Lecture 1: Perturbative renormalization

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This lecture is a very elementary introduction to renormalization of Feynman integrals.

1.1. Perturbative expansion of a 2-point correlation function.

Consider a quantum field theory with one scalar bosonic field $\phi(x)$ on a Minkowski space $(V, (,))$ of signature (1,n-1) (see Kazhdan's lectures). Let V_+, V_- be the closures of the upper and the lower part of the full cone of time-like vectors. Let $\mathcal{H}$ be the Hilbert space of the theory, $\mathcal{D} \subset \mathcal{H}$ a dense subspace, and $\varphi : \mathcal{S}(V) \to \mathrm{End}(\mathcal{D})$ be the quantization map. We assume that the triple $(\mathcal{H}, \mathcal{D}, \phi)$ satisfies Wightman axioms. In this case we can define quantum fields $\phi(x) := \varphi(\delta_x)$, which are distributions on V with values in $\mathrm{End}(\mathcal{D})$. When it does not cause confusion, we will treat them as usual $\mathrm{End}(\mathcal{D})$-valued functions.

We can assume without loss of generality that the 1-point Wightman function of the theory vanishes. Indeed, the 1-point function is a constant C, and we can redefine the map φ by setting $\varphi'(f) = \varphi(f) - C\int f dv$. The map φ' satisfies Wightman axioms as well and has a zero 1-point function.

We look at the time ordered 2-point Wightman function

1.1 $$\mathcal{W}_2^T(x,y) = \langle \Omega | T(\phi(x)\phi(y)) | \Omega \rangle$$

Here the time ordering T means the following. By the definition,

$$T(\phi(x)\phi(y)) = \phi(y)\phi(x)$$

if $x - y \in V_-$, and

$$T(\phi(x)\phi(y)) = \phi(x)\phi(y)$$

otherwise. Since $\phi(x)$ commutes with $\phi(y)$ when $x - y$ is space-like, the function (1.1) is even. Because of the Poincare invariance axiom, (1.1) is a function of $x - y$. Denote this function by $W(x)$. It follows from the Poincare invariance that the function $W(x)$ actually depends only on x^2.

Let $\mathcal{H}_1$ be the closed subspace of $\mathcal{H}$ spanned by vectors of the form $\phi(f)\Omega$, where f is a Schwartz function on V. The space $\mathcal{H}_1$ is a representation of the Poincare group, and all its irreducible components have spin zero, i.e.

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have the form $L^2(\mathcal{O}_m^+)$, where $\mathcal{O}_m^+$ is the upper sheet of the two-sheeted hyperboloid $k^2 = m^2$ in the dual space V^*. This happens because we have a homomorphism of representations $\mathcal{S}(V) \to \mathcal{H}_1$, given by $f \to \phi(f)\Omega$, whose image is dense in $\mathcal{H}_1$.

Let $\rho_1 : P \to \mathrm{Aut}(\mathcal{H}_1)$ be the action of the Poincare group in $\mathcal{H}_1$. If $x \notin V_-$, we can write $W(x)$ as

1.2 $$W(x) = \lim_{\varepsilon\to 0}\langle\rho_1(x)v_\varepsilon, v_\varepsilon\rangle, v_\varepsilon = \varphi(\delta_\varepsilon)\Omega,$$

where δ_ε is a family of smooth Schwartz functions tending to the δ-function.

Assume first that $\mathcal{H}_1$ is an irreducible representation, i.e.

$$\mathcal{H}_1 = L^2(\mathcal{O}_m^+).$$

In this case, it is very easy to evaluate (1.2) explicitly. Indeed, the operator $\rho_1(x)$ in this case is just the operator of multiplication by the function

$$e^{i(k,x)},\ (k \in \mathcal{O}_m^+ \subset V^*).$$

It is easy to see that there exists a limit as $\varepsilon \to 0$ of v_ε in the sense of distributions. Because of P-invariance, this limit is an $O(V)$-invariant distribution on $\mathcal{O}_m^+$. Therefore, this limit equals to a constant function on $\mathcal{O}_m^+$. We can always normalize this constant to 1, by rescaling the map φ. Then from (1.2) we will get

1.3 $$W(x) = W_m(x) := \int_{\mathcal{O}_m^+} e^{i(k,x)}dk, x \notin V_-; W(-x) = W(x),$$

where dk is the $O(V)$-invariant measure on $\mathcal{O}_m^+$: if $k = (k_0, k_1)$, $k_0 \in \mathbb{R}$, $k_1 \in \mathbb{R}^{n-1}$, then $dk = dk_1/\sqrt{k_1^2 + m^2}$, and dk' is the Lebesgue measure on $\mathbb{R}^{n-1}$.

