the ungraded case (Bourbaki Alg. Ch. 8, §7 n^o 3, cor. 2 to prop. 3).

1.8. An algebra A is *central simple* if it is semi-simple with center k. The condition "center k" implies that $A \neq 0$. It does not imply that the underlying ungraded algebra $|A|$ has a center reduced to k. Example: the algebra $k[\varepsilon]$ with ε odd, $\varepsilon^2 = 1$ is a (super) division algebra, hence is semi-simple. The non vanishing of the super commutator $[\varepsilon, \varepsilon] = 2\varepsilon^2$ shows that its center is reduced to k: it is central simple, but $|A|$ is not.

Corollary 1.9. *If k' is a separable extension of k, A is central simple if and only if the k'-algebra $A \otimes_k k'$ is.*

Proof. Results from 1.7, and from the fact that taking the center is compatible with extension of scalars. Indeed, the center is the intersection of the kernel of the super commutators maps adx, for x running over a system of generators of the algebra.

Remark. The conclusion of 1.9 holds for any field extension k'/k. This results from 1.14.

1.10. Let D be a division algebra. If D is purely even, D has two isomorphism classes of simple modules: D, viewed as purely even, or as purely odd. If D is not purely even, D has only one isomorphism class of simple modules. In the purely even case, $M_{r,s}(D)$ is correspondingly isomorphic to $M_{r+s}(D)$. By "matrix algebra over D", we mean in all cases an algebra $M_{r,s}(D)$, with $r+s>0$.

Proposition 1.11. *An algebra A is central simple if and only if it is isomorphic to a matrix algebra over some division algebra D with center k.*

Proof. Let A be a k-algebra. If F is a composition series of the A-module A, any simple A-module occurs, up to parity change, in the successive quotients of F. It follows that they are only a finite number of isomorphism classes of simple modules. Let $S_1,\dots,S_n$ be a system of representatives of them, up to parity change. Let D_i be the commutant of S_i. If D_i is not purely even, the D_i-module S_i is isomorphic to D_i^n for some $n \neq 0$. if D_i is purely even, $D_i^{n|m}$ occurs as well. By 1.6,

$$A \longrightarrow \prod \underline{\mathrm{End}}_{D_i}(S_i)$$

is onto. It is an isomorphism if A is semi-simple.

If A is central simple, we have $n=1$: up to parity change, A has exactly one isomorphism class of simple modules S. If D is its commutant, we have

$$A \xrightarrow{\sim} \underline{\mathrm{End}}_D(S) = M_n(D^0) \quad \text{or}$$
$$M_{n,m}(D^0)$$

for D^0 the opposite algebra. Indeed, $\mathrm{End}_D(D) = D^0$. By 1.6, the center $A \cap D$ of A is also the center of D: if A is central simple, the center of D is reduced to k and A has the required form. The converse is left to the reader. The proof gave the

Complement 1.12. The algebra D of 1.11 is $(\mathrm{End}_A(S))^0$ for S the unique simple A-module (up to isomorphism and parity change). It is hence determined, up to isomorphism, by A.

Proposition 1.13. *If k is separably closed, the only division algebras D with center k are, up to isomorphism, k and $k[\varepsilon]$ with ε odd and $\varepsilon^2 = 1$.*

Proof. The even part D^+ of D is an ordinary division algebra. Let Z be its center. The center of D being reduced to k, it follows that the fixed point set of the automorphisms xzx^{-1} ($x \in D^-$, $x \neq 0$) of Z is k. We see that Z is a Galois extension of k. As k is separably closed, $Z = k$ and $D^+ = Z = k$, because $\mathrm{Br}(k) = 0$.

If D is purely even, we get $D = k$. If not, pick $\varepsilon \neq 0$ in D^-. It is invertible. It follows that $D^- = \varepsilon D^+$ and that $\varepsilon^2 \neq 0$. As k is separably closed of characteristic $\neq 2$, ε^2 has a square root λ in k. Replacing ε by $\lambda^{-1}\varepsilon$, we obtain that D is isomorphic to the $k[\varepsilon]$ of 1.13.

Corollary 1.14. *An algebra $A \neq 0$ is central simple if and only if*

$$A \otimes A^0 \xrightarrow{\sim} \mathrm{End}_k(A). \tag{1.14.1}$$

The map is defined by the commuting A and A^0-module structures of A, the A^0-module structure coming from $A^0 = \mathrm{End}_A(A)$.

If A is not semi-simple (resp. with center k), the image of $A \otimes A^0$ in $\mathrm{End}_k(A)$ respects the radical of A (resp. commutes with the center), hence cannot be all of $\mathrm{End}_k(A)$.

We now assume A central simple and prove that $A \otimes A^0 \xrightarrow{\sim} \mathrm{End}_k(A)$. Extending the scalars, we may assume that k is separably closed. By 1.11 and 1.13, one has then $A \sim M_{r,s} \otimes D$, with D as in 1.13. this reduces us to the cases of $M_{r,s}$, left to the reader, and D. If $D = k$, the statement is clear. For $K = k[\varepsilon]$, in the basis $(1, \varepsilon)$, the left multiplication by $\varepsilon \in D$ (resp. $\varepsilon \in D^0$) is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{(resp. } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{)}$$

which suffices to generate the 2×2 matrix algebra.

Corollary 1.15. *The tensor product of two non zero algebras is central simple if and only if each is.*

Proof. The map (1.14.1) for a tensor product is the tensor product of the maps (1.14.1) for each factor.

Definition 1.16. (i) *Two central simple algebras are* similar *if they are matrix algebras over isomorphic division algebras (cf. 1.11, 1.12).*

(ii) *The* super Brauer group $\mathrm{sBr}(k)$ *is the set of similarity classes of central simple algebras over k, with composition law induced by $\otimes$.*

The composition law is associative and commutative, by associativity and commutativity of the tensor product. It has a unit: the class of matrix algebras. By 1.14, the class of A^0 is inverse to that of A: $\mathrm{sBr}(k)$ is a commutative group.

1.17 Example. By 1.13, if k is separably closed, then $\mathrm{sBr}(k) = \mathbb{Z}/2$.

The trivial element in $\mathrm{sBr}(k)$ corresponds to the central simple algebras $A = \underline{\mathrm{End}}_k(S)$, for S a (super) vector space. The commutant Z of A^+ in A is contained in A^+: it is the center Z of A^+ and consists of those endomorphisms of S which act as scalars on S^+ and on S^-. If A is not purely even, i.e. if S is not homogeneous, $Z \simeq k \times k$, and the two homomorphisms $\chi\colon Z \to k$ correspond to the two isomorphism classes of simple A-modules T (permuted by parity change) by $T \mapsto \chi$ such that $zv = \chi(z)v$ for $v \in T^+$.

The non trivial element in $\mathrm{sBr}(k)$ corresponds to the central simple algebras $A = M_n(D)$, $D = k[\varepsilon]$ as above. Here, A^+ is central simple and its commutant Z in A is of dimension $(1,1)$ (isomorphic to D). One has $A^+ \otimes Z \xrightarrow{\sim} A$.

Proposition 1.18. *the super Brauer group of $\mathbb{R}$ is $\mathbb{Z}/8$. It is generated by the super division algebra $\mathbb{R}[\varepsilon]$ with ε odd and $\varepsilon^2 = 1$.*

Proof. We first enumerate all super division algebras D over $\mathbb{R}$. If, after extending scalars to $\mathbb{C}$, D becomes isomorphic to a matrix algebra over $\mathbb{C}[\varepsilon]$ (non trivial class in $\mathrm{sBr}(\mathbb{C})$), then D^+ is a division algebra, its commutant Z is a division algebra of dimension $(1,1)$ and $D^+ \otimes Z \xrightarrow{\sim} D$. Indeed, all those assertions hold after extension of scalars to $\mathbb{C}$, and are invariant by extension of scalars. This gives four possibilities: $D^+ = \mathbb{R}$ or $\mathbb{H}$, and $Z = \mathbb{R}[\varepsilon]$ with ε odd and $\varepsilon^2 = \pm 1$. It follows already that $\mathrm{sBr}(\mathbb{R})$ is of order 8.

For D purely even, we get the usual Brauer group: $D = \mathbb{R}$ or $\mathbb{H}$.