Let δ_m^+ denote the delta function of the upper sheet of the hyperboloid $k^2 = m^2$, and δ_m^- the delta function of its lower sheet. Then (1.3) says that $W_m(x)$ equals to the Fourier transform of δ_m^+ in V_+, of δ_m^- in V_-, and to both of them in the rest of V (these Fourier transforms are equal outside of $V_+ \cup V_-$).

It is clear from this description that it is more convenient to work with the Fourier transform of the function $W_m(x)$ than with this function itself. The Fourier transform of $W_m(x)$ is obtained by a direct computation, similar to one from Kazhdan's lectures. The answer is called the Klein-Gordon propagator:

1.4 $$\tilde{W}_m(k) = w_m(k^2) = \frac{i}{k^2 - m^2 + i\varepsilon},$$

where by definition $\frac{1}{k^2-m^2+i\varepsilon}$ is the distribution on V^* obtained as the weak limit of $\frac{1}{k^2-m^2+ia}$ as $a \to +0$.

Now we turn to the general case, when $\mathcal{H}_1$ is not an irreducible representation of P. We will define the spectral measure μ of $\mathcal{H}_1$ (on $\mathbb{R}$) by the following rule. Let $\alpha \in \mathcal{S}(\mathbb{R})$ be a positive function. We define the integral $\int \alpha(s)d\mu(s)$ by

1.5 $$\int \alpha(s)d\mu(s) = \langle \varphi(f), \varphi(f) \rangle,$$

where f is any function in $\mathcal{S}(V)$ such that $\int_{\mathcal{O}_s^+} |\tilde{f}|^2 dk = \alpha(s)$, (here $\tilde{f}$ is the Fourier transform of f). It is easy to show that the r.h.s. of (1.4) does not depend on the choice of f, so the measure $\mu(s)$ is well defined. By the definition of $\mu(s)$, the function $W(x)$ can be expressed in the form

$$W(x) = \int W_s(x)d\mu(s).$$

Therefore, the Fourier transform of $W(x)$ has the form

$$\tilde{W}(k) = w(k^2) = \int \frac{id\mu(s)}{k^2 - s^2 + i\varepsilon}.$$

It follows from scattering theory (See Kazhdan's lectures) that if $\mathcal{H}$ has a subrepresentation of P isomorphic to $L^2(\mathcal{O}_m^+)$ then it has components of the form $L^2(\mathcal{O}_s^+)$, occuring as continuous spectrum, for all $s \geq 2m$. Therefore, one expects that for a sufficiently generic quantum field theory, the space $\mathcal{H}_1$ will already contain some of this spectrum, and therefore it will be possible to see it looking at the measure μ, i.e. at the function $W(x)$.

In the case when the field theory φ is a small perturbation of the theory φ_0 of a free scalar field of mass m_0, it is expected thet $\mathcal{H}$ has a P-invariant subspace isomorphic to $L^2(\mathcal{O}_m^+)$ for m close to m_0, a continuous spectrum from $2m$ to ∞, and, possibly, a finite number of discrete components. This assumtion can be tested by looking at μ: it would mean that μ is supported at $\{m\} \cup [2m, \infty)$, has an atom at $s = m$, and is absolutely continuous with respect to the Lebesgue measure for $s \geq 2m$ (the new discrete spectrum makes an exponentially small contribution to μ, with respect to the deformation parameter, and so it is not seen in the perturbation expansion).

In terms of the function $w(s)$, this means the following. Let us analytically continue $w(s)$ to a complex analytic function. Then, the conjectural behavior of the spectrum that we described above would mean that $\tilde{w}(s)$ has a pole at $s = m^2$ and a cut from $4m^2$ to $+\infty$, with jump $-2\pi d\mu/ds$ when crossing the cut from up to down at the point s.

Now we will take a concrete quantum field theory and compute a perturbative expansion of the function $w(k^2)$, in order to find out if it really has such analytic properties.

1.2. The ϕ^3-theory.

Now we consider the quantum field theory with the Lagrangian

1.6
$$\mathcal{L} = \int \left(\frac{1}{2}(\nabla\phi)^2 - \frac{m^2}{2}\phi^2 + \frac{g}{3!}\phi^3 \right) d^n x.$$

This theory is a perturbation of the theory of a free scalar field of mass m with respect to a small parameter g. It is called the ϕ^3-theory.

Remark. From physical point of view, the ϕ^3-theory is unsatisfactory, since the energy in this theory is not bounded below, for any finite nonzero value of g. However, one can consider this theory perturbatively, i.e. regard g as an infinitesimal formal parameter, which in algebraic terms means that we work over the ring $\mathbb{C}[[g]]$. Using this theory as an example, we will do some Feynman diagrams computations, which are done in a similar manner in more complicated but more physically meaningful theories.

According to the rules of quantum mechanics, if the ϕ^3-theory actually existed, the correlation functions of this theory would be given by the formal expression

1.7
$$\langle \Omega | \phi(x_1)...\phi(x_N) | \Omega \rangle = \frac{1}{Z} \int \phi(x_1)...\phi(x_n) e^{i\mathcal{L}(\phi)} D\phi,$$

where

1.8
$$Z = \int e^{i\mathcal{L}(\phi)} D\phi$$

is the partition function. As usual, these "formulas" do not apriori make sense, as the formal expression $e^{i\mathcal{L}(\phi)} D\phi$ does not represent a measure on the space of fields. However, if $g = 0$ (the free theory), we can use (1.7) as a definition: define the integral

1.9
$$\int \phi(x_1)...\phi(x_n) e^{i\mathcal{L}(\phi)} D\phi$$

to be equal to $\langle \Omega | \phi(x_1)...\phi(x_N) | \Omega \rangle$. Further, for $g \neq 0$, we can expand (1.7) in a formal series in powers of g, and successive coefficients will be expressed as finite-dimensional integrals of expressions of the form (1.9). If we can compute these finite-dimensional integrals, we can get natural definition of (1.7). This computation is done using the Feynman diagrammatic techniques. Unfortunately, it turns out that some of these integrals are divergent and need to be renormalized.

Remark. Strictly speaking, in ϕ^3-theory as we stated it, the 1-point function does not vanish. However, as we explained before, this problem can be removed by shifting the quantization map φ. In the language of functional integral, it corresponds to adding an auxiliary linear term to the Lagrangian, and in the language of Feynman diagrams it corresponds to ignoring graphs with one external vertex, and all graphs which contain such graphs as subgraphs. In fact, this shifting procedure is not so trivial, since the integral representing the 1-point function is divergent for $n \geq 1$. Thus the shifting procedure requires renormalization of some graphs with 1 external vertex. Therefore, we will come back to this topic at the end of the lecture. Until then, we will assume that the 1-point function has been normalized to zero.

1.3. Perturbative expansion of Feynman integrals

In this part of the lecture we will remind how to compute the perturbative expansion of Feynman integrals. For simplicity consider the finite-dimensional case (Kazhdan's lectures). Suppose we have a finite dimensional real vector space $\mathcal{S}$ with a positive definite symmetric bilinear form B. Let dv be a Lebesgue measure on $\mathcal{S}$ such that

1.10 $$\int_{\mathcal{S}} e^{-B(v,v)/2} dv = 1.$$

We want to learn to compute the integral

1.11 $$\int_{\mathcal{S}} P(v) e^{-B(v,v)/2} dv,$$

where $P : \mathcal{S} \to \mathbb{R}$ is a polynomial. This integral is a sum of integrals of the form

1.12 $$\langle f_1 ... f_N \rangle_0 := \int_{\mathcal{S}} f_1(v) ... f_N(v) e^{-B(v,v)/2} dv,$$

where $f_1, ..., f_N \in \mathcal{S}^*$. It is clear that if N is odd, this integral vanishes, as the integrand is an odd function. Thus, it is enough to consider the case when $N = 2K$. In this case, the answer is given by the following formula, which is called the Wick formula.

Proposition 1.1

1.13 $$\langle f_1 ... f_{2N} \rangle_0 = \sum_{s \in S_{2K}/\sim} B^{-1}(f_{s(1)}, f_{s(2)}) ... B^{-1}(f_{s(2K-1)}, f_{s(2K)}),$$

where B^{-1} is the inverse form to B on $\mathcal{S}^*$, S_{2K} is the symmetric group, and $s_1 \sim s_2$, $s_1, s_2 \in S_{2K}$ if they define the same term in (1.13).

The proof is obvious: the right hand side of (1.13) is the only polylinear expression in f_i invariant under the group $O(\mathcal{S}) \times S_{2K}$, up to a factor, and the normalization is deduced from the case $f_1 = ... = f_{2K} = f$.

Remark. The set $S_{2K}/\sim$ is the set of pairings, i.e. splittings of the set $\{1, ..., 2K\}$ into pairs. Thus, terms of the r.h.s. of (1.13) are in 1-1 correspondence with pairings. In particular, the number of these terms is equal to $(2m)!/2^m m!$.