If D is not purely even and becomes a matrix algebra over $\mathbb{C}$, the center of D^+ is a non trivial extension Z of $\mathbb{R}$, as one checks after extension of scalars to $\mathbb{C}$. It follows that one has $C \simeq \mathbb{C}$ and D^+, central simple over Z, is reduced to $\mathbb{C}$ as well. Fix $\varepsilon \neq 0$ in D^-. One has $D^- = D^+.\varepsilon$. the center of D being reduced to $\mathbb{R}$, the automorphism $\varepsilon z \varepsilon^{-1}$ of $\mathbb{C}$ must be non trivial: $\varepsilon z \varepsilon^{-1} = \bar{z}$. It fixes ε^2: $\varepsilon^2 \in \mathbb{R}$. Changing ε to εz changes ε^2 to $\varepsilon^2 z \bar{z}$. This gives the two last cases

$$D = \mathbb{C}[\varepsilon] \quad \text{with } \varepsilon \text{ odd, } \varepsilon z \varepsilon^{-1} = \bar{z} \text{ and } \varepsilon^2 = \pm 1.$$

We now check that the map from $\mathbb{Z}/8$ to the group $\mathrm{sBr}(\mathbb{R})$ defined by $1 \mapsto$ (class of $\mathbb{R}[\varepsilon]$, ε odd, $\varepsilon^2 = 1$), is as follows:

$$\begin{array}{ll}
0 \longmapsto \mathbb{R} & 4 \longmapsto \mathbb{H} \\
1 \longmapsto \mathbb{R}[\varepsilon], \varepsilon^2 = 1 & 5 \longmapsto \mathbb{H}[\varepsilon], \varepsilon h = h\varepsilon, \varepsilon^2 = 1 \\
2 \longmapsto \mathbb{C}[\varepsilon], \varepsilon z = \bar{z}\varepsilon, \varepsilon^2 = 1 & 6 \longmapsto \mathbb{C}[\varepsilon], \varepsilon z = \bar{z}\varepsilon, \varepsilon^2 = -1 \\
3 \longmapsto \mathbb{H}[\varepsilon], \varepsilon h = h\varepsilon, \varepsilon^2 = -1 & 7 \longmapsto \mathbb{R}[\varepsilon], \varepsilon^2 = -1
\end{array}$$

The image of 2 is the class of $\mathbb{R}[\varepsilon_1, \varepsilon_2]$ with $\varepsilon_i^2 = 1$, $\varepsilon_1\varepsilon_2 = -\varepsilon_2\varepsilon_1$. One has $(\varepsilon_1\varepsilon_2)^2 = -1$ and the claim for 2 follows. Passing to the opposite algebra, we get the image of 6. It is not the same as the image of 2, and it follows that $\mathrm{sBr}(\mathbb{R})$ is indeed cyclic of order 8. As the class of $\mathbb{H}$ is of order 2, it must be the image of 4. One concludes by using that $5 = 4 + 1$ and using passages to the opposite algebra.

Remark 1.19. Let A be a central simple algebra over $\mathbb{R}$ with a class n in $2.\mathbb{Z}/8$, i.e. such that $D_{\mathbb{C}}$, deduced from A by extension of scalars, is a matrix algebra. The algebra $A_{\mathbb{C}}$ has two isomorphism classes of simple modules, permuted by parity change.

Each class is defined over $\mathbb{R}$, i.e. is fixed by $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$, if and only if the center Z of A^+ is $\mathbb{R}$ (if A is purely even) or $\mathbb{R} \times \mathbb{R}$. This happens if and only if $n \equiv 0(4)$. if $n \equiv 0(8)$, each isomorphism class can be realized over $\mathbb{R}$: A is a matrix algebra. If $n \equiv 4(8)$, it cannot: A admits only a module M which by extension of scalars becomes twice a simple module over $A_{\mathbb{C}}$. Its commutant is $\mathbb{H}$.

If $n = 2(8)$, the ungraded algebra A admits an ungraded module M which, after extension of scalars, becomes the underlying ungraded module to a simple $A_{\mathbb{C}}$-module. The even and odd components of $M_{\mathbb{C}}$ are permuted by complex conjugation.

If $n \equiv 6(8)$, one can only find M which, after extension of scalars, becomes twice such a module. Again, the even and odd part are permuted by complex conjugation. The commutant of M (as ungraded module) is $\mathbb{H}$.

If A is central simple with a class n in $\mathbb{Z}/8$ which is odd, A^+ is a matrix algebra if $n = 1$ or $7 \mod 8$, and a matrix algebra over $\mathbb{H}$ otherwise. In the first case, the unique simple $A^+_{\mathbb{C}}$-module can be realized over $\mathbb{R}$. In the second, it cannot. The centralizer Z of A^+ in A is isomorphic to $\mathbb{R} \times \mathbb{R}$ if $n \equiv 1$ or $5(8)$, to $\mathbb{C}$ otherwise.

1.20 Remark. For A a central simple algebra over $\mathbb{R}$, let t be the trace for the underlying ungraded algebra, sgn^+ and sgn^- be the signatures of the quadratic forms $t(xy)$ on A^+ and A^-, respectively, and define

$$s(A) := \mathrm{sgn}^+ + \mathrm{sgn}^- i \in \mathbb{Z}[i] \subset \mathbb{C}$$

For a tensor product $A_1 \otimes A_2$, t vanishes on the $A_1^\pm \otimes A_2^\pm$ other than $A_1^+ \otimes A_2^+$, where it is the tensor product of t for A_1 and A_2. On $A_1^\pm \otimes A_2^\pm$, $t(xy)$ is related to the tensor product of the bilinear forms $t_1(x_1y_1)$ and $t_2(x_2y_2)$ by

$$\begin{aligned} A_1^+ \otimes A_2^+ &: \ t(xy) = t_1(x_1y_1) \otimes t_2(x_2y_2) \\ A_1^+ \otimes A_2^- &: \ \text{same} \\ A_1^- \otimes A_2^+ &: \ \text{same} \\ A_1^- \otimes A_2^- &: \ t(xy) = -t_1(x_1y_1) \otimes t_2(x_2y_2), \end{aligned}$$

the latter because $(x_1 \otimes y_1) \cdot (x_2 \otimes y_2) = -(x_1x_2) \otimes (y_1y_2)$.

The signature being multiplicative, and changing sign when the bilinear form is replaced by its opposite, we get

$$s(A_1 \otimes A_2) = s(A_1)s(A_2).$$

For a matrix algebra $M_{r,s}$, off diagonals contribute an isotropic form, and

$$s(M_{r,s}) = r + s$$

For $\mathbb{R}[\varepsilon]$ with ε odd and $\varepsilon^2 = 1$, one has $s = 1 + i$, of order 8 in $\mathbb{C}^* / (\mathbb{R}^+)^*$. This shows again that the group $\mathrm{sBr}(\mathbb{R})$, being of order 8, is cyclic with generator the division algebra $\mathbb{R}[\varepsilon]$ with ε odd and $\varepsilon^2 = 1$.

1.21 Construction. *The super Brauer group* $\mathrm{sBr}(k)$ *is an extension of* $H^0(\mathrm{Gal}(\bar{k}/k), \mathbb{Z}/2) = \mathbb{Z}/2$ *by* $H^1(\mathrm{Gal}(\bar{k}/k), \mathbb{Z}/2) = k^*/k^{*2}$ *by* $H^2(\mathrm{Gal}(\bar{k}/k)\bar{k}^*) = \mathrm{Br}(k)$.

In 1.21, $\bar{k}$ is a separable closure of k.

The map to $\mathbb{Z}/2$ is extension of scalars to $\bar{k}$, see 1.17. We now consider the kernel.

Let A be a central simple algebra whose class in $\mathrm{sBr}(k)$ is in the kernel of the map to $\mathbb{Z}/2$. Extending scalars to $\bar{k}$, we obtain $\bar{A}$, isomorphic to some $M_{r,s}$. As such, it has two isomorphism classes of simple modules, permuted by parity change. Let $\mathcal{S}(\bar{A})$ be the set of those isomorphisms classes. Having two elements, it can be viewed as a torsor over $\mathbb{Z}/2$. The Galois group $\mathrm{Gal}(\bar{k}/k)$ acts on $\mathcal{S}(\bar{A})$ by transport of structures, defining an homomorphism

$$h_A\colon \mathrm{Gal}(\bar{k}/k) \longrightarrow \mathbb{Z}/2,$$

i.e. an element $h_A \in H^1(\mathrm{Gal}(\bar{k}/k), \mathbb{Z}/2)$: the class of the equivariant $\mathbb{Z}/2$-torsor $\mathcal{S}(\bar{A})$.

For a tensor product $A_1 \otimes A_2$, simple $(A_1 \otimes A_2 \otimes \bar{k})$-modules are the tensor products over $\bar{k}$ of simple $A_1 \otimes \bar{k}$-modules and simple $A_2 \otimes \bar{k}$-modules, giving

$$\mathcal{S}(A_1 \otimes A_2^-) = \mathcal{S}(\bar{A}_1) + \mathcal{S}(\bar{A}_2)$$

(addition of torsors) and $h_A = h_{A_1} + h_{A_2}$.

For A a matrix algebra $M_{r,s}$, $\mathrm{Gal}(\bar{k}/k)$ acts trivially on $\mathcal{S}(\bar{A})$. it follows that $A \mapsto h_A$ factors through an homomorphism

$$h\colon \mathrm{Ker}(\mathrm{sBr}(k) \to \mathbb{Z}/2) \longrightarrow H^1(\mathrm{Gal}(\bar{k}/k), \mathbb{Z}/2).$$

Let us show that h is onto. By 1.17, if A is not purely even, $\mathcal{S}(\bar{A})$ is identified with the set of homomorphisms

$$\mathrm{Hom}(Z, \bar{k})$$

for Z the center of A^+. We need to show that any quadratic extension of k can be obtained as a Z. Indeed, for

$$(k[\varepsilon] \text{ with } \varepsilon^2 = a) \otimes (k[\varepsilon] \text{ with } \varepsilon^2 = -1),$$

$Z = A^+$ is $k(\sqrt{a})$.

It remains to compute the kernel of h. It corresponds to algebras A, with $\bar{A} \simeq M_{r,s}$, for which an isomorphism class of simple $\bar{A}$-module $\bar{S}$ can be defined over k. The automorphism group of $\bar{S}$ is $\bar{k}^*$. The obstruction to finding S giving $\bar{S}$ by extension of scalars is hence in $H^2(\mathrm{Gal}(\bar{k}/k)\bar{k}^*)$.

It is additive in A, and vanishes only if $A \simeq M_{r,s}$. This gives an injective morphism

$$\mathrm{Ker}(h) \longrightarrow H^2(\mathrm{Gal}(\bar{k}/k), \bar{k}^*).$$

It is onto, as shown by taking A to be purely even (usual Brauer group).

2. Variant.

The case of Clifford algebras suggests to consider a variant of the super Brauer group.

2.1. One considers central simple super algebras A, together with an even involution β: $A \to A$ which is an antiautomorphism of the underlying ungraded algebra:

$$\beta(xy) = \beta(y)\beta(x). \tag{2.1.1}$$

The tensor product is defined as follows: take the tensor product of super algebras, and define β as extending the involutions of the factors:

$$\beta(a \otimes b) = (-1)^{p(a)p(b)}\beta(a) \otimes \beta(b). \tag{2.1.2}$$

Tensor product is associative and commutative.

2.2 Example. Let S be a super vector space, and $A := \underline{\mathrm{End}}(S)$. Antiautomorphisms β of the underlying ungraded algebra correspond to non degenerate bilinear forms on S taken up to a factor by

$$(ax, y) = (x, \beta(a)y).$$

Involutive β correspond to symmetric or antisymmetric forms. Even β correspond to even or odd forms.

Suppose given two super vector spaces S_i $(i = 1, 2)$ provided with even or odd, symmetric or antisymmetric bilinear forms. Let (A, β) be the tensor product of the corresponding (A_i, β). One has

$$A = A_1 \otimes A_2 = \underline{\mathrm{End}}(S_1) \otimes \underline{\mathrm{End}}(S_2) \xrightarrow{\sim} \underline{\mathrm{End}}(S_1 \otimes S_2).$$

We leave it to the reader to check that the involution β of A correspond to the following bilinear form on $S_1 \otimes S_2$:

$$(s_1 \otimes s_2, t_1 \otimes t_2) + (-1)^{(p(s_1)+p(t_1))p(s_2)}(s_1, s_2)(t_1, t_2). \tag{2.2.1}$$

Its parity is the sum of the parities of the forms $(\quad,\quad)_i$ on the S_i. Its sign ($+$ for symmetric, $-$ for antisymmetric) is the product of the signs of the $(\quad,\quad)_i$, times (-1) at the power the product of the parities.

2.3. Let us call (A,β) neutral if A is of the form $\operatorname{End}(S)$ and if β is given by an even symmetric form $(\ \ ,\ \ )$ on S: the sum of symmetric forms on S^+ and S^-.

Define (A,β) and (B,β) to be *similar* if for suitable neutrals (M,β) and (N,β), one has

$$(A,\beta)\otimes(M,\beta)\simeq(B,\beta)\otimes(N,\beta).$$

Similarity is an equivalence relation, and is stable by tensor product. let $C(k)$ be the set of similarity classes. Tensor product induces on $C(k)$ a composition law associative, commutative and with unit. The following proposition shows it is group.

Proposition 2.4. *Let (A,β) be as in 2.1, and let* Tr *be the trace of the underlying ungraded algebra. Let α be the parity automorphism $a\mapsto(-1)^{p(a)}a$. Then, $(A,\beta)\otimes(A^0,\alpha\beta)$ is neutral. More precisely,*

$$A\otimes A^0\xrightarrow{\sim}\underline{\operatorname{End}}(A)$$

and its involution corresponds to the even symmetric form $\operatorname{Tr}(x\beta(y))$ on A.

Proof. We have

$$\operatorname{Tr}((a\otimes b\cdot x)\beta(y))=(-1)^{p(b)p(x)}\operatorname{Tr}(axb\beta(y));$$

as

$$\operatorname{Tr}(axb\beta(y))=\operatorname{Tr}(xb\beta(y)a)\quad\text{and that}$$
$$b\beta(y)a=\beta(\beta(a)y\beta(b))=(-1)^{p(b)p(y)}\beta(\beta(a)\otimes\beta(b).y),$$

this equals

$$=(-1)^{p(b)(p(x)+p(y))}\operatorname{Tr}(x.\beta(\beta(a)\otimes\beta(b)y).$$

Both sides are zero if $p(a)+p(b)+p(x)+p(y)$ is odd. We hence may replace $p(b)(p(x)+p(y))$ by $p(b)(p(x)+p(b))$, giving

$$=\operatorname{Tr}(x.\beta((-1)^{p(a)p(b)}\beta(a)\otimes\alpha\beta(b).y),$$

$a\otimes b\mapsto(-1)^{pa\,pb}\beta(a)\otimes\alpha\beta(b)$ is the tensor product of β and $\alpha\beta$.

Proposition 2.5. *The kernel of the "forgetting β" map from $C(k)$ to* $\mathrm{sBr}(k)$ *is cyclic of order* 4.

Proof. The (A, β) whose class is in the kernel are of the form $\mathrm{End}(S)$ of 2.2, with β given by a non degenerate bilinear form on S which is even or odd, as well as symmetric or antisymmetric. We label those cases by $2.\mathbb{Z}/8$ as follows:

0: even symmetric

2: odd symmetric

4: even antisymmetric

6: odd antisymmetric.

By 2.2, the tensor product of algebras of type i and j in $2.\mathbb{Z}/8$ is of type $i + j$. It follows that each type is a similarity class, and 2.5 follows.

Proposition 2.6. *If k is separably closed, $C(k) = \mathbb{Z}/8$, with generator the class of $(k[\varepsilon], \mathrm{Id})$ with ε odd and $\varepsilon^2 = 1$.*

The proposed generator maps to the non trivial element of $\mathrm{sBr}(k)$. It remains to see that its square is a generator of the kernel of $C(k) \to \mathrm{sBr}(k)$, computed in 2.5. More precisely,

Lemma 2.7. *For k separably closed, $(k[\varepsilon], \mathrm{Id}) \otimes (k[\varepsilon], \mathrm{Id})$ is of the type 2 of 2.5.*

The algebra $k[\varepsilon]$ is isomorphic to the opposite algebra $k[\eta]$, with $\eta^2 = -1$, and 2.6 follows from the more general lemma, valid over any k

Lemma 2.8. *The tensor product of $k[\varepsilon], \mathrm{Id})$ and $(k[\eta], \mathrm{Id})$, with β the identity, is of the type 2 of 2.5.*

We identify the tensor product with the 2×2 matrix algebra, by taking $\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\eta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Those matrices are fixed by the transposition rel. the quadratic form xy, which corresponds to an odd symmetric form.