Now consider a perturbation of this situation. Let

$$Q(v) = \sum_{n \geq 1} g_m Q_m(v^{\otimes m})/m!,$$

where g_m are formal variables and $Q_m \in S^m \mathcal{S}^*$. Consider the integral

1.14 $$\langle f_1 ... f_N \rangle = \int_{\mathcal{S}} f_1(v) ... f_N(v) e^{-B(v,v)/2 + Q(v)} dv \in \mathbb{C}[[g_1, g_2, g_3, ...]].$$

This integral can be computed using Feynman graphs, as follows.

Let $\mathfrak{n} = \{n_1, n_2, n_3, n_4, ...\}$ be any sequence of nonnegative integers which is eventually zero. Let $G(N, \mathfrak{n})$ be the set of equivalence classes of all graphs which have N 1-valent vertices labeled by $1, ..., N$, and n_i unlabeled i-valent vertices, $i \geq 1$. The labeled vertices are called external, and the unlabeled ones internal.

Any graph $\Gamma \in G(N, \mathfrak{n})$ defines a polylinear function F_Γ of $f_1, ..., f_N$ which is defined as follows. At the 1-valent vertex of Γ labeled by i one places the vector f_i, at every m-valent vertex one places the tensor Q_m, and then takes contractions along edges using the form B^{-1} (see Kazhdan's lectures). Then one has

Theorem 1.2

1.15 $$\langle f_1, ..., f_N \rangle = \sum_{\mathfrak{n}} \prod_i g_i^{n_i} \sum_{\Gamma \in G(N, \mathfrak{n})} |Aut(\Gamma)|^{-1} F_\Gamma(f_1, ..., f_N),$$

where $Aut(\Gamma)$ denotes the group of automorphisms of Γ which fix the external vertices.

The proof is easy. First one observes that (1.13) is a special case of (1.15), when $\mathfrak{n} = 0$. Next, one can think of each i-valent vertex of a graph Γ as a collection of i 1-valent vertices which are situated very close to each other. Then it is clear that any graph $\Gamma \in G(N, \mathfrak{n})$ defines as many as $|\mathrm{Aut}(\Gamma)|^{-1} \prod i!^{n_i} \prod n_i!$ different pairings of such 1-valent vertices. Thus, formula (1.13) implies (1.15).

1.4. Computation of a Feynman integral over functions on a Minkowski space.

Now we will try to compute the function $w(k^2)$. Using formula (1.7), we get

1.16 $$W(x) = Z^{-1}\int T(\phi(x)\phi(0))e^{i\mathcal{L}(\phi)}D\phi$$

Applying formula (1.15), we express the r.h.s. of (1.16) as a sum over graphs.The function $Q(\phi)$, which was considered above for the finite dimensional case, is of the form $\frac{g}{3!}Q_3(\phi)$, where Q_3 is a cubic form given by

$$Q_3(\phi) = \langle \phi(a)\otimes\phi(b)\otimes\phi(c), i\delta_{a=b=c}\rangle.$$

Thus, we get a sum over all graphs with two external 1-valent vertices, and a number of internal trivalent vertices.

Graphs which have components without external vertices will not occur in this sum, since we have divided by Z. So there are two remaining types of graphs: connected and disconnected. Disconnected graphs have two components, each having one external vertex.

It is easy to see that the sum over all disconnected graphs equals

$$\langle\Omega|\phi(0)|\Omega\rangle^2 = 0.$$

Therefore, disconnected graphs can be ignored.

We see that the function $w(k^2)$, can be written as a sum over connected graphs with two external vertices. So, we can represent $w(k^2)$ in the form

1.17 $$w(k^2) = \sum_{j=0}^{\infty} w^{(j)}(k^2),$$

where $w^{(j)}(k^2)$ is the sum over all connected graphs which are chains of j 1-particle irreducible graphs. We have $w^{(0)}(k^2) = w_m(k^2) = \frac{i}{k^2-m^2+i\varepsilon}$ (the free propagator), and

1.18 $$w^{(j)}(k^2) = w^{(0)}(k^2)\left(\frac{w^{(1)}(k^2)}{w^{(0)}(k^2)}\right)^j.$$

Set

1.19 $$\Sigma(k^2) = i\frac{w^{(1)}(k^2)}{w^{(0)}(k^2)^2}.$$

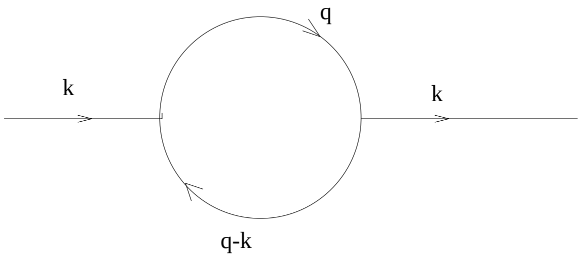
(It is easy to see that $\Sigma(k^2)$ is a real function). Summing the geometric progression in (1.17), we get

1.19 $$w(k^2) = \frac{i}{k^2-m^2-\Sigma(k^2)+i\varepsilon}.$$

Thus, it remains to compute $\Sigma(k^2)$. We hope to find that $\Sigma(k^2)$ is analytic near $k^2 = m^2$, and has a cut at $k^2 \geq 4m^2$. This would confirm the expected analytic behavior of $w(k^2)$.