2.9 Remark. Suppose (A, β) over k separably closed is non trivial in $\mathrm{sBr}(k)$. The simple A-modules are then obtained as follows: start from a A^+-simple module S^+, and extend scalars to A. The involution β, restricted to A, corresponds to a non degenerate bilinear form $(\ ,\ )$ on S^+, symmetric or antisymmetric, unique up to a scalar factor. It is even or odd depending on the class of A in $\mathbb{Z}/8$. The class of (A,β) in $\mathbb{Z}/8$ detect if it is even or odd, and if the restriction of β to the (2-dimensional) centralizer Z of A^+ is the identity or not:

$$\begin{array}{lll} 1: & \text{symmetric} & ,\quad \beta|Z = \mathrm{Id} \\ 3: & \text{antisymmetric} & ,\quad \beta|Z \neq \mathrm{Id} \\ 5: & \text{antisymmetric} & ,\quad \beta|Z = \mathrm{Id} \\ 7: & \text{symmetric} & ,\quad \beta|Z \neq \mathrm{Id} \end{array}$$

Determination of $\beta|Z$: As $(Z,\beta|Z)$ is unchanged by tensoring with a (B,β) with B a matrix algebra, if suffices to consider the case of the tensor product of n $k[\varepsilon]$ with $\beta = \mathrm{Id}$. In this case, Z is generated by the product η of the ε, and $\beta(\eta) = (-1)^{n(n-1)/2}\eta$.

Parity of $(\ ,\ )$: Let (B,β) be the tensor product of (A,β) with $(k[\varepsilon],\mathrm{Id})$ (resp. $(k[\varepsilon],\beta)$ with $\beta(\varepsilon) = -\varepsilon$). Its invariant is respectively 2, 4, 6, 0 (resp. 0, 2, 6, 6). If S is a simple B-module, its restriction to A continues to be simple. If the restriction to S^+ of the form $(\ ,\ )$ on S is non degenerate, it is the form $(\ ,\ )$ we are looking for, and the table above results from that of 2.5

2.10. For a general field k, extension of scalars to a separable closure and 2.6 gives a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & C(k) & \longrightarrow & \mathrm{sBr}(k) & & \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & \mathbb{Z}/8 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \end{array} \tag{2.10.1}$$

For $k = \mathbb{R}$, (1.18) shows that $C(\mathbb{R})$ maps onto $\mathrm{sBr}(k)$ and (2.10.1) shows that the maps

$$C(\mathbb{R}) \longrightarrow \mathbb{Z}/8 = C(\mathbb{C}) \qquad \text{and}$$

$$C(\mathbb{R}) \longrightarrow \mathbb{Z}/8 = \mathrm{sBr}(\mathbb{R})$$

induce an isomorphism from $C(\mathbb{R})$ to the fiber product of $\mathbb{Z}/8$ and $\mathbb{Z}/8$ over $\mathbb{Z}/2$:

$$0 \to C(\mathbb{R}) \to \mathbb{Z}/8 \times \mathbb{Z}/8 \to \mathbb{Z}/2 \to 0.$$

3. The Clifford algebra and spin-modules.

3.1. Let k be a commutative ring, V be a k-module and Q be a quadratic form on V. The *Clifford algebra* $C(Q)$ is the k-algebra generated by the k-module V with the relations

$$x^2 = Q(x).1 \tag{3.1.1}$$

for $x \in V$. This is Bourbaki's definition (Alg. Ch. 9 §9, n^o 1). Some authors prefer to use as defining relations $x^2 = -Q(x).1$. We will sometimes write $C(V)$, or $C(V, Q)$, instead of $C(Q)$.

It results immediately from this definition that

(A) $C(Q)$ is mod 2-graded, the image of V being odd. In other words: $C(Q)$ is a super algebra. Indeed, the defining relations are in the even part of the tensor algebra on V.

(B) The algebra $C(Q)$ admits a unique antiinvolution β which is the identity on the image of V. Indeed, the opposite algebra $C(Q)^0$ is a solution to the same universal problem as $C(Q)$ is. Bourbaki's terminology: β is the principal antiautomorphism of $C(Q)$. By definition, $\beta(xy) = \beta(y)\beta(x)$. That there is no sign $(-1)^{p(x)p(y)}$ is not a misprint.

If we apply (3.1.1) to $x + y$, x and y and take a difference, we obtain the polarized form of (3.1.1):

$$xy + yx = \phi(x, y).1 \tag{3.1.2}$$

for ϕ the bilinear form $Q(x + y) - Q(x) - Q(y)$ associated to Q.

When 2 is invertible in k, (3.1.1) is equivalent to (3.1.2): take $x = y$ in (3.1.2). In general, if $X \subset V$ generates V, (3.1.1) is implied by (3.1.1) for $x \in X$, and (3.1.2) for $x \neq y$ in X. This makes it clear that

(C) The formation of $C(Q)$ is compatible with extension of scalars.

(D) If (V, Q) is the orthogonal direct sum of (V', Q') and (V'', Q''), the Clifford algebra $C(Q)$ is the tensor product, in the sense of super algebras, of $C(Q')$ and $C(Q'')$. the principal antiautomorphism β of $C(Q)$ is the tensor product, in the sense of 2.1, of those of $C(Q')$ and $C(Q'')$.

Indeed, $C(Q)$ is generated by the k-modules V' and V'', with the relations (3.1.1) for $x \in V'$ or $x \in V''$ and the relations (3.1.2) for $x \in V'$ and $y \in V''$. For $x \in V'$ and $y \in V''$, (3.1.2) reduces to a commutation relation (in the super sense)

$$xy = (-1)^{\deg(x)\,\deg(y)} yx$$

and (D) follows.

(E) The identity of V extends to an isomorphism from the opposite of the super algebra $C(Q)$ to the super algebra $C(-Q)$.

Indeed, each defining relation $x.x = Q(x).1$ is replaced by its opposite $x.x = -Q(x).1$.

Proposition 3.2. *Assume that V is finite dimensional over a field k of characteristic $\neq 2$. If the form Q is non degenerate, the super algebra $C(Q)$ is central simple (in the super-sense).*

Proof. V is the orthogonal direct sum of 1-dimensional subspace. By 3.1 (D) and 1.15, we are reduced to the case where $\dim(V) = 1$. In that case, $C(Q)$ is a super division algebra with center k.

Corollary 3.3. *The construction $(V, Q) \mapsto C(Q)$ induces a morphism from the Witt group of k to its super Brauer group.*

Proof. By 3.2 and 3.1 (D), it remains only to check that if V is hyperbolic, then $C(Q)$ is a matrix algebra. Indeed, an hyperbolic quadratic space of characteristic $\neq 2$ can be presented as an orthogonal direct sum $(W, Q) \oplus (W, -Q)$ and one uses 3.1 (E) and 1.14. Another proof will be given in 3.7.

Corollary 3.4. *The class of $(C(Q), \beta)$ in the variant $C(k)$ (2.3) of $\mathrm{sBr}(k)$ depends only on $\dim(V) \mod 8$ and the class of (V, Q) in the Witt group $W(k)$.*

This results from 2.10.1, as $\dim(V) \mod 8$ determines the similarity class of $(C(Q), \beta)$ on the separable closure. The class in $W(k)$ determines the dimension of $V \mod 2$, and

3.3, 3.4 give rise to a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 2.\mathbb{Z}/8 & \longrightarrow & \mathrm{Ker}(W(k)\times\mathbb{Z}/8\to\mathbb{Z}/2) & \longrightarrow & W(k) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 2.\mathbb{Z}/8 & \longrightarrow & C(k) & \longrightarrow & \mathrm{sBr}(k) & & \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 2.\mathbb{Z}/8 & \longrightarrow & C(\bar{k}) = \mathbb{Z}/8 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0\,.
\end{array}$$

3.5. Let F be the filtration of $C(Q)$ defined by

$$F_n = \text{image of } \bigoplus_{0\le i\le n} V^{\otimes i}.$$

If V is a free k-module, one has (Bourbaki Alg. Ch. 9 §9, n° 3, Th. 1)

$$\overset{*}{\wedge} V \xrightarrow{\sim} \mathrm{Gr}^F C(Q). \tag{3.5.1}$$

We now give an "explanation" of (3.5.1), assuming that k is a field of characteristic $\neq 2$. As 2 is invertible, $C(Q)$ is generated by V with the relations (3.1.2) (instead of (3.1.1)).

If $\mathfrak{L}$ is a super Lie algebra, the Poincaré-Birkhoff-Witt theorem says that if F is the filtration of the universal enveloping algebra $\mathfrak{U}(\mathfrak{L})$ given by

$$F_n = \text{image of } \bigoplus_{0\le i\le n} \mathfrak{L}^{\otimes i},$$

then

$$\mathrm{Sym}^*(\mathfrak{L}) \xrightarrow{\sim} \mathrm{Gr}^F \mathfrak{U}(\mathfrak{L})$$

where Sym^* is taken in its super sense: it is $\mathrm{Sym}^*(\mathfrak{L}^+)\otimes\overset{*}{\wedge}(\mathfrak{L}^-)$. If $i\colon k\hookrightarrow\mathfrak{L}$ is central, we have a variant of $\mathfrak{U}(\mathfrak{L})$:

$$\mathfrak{U}_0(\mathfrak{L}) := \mathfrak{U}(\mathfrak{L})/\text{relation } i(1)=1.$$

A $\mathfrak{U}_0(\mathfrak{L})$-module can be identified with a super Lie algebra module on which $i(1)$ acts as the identity. A variant of Poincaré-Birkhoff-Witt says that for the quotient filtration F of $\mathfrak{U}_0(\mathfrak{L})$,

$$\mathrm{Sym}^*(\mathfrak{L}/k) \xrightarrow{\sim} \mathrm{Gr}^F \mathfrak{U}_0(\mathfrak{L}) \tag{3.5.2}$$

The Theorem (3.5.1) is a particular case of (3.5.2): take $\mathfrak{L} := k \oplus V$, with V odd, k even and central, and the super commutator

$$[x, y] = \phi(x, y).1$$

for $x, y \in V$. The definition of $C(Q)$ using (3.1.2) amounts to

$$C(Q) = \mathfrak{U}_0(\mathfrak{L}),$$

making (3.5.1) a particular case of (3.5.2).

3.6. Suppose (V, Q) is isotropic: $V$is a direct sum $L^* \oplus L$ with

$$Q(\omega + \ell) = \langle \omega, \ell \rangle .$$

The exterior algebra $\overset{*}{\wedge} L$ is then turned into a $C(Q)$-module by requiring $\ell \in L$ (resp. $\omega \in L^*$) to act by exterior multiplication $\ell\wedge$ (resp. contraction by ω). It is a super module, if we take the parity of $\overset{i}{\wedge} L$ to be that of i.

Proposition 3.7. *With the notations of 3.4, if L is a free module of finite rank, one has*

$$C(Q) \xrightarrow{\sim} \underline{\mathrm{End}}\left(\overset{*}{\wedge} L\right). \tag{3.7.1}$$

Proof. Let $L = \oplus L_i$ be a decomposition of L as a direct sum of free modules of rank one. It defines a corresponding decomposition of V in the orthogonal direct sum of the $(L_i^* \oplus L_i)$. Both sides of (3.7.1) are the tensor product of the corresponding $C(L_i^* \oplus L_i)$ and $\underline{\mathrm{End}}(\overset{*}{\wedge} L_i)$. This reduces (3.7.1) to the case $L = k$, which is left to the reader.

3.8. One can view 3.7 as an odd analogue of the Schrödinger representation of Heisenberg groups. Indeed, let (V, ψ) be a symplectic, rather than orthogonal, finite dimensional vector space over k, assumed of characteristic zero. Provide $k \oplus V$ with the Lie algebra structure for which k is central and $[v_1, v_2] = \psi(v_1, v_2)$ for v_1, v_2 in V: the Heisenberg Lie algebra attached to V. If V is a direct sum $L^* \oplus L$, with $\psi((\omega', \ell'), (\omega'', \ell'')) = \omega'(\ell'') - \omega''(\ell')$, the symmetric algebra $\mathrm{Sym}^*(L)$ is turned into a $\mathfrak{U}_0(k \oplus V)$-module by requiring ℓ in L to act by multiplication, and $\omega \in L^*$ to act by contraction. If L is the dual of E, $\mathrm{Sym}^*(L)$

is the ring of polynomial functions on E, the action of $\ell \in L$ is multiplication by a linear function, and that of $\omega \in L^*$ — identified with a constant vector field on E — is ∂_ω. This representation identifies $\mathcal{U}_0(k \oplus V)$ with the ring $\mathcal{D}_E$ of all linear differential operators on E with polynomial coefficients.

In 3.6, $\overset{*}{\wedge} L$ is similarly the ring of function on the odd super affine space defined by E, with $\ell\wedge$ and $\omega\lrcorner$ multiplication and derivation operators, respectively.

3.9. Under the assumptions of 3.7, with k a field, and $V \neq 0$, the algebra $C(Q)$ is a matrix algebra by 3.7, hence has up to isomorphism a unique simple module S. The automorphism group of S is k^*. This module is even a super module over the super algebra $C(Q)$. Its grading is, however, unique only up to parity change.

It follows that any automorphism g of $C(Q)$ can be extended to an automorphism of $(C(Q)$, module $S)$. the extension is unique up to an automorphism $\lambda \in k^*$ of S. If g respects the grading of $C(Q)$, the extension respects or permutes the homogeneous components $S^\pm$ of S, depending on the action of g on the center of $C^+(Q)$.

In particular, $O(Q)$ acts projectively on S. The parity respecting subgroup is $\mathrm{SO}(Q)$.

Similar arguments can be made for k any local ring. Taking the dual numbers $k[x]/(x^2)$, and interpreting the Lie algebra $\mathfrak{so}(Q)$ as $\ker(O(Q)(k[x]/(x^2)) \to O(Q)(k))$, one gets a projective action of the Lie algebra $\mathfrak{so}(Q)$ on S. As $C(Q) \xrightarrow{\sim} \mathrm{End}_k(S)$, this action is given by a Lie algebra morphism

$$\bar{f}\colon \mathfrak{so}(Q) \longrightarrow C(Q)/k$$

with $[\bar{f}(x), v]$ being the Lie algebra action of $\mathfrak{so}(Q)$ on V. As $C(Q)$ is the Lie algebra product of k and $[C(Q), C(Q)]$ (usual brackets), $\bar{f}$ lifts uniquely to a morphism f from $\mathfrak{so}(Q)$ to $[C(Q), C(Q)] \subset C(Q)$. Here is a direct construction of this lifting. Identify $\mathfrak{so}(Q)$ to $\overset{2}{\wedge} V$ by

$$x \wedge y \longmapsto \text{ endomorphism } \phi(y, v)x - \phi(x, v)y.$$

Map $x \wedge y$ to $\frac{1}{2}(xy - yx)$.

Check: as $xy + yx$ is scalar,

$$\left[\tfrac{1}{2}(xy - yx), v\right] = [xy, v] = x\{y, v\} - \{x, v\}y$$

for $\{\quad\}$ the super bracket

$$= \phi(y,v)x - \phi(x,v)y.$$

The argument we used to see the existence and unicity of f used that V is split and even dimensional. The actual construction used neither, and holds for any non degenerate quadratic form. It turns any $C(Q)$-module into a $\mathfrak{so}(Q)$-module.

3.10. The argument used in 3.9 is analogue to the one by which, using the unicity of an irreducible representation of the Heisenberg commutation relations, one gets on the representation space a projective action of a real symplectic group — or an actual representation of its double covering, the metaplectic group. This is, however, an analytic story, involving infinite dimensional Hilbert space, as evidenced by the fact that the metaplectic double covering of the real symplectic group is not algebraic. The Lie algebra story, however, has a purely algebraic analogue. With the notations of 3.8, the symmetrized product gives a vector space isomorphism

$$\mathrm{Sym}^*(V) \xrightarrow{\sim} \mathfrak{U}_0(k \oplus V) \tag{3.10.1}$$

The image of $\mathrm{Sym}^2(V)$ by this map is a Lie algebra normalizing V, identified by its action on V with the Lie algebra of the symplectic group.