In our setup, the bilinear form B^{-1} from the previous section has the kernel $\frac{i}{(k_1-k_2)^2-m^2+i\varepsilon}$, and the 3-tensor Q_3 has the kernel $i\delta(k_1+k_2+k_3)$. Therefore, the term corresponding to each graph Γ in the sum for $\Sigma(k^2)$ is an integral of a rational function over the space $H^1(\Gamma,\mathbb{R})\otimes V^*$. So the number of integrations is proportional to the number of loops in the graph. Therefore, these integrals, over loop parameters, are called loop integrals.

Now we will compute the first nontrivial coefficient of the g-expansion of Σ. We have $\Sigma = g^2\Sigma_2 + O(g^3)$, where Σ_2 is the sum over all connected graphs with two trivalent vertices (there is no graph with only one trivalent vertex). It is easy to see that there is only one such graph, namely the following one:



So we must put an integration parameter $q \in V^*$ on the only loop of the graph. Thus, the function $\Sigma_2(k^2)$ can be represented as a single loop integral

1.19 $$\Sigma_2(k^2) = \frac{i}{2}\int_V \frac{d^nq/(2\pi)^n}{(q^2-m^2+i\varepsilon)((q-k)^2-m^2+i\varepsilon)}.$$

(division by 2 arises from the automorphism group of Γ, which has order 2).

Remark 1. When we compute the group of automorphisms of a graph, we do not take into account the orientation of edges. The arrows are put on edges arbitrarily, in order to balance the momenta which are attached to edges. The distribution of momenta should satisfy the condition that the sum of incoming momenta at each internal vertex should equal the sum of outgoing ones.

Remark 2. It may appear that the integral in (1.19) is real, but in fact it is not. It is easy to see that for real negative k^2 the integral is imaginary, so the r.h.s. of (1.19) is real. This happens because of addition of $i\varepsilon$ in the denominator.

Now we try to compute integral (1.19). The integrand is a fraction where the denominator is a product of two different factors. For a general graph, the integrand will have a product of many different factors in the denominator, which causes inconvenience. There is a remarkable trick, which allows to convert this integral into an integral of a function whose denominator is a power of a single factor. This trick is the "Feynman famous formula":

Proposition 1.3

1.20 $$\frac{1}{A_1 ... A_N} = \int_{\alpha_1+...+\alpha_N=1} \frac{1}{(\sum \alpha_i A_i)^N} d\sigma,$$

where $d\sigma$ is a Lebesgue measure on the simplex with volume 1.

Proof We prove the statement by induction in N. For $N = 1$, the statement is obvious. Let $N > 1$.

Denote the r.h.s. of (1.20) by $I_N(A_1, ..., A_N)$. Then from a homogeneity argument it follows that

1.21 $$\int_{s \le \alpha_1+...+\alpha_N \le t} \frac{1}{(\sum \alpha_i A_i)^N} d\alpha_1 \wedge ... \wedge d\alpha_N = \frac{1}{(N-1)!} I_N(A_1, ..., A_N) \ln(t/s).$$

Now observe that the N-form under the integral in (1.21) has the form $d\omega$, where

1.22 $$\omega = \sum_{j=1}^{N} \frac{(-1)^j}{N(N-1)A_j(\sum \alpha_i A_i)^{N-1}} d\alpha_1 \wedge ... \wedge \widehat{d\alpha_j} \wedge ... \wedge d\alpha_N.$$

Therefore, using Stokes' formula, we get

1.23 $$\sum_{j=1}^{N} \frac{1}{N(N-1)A_j} \frac{1}{(N-2)!} I_{N-1}(A_1, ..., \hat{A}_j, ..., A_N) \ln(t/s)$$
$$= \frac{1}{(N-1)!} I_N(A_1, ..., A_N) \ln(t/s).$$

(the integrals of ω over simplices $\sum \alpha_i = s, t$ cancel each other, and only the integrals over N truncated simplices remain). Using the induction assumption, we obtain (1.20).□

We will use formula (1.20) for $N = 2$. In this case it has the form

1.24 $$\frac{1}{AB} = \int_0^1 \frac{d\alpha}{(\alpha A + (1-\alpha)B)^2}.$$

Applying this formula to (1.19), and shifting the integration variable, we obtain