Assume V is the dual of E, so that $\mathrm{Sym}^*(V)$ is the polynomial functions on E. If we replace ψ by $t\psi$, and transport the product on the corresponding $\mathfrak{U}_0(k\oplus V)$ to $\mathrm{Sym}^*(V)$ by (3.10.1), we obtain on $\mathrm{Sym}^*(V)$ a product $*_t$ depending on t. It comes from a $k[t]$-algebra structure on $\mathrm{Sym}^*(V) \otimes k[t]$.

One has

$$f *_t g = fg + \tfrac{1}{2}\{f,g\}t + O(t^2),$$

for $\{\quad\}$ the Poisson bracket on the symplectic manifold E. A weakened version of the fact that $\mathrm{Sym}^*(V) \subset \mathfrak{U}_0(k\oplus V)$ acts on V as $\mathrm{Lie}\ \mathrm{Sp}(V)$ is the fact that the hamiltonian vector fields on E given by quadratic functions are the infinitesimal symplectic transformations.

We now assume that k is a field of characteristic $\neq 2$, that V is a finite dimensional vector space, and Q a non degenerate quadratic form on V.

Proposition 3.11. *The sub Lie algebra of $C^+(Q)$ normalizing $V \subset C(Q)$ is $k \oplus f(\mathfrak{so}(Q))$, and $f(\mathfrak{so}(Q))$ the subspace on which the principal antiautomorphism of $C(Q)$ acts by -1.*

Proof. If x in $C^+(Q)$ normalizes V, $adx|V$ is an infinitesimal orthogonal transformation:

$$\phi([xv], v) = [xv]v + v[xv] = [x, v^2] = [x, Q(v)] = 0.$$

Let us write $x = y + f(z)$ with $z \in \mathfrak{so}(V)$. By construction, y centralizes V, hence is in the center of $C(V)$. Being even, it is in the center of the super algebra $C(V)$, hence a scalar by 2.2. This proves the first assertion.

The principal antiautomorphism β transform $\frac{1}{2}(vw - wv)$ $(v, w \in V)$ into $\frac{1}{2}(wv - vw)$. This checks the second assertion.

Corollary 3.12. *Let S be a $C^+(Q)$-module and $(\quad,\quad)$ a bilinear form on S for which*

$$(xs, t) = (s, \beta(x)t) \tag{3.12.1}$$

for x in $C^+(Q)$ and s, t in S. For the induced structure of $\mathfrak{so}(Q)$-module, the bilinear form $(\quad,\quad)$ is invariant.

Proposition 3.13. *Let S be an absolutely simple super $C(Q)$-modules.*

(i) *If* $\dim(V)$ *is even, the representations S^+ and S^- of $\mathfrak{so}(Q) \subset C^+(V)$ are non isomorphic irreducible representations.*

(ii) *If* $\dim(V)$ *is odd, the representations S^+ and S^- are irreducible and isomorphic.*

Isomorphy in (ii) result from the decomposition

$$C(Q) = C^+(Q) \otimes (\text{centralizer } Z \text{ of } C^+(Q) \text{ in } C(Q)) :$$

an odd generator of Z provides an isomorphism.

As $\mathfrak{so}(Q) = \overset{2}{\wedge} V \subset C^+(Q)$ generates the algebra $C^+(Q)$, (i) and (ii) results from the structure of $C^+(Q)$ when k is separably closed: a matrix algebra over $k \times k$ in case (i), a matrix algebra over k in case (ii).

3.14. Whether there exist an absolutely simple super $C(Q)$-module S (i.e. S remains simple after extension of scalar to $\bar{k}$) depends on the class of $C(Q)$ in $\mathrm{sBr}(k)$, itself depending only on the class of Q in the Witt group. The kind of bilinear forms on S for which β is the transposition depends on the class of $(C(Q), \beta)$ in the variant $C(k)$ (2.3) of $\mathrm{sBr}(k)$.

Until the end of this §, we assume k separably closed.

For $\dim(V)$ even (resp. odd), $C(Q)$ (resp. $C^+(Q)$) is a matrix algebra, with involution β. If S is a simple $C(Q)$- (resp. $C^+(Q)$-) module, there is up to a scalar factor a unique bilinear form $(\ ,\ )$ on S such that

$$(as, t) = (s, \beta(a)t). \tag{3.14.1}$$

For $\dim V$ even, there is on S a $\mathbb{Z}/2$-grading, unique up to parity change, turning S into a super module.

A space of *spinors* is a super $C(Q)$-module (resp. a $C^+(Q)$-module) S provided with $(\ ,\ )$ obeying (3.14.1). it is unique up to isomorphism, with ± 1 as the only automorphisms. The Lie algebra of infinitesimal automorphisms of (V, S) provided with Q, the module structure of S and $(\ ,\ )$ is $\mathfrak{so}(Q)$.

For $\dim(V)$ even, S^+ and S^- are the spaces of *semi-spinors*. By 3.7, if L is a maximal isotropic subspace of V, there is up to a scalar factor a unique semi-spinor $s(L)$ such that $L.s(L) = 0$ for the $C(Q)$-module structure of S. The two types of maximal isotropic subspaces are distinguished by whether $s(L)$ is in S^+ or in S^-. For the semi-spinors $s(L)$, one can check that the vanishing of $(\ ,\ )$ has a simple interpretation

$$(s(L'), s(L'')) = 0 \Leftrightarrow L' \cap L'' \neq 0 \tag{3.14.2}$$

(C. Chevalley, the algebraic theory of spinors, Columbia Univ. Press 1954, III 2.4, p. 79).

The class of $(C(Q), \beta)$ in the group $C(k) = \mathbb{Z}/8$ of 2.6 is the dimension of V modulo 8: write V as an orthogonal direct sum of lines, and apply 3.1 (D). By the tables of 2.5

and 2.7, the form (,) is hence as follows, according to the vaue of $\dim(V)$ mod 8:

$$\begin{array}{ccl} 0 & : & \text{even symmetric} \\ 1 & : & \qquad\ \ \text{symmetric} \\ 2 & : & \text{odd symmetric} \\ 3 & : & \qquad\ \ \text{antisymmetric} \\ 4 & : & \text{even antisymmetric} \\ 5 & : & \qquad\ \ \text{antisymmetric} \\ 6 & : & \text{odd antisymmetric} \\ 7 & : & \qquad\ \ \text{symmetric}. \end{array}$$

If $\dim(V)$ is even, the even or odd character of (,) can be read from (3.14.2), using that L and L'' are of the same type if and only if $L' \cap L''$ is of even codimension in L' (and L''): if $\dim(V)/2$ is even (resp. odd), and L', L'' of different (resp. same) type, $L' \cap L'' \neq 0$ and $(s(L'), s(L'')) = 0$.

If $e_1, \ldots, e_n$ is an orthogonal basis of V, the product $z = e_1 \ldots e_n$ generates the commutant Z of $C^+(Q)$. One has

$$\beta(z) = (-1)^{n(n-1)/2} z \tag{3.14.3}$$

If $\dim(V)$ is even, $\beta|Z$ is trivial if and only if (,) is even. By (3.14.3), this shows again that the parity of (,) is that of $\dim(V)/2$.

3.15. Consider $(k[\varepsilon], \beta)$ with ε odd, $\varepsilon^2 = 1$ and $\beta(\varepsilon) = \pm\varepsilon$: sign $\pm$. The class of $(A, \eta) := (C(V), \beta) \otimes (k[\varepsilon], \beta)$ is $\dim V \pm 1 \mod 8$. If V is odd dimensional, and S a simple super module over A, S remains a simple super module over $C(V)$, and S^+ is a simple $C^+(V)$-module. If the form (,) on the A-module S obeying (3.14.1) is even (resp. odd), a form (,) on the $C^+(V)$-module S^+ obeying (3.14.1) is

$$\begin{aligned} (s, t) &= (s, t), \text{ respectively} \\ &= (s, \varepsilon t) \end{aligned}$$

with sign that of $(\ ,\ )$ (resp. $\pm$ that of $(\ ,\ )$). This shows again that if $2n \in \mathbb{Z}/8$ corresponds to even forms, then the forms for $2n$ and $2n \pm 1$ have the same sign. If $2n$ corresponds to odd forms, the sign is the same for $2n-1$, opposite for $2n+1$.