1.25 $$\Sigma_2 = \frac{i}{2} \int_0^1 \int_V \frac{1}{(q^2 + \alpha(1-\alpha)k^2 - m^2 + i\varepsilon)^2} \frac{d^n q}{(2\pi)^n} d\alpha.$$

It is convenient now to perform a Wick rotation of the integration cycle. Write any vector $q \in V$ as (q_0, q_1), $q_0 \in \mathbb{R}$, $q_1 \in \mathbb{R}^{n-1}$, so that $q^2 = q_0^2 - q_1^2$ (where q_1^2 is the usual squared norm). Consider the 1-parameter family of integration cycles, $C(t) = \{(e^{\pi i t/2} q_0, q_1) \in V_{\mathbb{C}} | (q_0, q_1) \in V\}$, $0 \le t \le 1$. The integral (1.25) is over $C(0)$. It is clear that during the deformation $C(t)$ we do not pick up any poles of the integrand in (1.25) (since $\varepsilon > 0$). Therefore, if (1.25) converges, the integral over $C(0)$ (which is Σ_2) equals to the integral over $C(1)$.

The cycle $C(1)$ is a real subspace of $V_{\mathbb{C}}$ which carries a natural positive definite metric, $|q|^2 = -q^2$. Thus, integral (1.25) can be written as

1.26
$$\Sigma_2(k^2) = \frac{i}{2} \int_0^1 \int_{\mathbb{R}^d} \frac{1}{(|q|^2 - \alpha(1-\alpha)k^2 + m^2)^2} \frac{d^n q}{(2\pi)^n} d\alpha.$$

($i\varepsilon$ is no longer necessary, as the integrand is now smooth for negative k^2).

It is now an exercise to compute this integral explicitly in elementary functions. However, we are more interested in qualitative properties of the answer. Namely, from formula (1.26) it is obvious that

1) Σ_2 is analytic near $k^2 = m^2$.

2) Σ_2 has a cut at the set of those values of k^2 for which the integrand is singular.

Since the function $\alpha(1-\alpha)$ varies between 0 and 1/4 as α varies from 0 to 1, the cut occurs at $k^2 \ge 4m^2$. Thus, the function Σ_2 has the expected analytic behavior.

Remark. We could have considered from the very beginning a Euclidean theory. This means, V is a Euclidean space, the Lagrangian is

1.27
$$\int (\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3) d^n x,$$

The correlation functions are

1.28
$$\langle \phi(x)\phi(0) \rangle = \int_{S(V)} \phi(x)\phi(0) e^{-\mathcal{L}(\phi)} D\phi,$$

the propagator is $\frac{1}{k^2+m^2}$, the cubic functional corresponding to a trivalent vertex is $-\frac{1}{3!}\delta(k_1+k_2+k_3)$, and the Fourier transform of (1.28) is $\frac{1}{k^2+m^2+\Sigma}$, where $-\Sigma$ is computed as a sum over 1-particle irreducible connected graphs. If we try to compute Σ_2, we will get exactly the same answer as given by (1.26), i.e. $\Sigma_2^{\text{Euclidean}}(|k|^2) = \Sigma_2^{\text{Minkowski}}(-k^2)$. This is a general phenomenon: the Wick rotated answer for a theory in the Minkowski space coincides with the answer for the corresponding Euclidean theory.

1.5. Renormalization of divergent graphs.

Unfortunately, the integrals (1.25), (1.26) are divergent for $n \geq 4$, since the integrand does not decay rapidly enough at infinity. Renormalization techniques help to give meaning to these integrals anyway.

From now until the end of the lecture we consider the case $n = 4$, and the Euclidean picture. In this case, the easiest way to make sense of (1.26) is to differentiate with respect to k^2. If we differentiate under the integral sign, we will obtain a convergent integral, and then we can integrate it back, which will define Σ_2 up to a constant. However, this is, in general, not the best way to proceed. The method which is usually applied in renormalization theory is the following.

We will replace the propagator $\frac{1}{k^2+m^2}$ with a more rapidly decreasing propagator depending on a parameter Λ, of the form $P_\Lambda = \frac{\chi(k^2,\Lambda^2)}{k^2+m^2}$, where $\chi(k^2,\Lambda^2)$ is a smooth function with sufficiently rapid decay at $k^2 \to \infty$ for a fixed Λ, which tends to 1 at $\Lambda \to \infty$ (it is called the cutoff function). For instance, one can take $\chi(k^2,\Lambda^2) = \frac{\Lambda^4}{(k^2+\Lambda^2)^l}$, $l \in \mathbb{N}$. Computing Σ_2 for this new propagator, we will obtain a convergent integral for each finite value of Λ, and the answer will depend on Λ in the following way:

1.29 $$\Sigma_2(k^2,\Lambda,m) = -A\ln(\Lambda/m) + O(1), \Lambda \to \infty,$$

where A is a positive constant. Of course, the limit of (1.29) as $\Lambda \to \infty$ (which would be the value of (1.26)) does not exist: one says that the integral is logarithmically divergent.