3.16. Let S be a space of spinors. If dim V is even, the morphism of $\mathfrak{so}(Q)$-modules

$$V \otimes S \longrightarrow S$$

given by the $C(V)$-module structures transposes to a morphism

$$\{\ \ \}:\ S \otimes S \longrightarrow V$$

characterized by the equality of inner products

$$(w, \{s, t\}) = (ws, t) \tag{3.16.1}$$

in V, resp. S. The parity of $\{\ \ \}$ is opposite to that of $(\ ,\ )$ on S. The sign is the same. Depending on dim $V \mod 8$, $\{\ \ \}$ is

0	odd symmetric
2	even symmetric
4	odd antisymmetric
6	even antisymmetric

If $\dim(V)$ is odd, choosing an odd generator z of the centralizer Z of $C^+(Q)$, one similarly defines $\{\ \ \}:\ S \otimes S \to V$ by

$$(w, \{s, t\}) = (zws, t). \tag{3.16.2}$$

The sign of $\{\ \ \}$ is that of $(\ ,\ )$ on S, times that of β acting on z: by 2.9, it is depending on $\dim(V) \mod 8$:

1	symmetric
3	symmetric
5	antisymmetric
7	antisymmetric.

3.17. If V_1 and V_2 are even dimensional, and if S_1 and S_2 are spaces of spinors for V_1 and V_2, then $S_1 \otimes S_2$ is a space of spinors for the orthogonal direct sum of V_1 and V_2. its module structure is given by the tensor product, in the super sense, of those of S_1 and S_2, and the form $(\quad,\quad)$ is given by (2.2.1).

Suppose now that $v \neq 0$ is an isotropic vector in V, supposed to be even dimensional. Define

$$V_1 := v^{\perp}/kv,$$

of dimensional two less. If S is a space of spinors for V, v acting on S supercommutes with $v^{\perp}$. The kernel $\mathrm{Ker}(v)$ is hence stable by $v^{\perp}$. The action of $v^{\perp}$ factors through V_1 and turns $\mathrm{Ker}(v)$ into a $C(V_1)$-module.

Proposition 3.18. *With the notations of 3.17*

(i) $\mathrm{Ker}(v) = \mathrm{Im}(v)$ *is an irreducible* $C(V_1)$*-module.*

(ii) *The form* $(\quad,\quad)$ *vanishes on* $\mathrm{Ker}(v)$.

(iii) *The form* $(\quad,\quad)_1$ *on* $\mathrm{Ker}(v)$ *defined by* $(s, vt)_1 = (-1)^{p(s)}(s,t)$ *turns* $\mathrm{Ker}(v)$ *into a space of spinors for* V_1.

Proof. (i) Lift V_1 in $v^{\perp}$ and let H be the orthogonal complement of the lifting: V becomes the orthogonal direct sum of V_1 and H, with $\dim(H) = 2$ and $v \in H$. The space of spinors for V_1 becomes the tensor product of one for V_1 and one for H, with the action of v coming from its action on spinors for H. This reduces us to the two dimensional case, where v acts as $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ in a suitable basis. The claim follows.

(ii) results from $\mathrm{Ker}(v) = \mathrm{Im}(v)$:

$$(vs, vt) = (s, vvt) = 0$$

as $v^2 = Q(v) = 0$. In (iii), $t \mod \mathrm{Ker}(v)$ is determined by vt, so that, by (i) and (ii), $(\quad,\quad)$ is well defined.

(iii) It remains to check that $(\quad,\quad)_1$ obeys (3.14.1). Indeed, if $t_1 = vt$ and $w \in v^{\perp}$, one has $wt_1 = -vwt$ and

$$(ws, t_1)_1 = (-1)^{p(ws)}(ws, t) = (-1)^{p(ws)}(s, wt) = (s, wt_1)_1$$

Proposition 3.19. *Under the same assumptions, if* $s, t \in \mathrm{Ker}(v)$*, one has in* V

$$\{s, t\} = (-1)^{p(s)}(s, t)_1 v. \tag{3.19.1}$$

Proof. We check that both sides have the same inner product with any $w \in V$. If $t = vx$, the left side indeed gives

$$\begin{aligned}(w, \{s, x\}) &= (ws, vx) = (vws, x) = ((vw + wv)s, x) \\ &= (v, w)(s, x) = (-1)^{p(s)}(v, w)(s, t)_1\end{aligned}$$

4. The group Spin.

4.1. Let $V, Q)$ be a finite dimensional quadratic vector space over k of characteristic $\neq 2$. Let G be the group of invertible elements in $C^+(Q)$ or $C^-(Q)$ which normalize V. We let it act on V by

$$\rho(g): \ v \longmapsto (-1)^{p(g)} g v g^{-1}.$$

This action respects the quadratic form $Q(v) = v^2$ on V. If g fixes V, g is in the center (in the super sense) of $C(Q)$, hence is a scalar: we have an exact sequence

$$1 \longrightarrow k^* \longrightarrow G \longrightarrow O(V). \tag{4.1.1}$$

Elements $w \in V$ with $Q(w) \neq 0$ are in $G^- := G \cap C^-(Q)$, and $\rho(w)$ is the reflection relative to the hyperplane orthogonal to w. As each element of $O(V)$ is a product of reflections, the sequence (4.1.1) is exact ot the right. As a product of k reflections has determinant $(-1)^k$, it follows from (4.1.1) that $G^+ := G \cap C^+(Q)$ maps onto $\mathrm{SO}(V)$, while G^- maps onto $O(V) - \mathrm{SO}(V)$. We have

$$1 \longrightarrow k^* \longrightarrow G^+ \longrightarrow \mathrm{SO}(V) \longrightarrow 1. \tag{4.1.2}$$

4.2. If $g \in G$, applying β to the defining relation

$$gv = (-1)^{p(g)} vg,$$

we see that $\beta(g) \in G$ with $\rho(\beta(g)) = \rho(g)^{-1}$: $g\beta(g)$ is in k^*. The *Spin group* is the kernel of

$$g\beta(g): \ G^+ \longrightarrow k^*. \tag{4.2.1}$$

On $k^* \subset G^+$, $g\beta(g)$ is squaring. The sequence (4.1.2) is an exact sequence of algebraic groups, and it follows that it induces a commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d] & \mathrm{Spin} \arrow[r] \arrow[d] & \mathrm{SO} \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[dr, "\lambda^2"'] & G^+ \arrow[r] \arrow[d, "g\beta(g)"] & \mathrm{SO} \arrow[r] & 1 \\
 & & \mathbb{G}_m & &
\end{tikzcd} \tag{4.2.2}$$

with $\mathbb{G}_m$ = multiplicative group, $\mu_2 = \{\pm 1\}$. In (4.2.2), the first two rows are exact sequences of algebraic groups. On the level of points, the first row may not be exact: it gives rise to a coboundary with values in $H^1(\mathrm{Gal}(\bar{k}/k), \mu_2)$.

4.3. If S is a $C^+(Q)$-module, and $(\ ,\ )$ a bilinear form on S obeying (3.14.1), the action of Spin $\subset C^+(Q)^*$ on S respects the form $(\ ,\ )$:

$$(gs, gt) = (s, \beta(g)gt) = (s, t).$$

This is the counterpart of 3.12 for groups.

For k separably closed, and S a space of spinors, the group Spin is the group of automorphisms of (V, S) provided with the following structure:

For $\dim(V)$ even: the map $V \otimes S \to S$, the mod 2 grading of S and $(\ ,\ )$ on S.

Indeed, from $V \otimes S \to S$ one recovers $Q(v) = v^2$ and the $C(V)$-module structure, the mod 2 grading of S cuts $O(V)$ down to SO, and the group of automorphisms, contained in $C(Q)^{+*}$, is a double covering of $\mathrm{SO}(Q)$.

For $\dim(V)$ odd: Q, a volume element, the $C^+(Q)$-module structure on S and $(\ ,\ )$. The proof is similar.

This is a counterpart of 3.14 for groups.