Now fix a constant $m = m_R > 0$ (the renormalized mass). Given Λ, we will adjust the mass m_0 of the theory in such a way that the Σ-function $\Sigma(k^2,\Lambda,m_0,g)$ for the theory with this mass and the propagator P_Λ has a pole exactly at $k^2 = -m_R^2$. That is, define $m_0(m_R,g,\Lambda) = m_R + O(g)$ by the equation

$$m_R^2 = m_0^2 + \Sigma(-m_R^2,\Lambda,m_0,g).$$

It is clear the solution of this equation (modulo g^3) has the form $m_0^2 = m_R^2 - g^2\Sigma_2(-m_R^2,\lambda,m_0)$, so by (1.29) it has the form $m_R^2 + g^2 A\ln(C\Lambda/m_R)$, where C is a constant which depends only on the cutoff function χ. It is easy to see that there exists a limit

$$\Sigma_2^R(k^2,m_R,g) = \lim_{\Lambda\to\infty} \Sigma(k^2,\Lambda,m_0(m_R,g,\Lambda),g).$$

This limit is called the renormalized Σ-function. Of course, it depends on the choice of m_R.

Let us now understand what this renormalization does to the Feynman diagram expansion. The new Lagrangian of the theory, with the mass m_0, differs from the old one (with mass $m = m_R$) by the additional quadratic term $\frac{1}{2}Ag^2 \ln(C\Lambda/m_R)\phi^2$. Therefore, according to the rules of Feynman calculus, the g^2-term of the perturbation expansion will now be the sum of terms corresponding to two Feynman diagrams:

At the 2-valent vertex of the second diagram, we put the tensor

$$-Ag^2 \ln(C\Lambda/m_R)\delta(k_1 - k_2).$$

This gives us an extra constant summand in $\Sigma_2(k^2, \Lambda, m_0)$, which compensates the divergence and ensures that there exists a limit Σ_2^R of

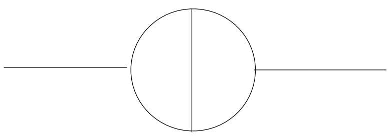
$$\Sigma_2(k^2, \Lambda, m_0(m_R, g, \Lambda))$$

as $\Lambda \to \infty$.

Remark. We have chosen our renormalization in such a way that the function $w(k^2)$ (modulo g^3) has a pole at $k^2 = -m_R^2$. This is the reason that the constant m_R is called the renormalized mass.

1.6. Renormalization in higher orders.

Now let us consider the terms our Feynman diagrams expansion which come with a power of g higher than 2. For example, consider the following graph,



which occurs with g^4. It is easy to see that this graph defines a convergent integral for $n = 4$. We will see that this is also the case for more complicated graphs. Namely, for $n = 4$, the more vertices a graph has, the better is the rate of convergence of the corresponding integral.

Let us analyse the situation for arbitrary n. It is convenient to introduce the following definitions.

Definition 1 *The superficial divergence index $div(\Gamma)$ of a graph Γ is the difference of the degrees of the numerator and denominator in the integrand of the corresponding integral.*

Remark. By the definition, the degree of the integration variables q_i and their differentials dq_i equals 1.

Definition 2 *A graph is called superficially divergent if $div(\Gamma) \geq 0$ and superficially convergent if $div(\Gamma) < 0$.*

Remark. Γ is called logarithmically, linearly, quadratically,... divergent if $div(\Gamma) = 0, 1, 2, ...$

Proposition 1.4 *In the ϕ^3-theory, $div(\Gamma) = (n-6)b_1 + 6 - 2E$, where b_1 is the number of loops, and E the number of external vertices.*

Proof Let M be the number of internal edges, and N the number of external vertices. Since Γ is connected and has only trivalent internal vertices, we have $M = \frac{3N-E}{2}$, $b_1 = \frac{N+2-E}{2}$. It is easy to see that the degree of the numerator in the integral corresponding to Γ is nb_1 (b_1 loop integrations over $\mathbb{R}^n$), and the degree of the denominator is $2M$ (M quadratic factors). Thus, $div(\Gamma) = nb_1 - 2M = (n-6)b_1 + 6 - 2E$.