5. The Minkowski case.

Theorem. *Let (V, Q) over $\mathbb{R}$ be of signature $(+, -, \ldots, -)$. Let $S_{\mathbb{R}}$ be a real representation of* $\mathrm{Spin}(V, Q)$*, irreducible as a real representation. Assume that after extension of scalars to $\mathbb{C}$, it becomes a sum of spinorial or semi-spinorial representations. The commutant Z of $S_{\mathbb{R}}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $^-$ be the standard anti-involution of Z.*

(i) *Up to a real factor, there exists a unique symmetric morphism $\{\ ,\ \}\colon S_{\mathbb{R}} \otimes S_{\mathbb{R}} \to V$ such that $\{zs, t\} = \{x, \bar{z}t\}$, i.e. which is invariant by the group of elements of norm 1 of Z.*

(ii) *For $v \in V$, if $Q(v) > 0$, the form $\langle v, \{s, t\}\rangle$ on $S_{\mathbb{R}}$ is positive or negative defined.*

The set of v for which $Q(v) > 0$ has two connected components. It follows from (ii) that for v in one of them, call it C, the form $< v, \{s, t\} >$ is positive defined and that for v in the other, i.e. for $-v$ in C, it is negative defined.

The class of $C(V)$ in $\mathrm{sBr}(\mathbb{R}) = \mathbb{Z}/8$ (1.18) is the signature modulo 8 (3.3), i.e. is $2 - d$ if V is of dimension d. The representation $S_{\mathbb{R}}$ is a simple $C^+(V)$-module and, by the table 1.18, depending on d modulo 8, Z and the complexification of $S_{\mathbb{R}}$ are given as follows. For d even, S^+ and S^- denote the semi-spinorial representations. For d odd, S denotes the spin representation.

$$\begin{array}{lll}
0: & \mathbb{C}\,, & S^+ + S^- \\
1: & \mathbb{R}\,, & S \\
2: & \mathbb{R}\,, & S^+ \text{ or } S^- \\
3: & \mathbb{R}\,, & S \\
4: & \mathbb{C}\,, & S^+ + S^- \\
5: & \mathbb{H}\,, & 2S \\
6: & \mathbb{H}\,, & 2S^+ \text{ or } 2S^- \\
7: & \mathbb{H}\,, & 2S
\end{array}$$

Proof. We first show that if d is congruent to 2 modulo 8, a morphism $\{\ \ \}$ with the listed properties exists. In this case, the signature is 0 modulo 8, so that $C(V)$ is a matrix algebra. Let $S = S^+ \oplus S^-$ be a simple $C(V)$-module. The possible $S_{\mathbb{R}}$ are S^+ and S^-. The pairing $(\ ,\ )$ on S (relative to which the principal antiautomorphism β is transposition) is odd (3.14). The corresponding $\{\ \}$: $S \otimes S \to V$ is even and symmetric (3.16).

Fix v with $Q(v) > 0$ and its orthogonal V_1. The graded module S remains simple as a graded $C(V_1)$-module, and S^+ remains absolutely irreducible as a representation of the group $\mathrm{Spin}(V_1)$. The bilinear form $\langle v, \{s,t\}\rangle$ on S^+ is not identically zero: if it were zero for one v, it would be zero for all v by $\mathrm{Spin}(V)$-invariance, and $\{\ ,\ \}$ would be zero. This form is invariant by the compact group $\mathrm{Spin}(V_1)$. As S^+ is an irreducible representation of $\mathrm{Spin}(V_1)$, it must be definite. The same applies to S^-.

We now show, in all dimensions, the existence of a morphism $S_{\mathbb{R}} \otimes S_{\mathbb{R}} \to V$ with the listed properties. Embed V in a Minkowski space V_0 of dimension congruent to 2 modulo 8: V is the orthogonal direct sum of V and of a negative defined quadratic space of a suitable dimension. Let pr be the orthogonal projection of V_0 onto V.

Let $S_0 = S_0^+ \oplus S_0^-$ be a simple $C(V_0)$-module. We dispose of $\{\ \}_0$: $S_0^\pm \otimes S_0^\pm \to V_0$, with the required properties for V_0. As $\mathrm{Spin}(V) \subset \mathrm{Spin}(V_0)$, we can also by restriction consider S_0 as a $\mathrm{Spin}(V)$-representation. The action of $\mathrm{Spin}(V)$ comes from the structure of $C(V)$-module of S_0, deduced from its structure of $C(V)$-module by $C(V_0) \hookrightarrow C(V)$. It follows that the irreducible constituents of S_0, as a $\mathrm{Spin}(V)$-representation, are of the type considered in the theorem. Further, S being a faithful $C(V_0)$-module, and hence a faithful $C(V_0)$-module, all representations of the type considered occur as a direct factor. There exists a morphism of $\mathrm{Spin}(V_0)$-representations $S_{\mathbb{R}} \hookrightarrow S_0^+$ or $S_{\mathbb{R}} \hookrightarrow S_0^-$. On $S_{\mathbb{R}}$, we now define $\{\ \}_1$ as the orthogonal projection

$$\{s,t\}_1 = \mathrm{pr}\{s,t\}_0$$

of $\{\ \}_0$ on V. From the same properties of $\{\ \}_0$, the form $\{\ \}_1$ is symmetric and if $Q(v) > 0$ with v in V, the form $\langle v, \{s,t\}_1\rangle = \langle v, \{s,t\}_0\rangle$ on $S_{\mathbb{R}}$ is positive or negative defined. It remains to average $\{\ \}_1$ over the compact group K of elements of norm 1 in Z to obtain the required

$$\{s,t\} = \int_K \{ks, kt\}_1 dk$$

We now prove the unicity, up to a real factor, of $\{\ \}$.

Case 1. $Z=\mathbb{R}$,i.e. the complexification $S_{\mathbb{C}} := S_{\mathbb{R}} \otimes \mathbb{C}$ of $S_{\mathbb{R}}$ is an irreducible representation. We note it S_0. It is spinorial for d odd, semi-spinorial for d even. If we exclude the case $d=2$, V is an absolutely irreducible representation, and it suffices to see that, over $\mathbb{C}$, $V_{\mathbb{C}}$ occurs at most once in $S_0 \otimes S_0$.

The case $d=2$ is easy to treat directly: V decomposes as $D_1 \oplus D_2$, for D_1 and D_2 the two isotropic lines, and the $S^{\pm}$ are of dimension one with tensor square isomorphic, respectively, to D_1 and D_2.

Case 2. $Z=\mathbb{H}$, i.e. $S_{\mathbb{C}}$ is twice an irreducible representation S_0: $S_{\mathbb{C}} = S_0 \otimes_{\mathbb{C}} W$ with $\dim(W)=2$. The invariance condition on $\{\quad\}$ amounts, after complexification, to $\mathrm{SL}(2,W)$-invariance: $\{\quad\}$ is to be the tensor product of $\{\quad\}_0$: $S_0 \otimes S_0 \to V_{\mathbb{C}}$ and of the unique (up to a factor) antisymmetric ψ: $W \otimes W \to \mathbb{C}$. Symmetry of $\{\quad\}$ amounts to antisymmetry of $\{\quad\}_0$. Again, it suffices to see that $V_{\mathbb{C}}$ occurs at most once in $S_0 \otimes S_0$.

Case 3. $Z=\mathbb{C}$, i.e. $S_{\mathbb{C}}$ is the sum of two inequivalent (complex conjugate) representations. This can happen only for d even, with $S_{\mathbb{C}} = S^+ \oplus S^-$ the sum of the two semi-spinorial representations. The invariance condition on $\{\quad\}$ amounts to say that, after complexification, $\{\quad\}$ is the sum of $S^+ \otimes S^- \to V$ and of its symmetric. It suffices to see that V occurs at most twice in$(S^+ \oplus S^-) \otimes (S^+ \oplus S^-)$.

Let us check those multiplicity statements, over $\mathbb{C}$. If d is odd, and for S the spinorial representation, $S \otimes S \simeq \underline{\mathrm{End}}(S) \simeq C^+(V) \simeq \overset{2*}{\wedge} V$. As $\overset{i}{\wedge} V \simeq \overset{d-i}{\wedge} V$ as a representation of the special orthogonal group, this can be rewritten

$$S \otimes S \simeq \bigoplus \overset{i}{\wedge} V \qquad (\text{sum for } 0 \leq i \leq (d-1)/2)\ .$$

The $\overset{i}{\wedge} V$ for $i \leq (d-1)/2$ are irreducible, non isomorphic (fundamental) representations, and V occurs only once.

If d is even, let $S = S^+ \oplus S^-$ be the sum of the semi-spinorial representations. We have now

$$S \otimes S \simeq \bigoplus \overset{i}{\wedge} V \qquad (0 \leq i \leq d)$$

and V occurs exactly twice. The unicity claim follows.