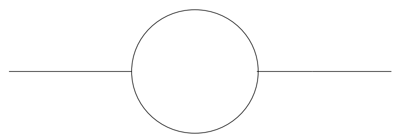
Proposition 1.4 implies the following.

1) If $n > 6$, divergence worsens as the number of vertices grow.

2) If $n = 6$, all graphs with $E = 2, 3$ are equally bad (have $div(\Gamma) = 6 - 2E$), while for $E \geq 4$ they are superficially convergent.

3) For $n \leq 3$, all graphs with $E \geq 2$ are superficially convergent.

4) For $n = 4$, the only superficially divergent graph with $E \geq 2$ is Γ_2:

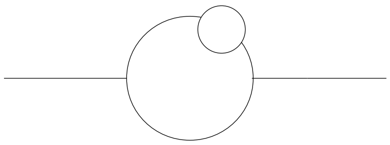


5) For $n = 5$, there are finitely many superficially divergent graphs.

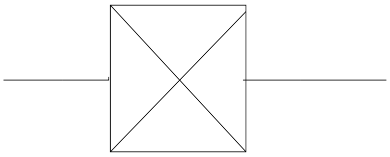
It is clear that superficial convergence of a graph is necessary but not sufficient for convergence of the corresponding integral. Indeed, a superficially convegent integral may have a subintegral that diverges. However, one can formulate a sufficient condition for actual convergence in terms of superficial convergence. This condition is given by Weinberg theorem.

Weinberg theorem *Let Γ be a graph such that the integral of the corresponding function over any subset of the set of loops of Γ is superficially convergent. Then the integral corresponding to Γ is convergent.*

Weinberg theorem gaurantees that in the ϕ^3-theory all integrals are convergent for $n \leq 3$. Let us see what happens if $n = 4$. In this case, we have one superficially divergent graph Γ_2. Of course, there are infinitely many superficially convergent but still divergent graphs, namely, all graphs which contain Γ_2 as a subgraph, e.g.

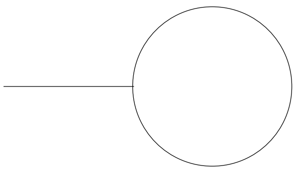
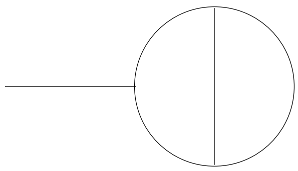


However, we have renormalized the graph Γ_2, i.e. compensated its divergence by adding another auxiliary graph of the form

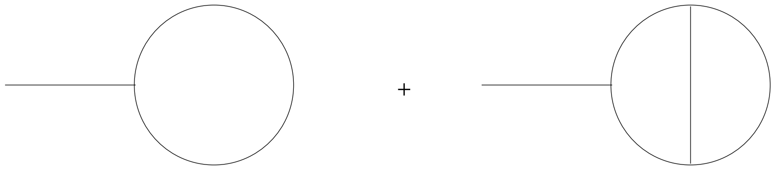


As a result, all graphs which are divergent because they contain Γ_2 will be renormalized automatically and become convergent. That is, the divergence of each graph containing Γ_2 will be compensated by a counterterm, in which Γ_2 will be replaced by the above auxiliary graph. The fact that this ensures convergence in all orders follows from the "Strong Weinberg theorem", which states, roughly, that if all graphs at orders $\leq N$ (in g) have been renormalized, then all superficially convergent graphs at order $N+1$ are actually convergent. Thus, after mass renormalization all correlation functions become well defined.

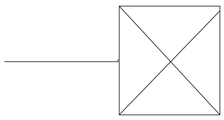
Remark. As we explain above, the procedure of setting the 1-point function to zero in ϕ^3-theory also requires renormalization. Namely, we have two graphs with one external vertex,

 , 

among which the first is quadratically divergent and the second logarithmically. Consider a new Lagrangian of the form $\mathcal{L}' = \mathcal{L} + B(m, g, \Lambda) \int \phi$, where $B(m, g, \Lambda) = a(m, g)\Lambda^2 + b(m, g)\Lambda + c(m, g)\ln(\Lambda/m) + d(m, g)$, and choose the constants a, b, c, d in such a way that the limit of the 1-point function for this Lagrangian, computed for the cutoff propagator at Λ, tends to 0. The constants a, b, c, d are determined by this condition uniquely, but they of course depend on the cutoff function χ. In the language of Feynman graphs, this corresponds to compensating the divergence in the sum



by adding a third summand



$B \cdot \delta(k)$

where at the vertex we put the linear functional $B \cdot \delta(k)$. This reduces the problem of renormalization of the ϕ^3 theory to renormalization of the graph Γ_2, which was done above.