Lecture 2: Perturbative renormalization (continued)

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2.1. Renormalizability of quantum field theories.

Last week we considered the ϕ^3 theory and found that its renormalizability (i.e. how many graphs are divergent, and how badly divergent they are) depends on whether $n < 6$, $n = 6$, or $n > 6$. Now we will see how to find that the critical value of n in the ϕ^3 theory is 6 without considering any graphs at all.

In order to find the critical value of n, we will define and compute the homogeneity degree of all terms in the Lagrangian. We recall that the Lagrangian is

$$\int (\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3) d^n x, \tag{2.1}$$

(we are considering the Euclidean theory; in the Minkowski version, the story is the same). Consider the dilation $x \to t^{-1}x$, $t \in \mathbb{R}$, $x \in V$. Under this transformation $d^n x \to t^{-n} d^n x$, $\nabla \to t\nabla$. If we want the integral (2.1) to be invariant under this dilation, we must impose the following three transformation laws: $(\nabla\phi)^2 \to t^n(\nabla\phi)^2$, $m^2\phi^2 \to t^n m^2\phi^2$, $g\phi^3 \to t^n g\phi^3$. From the first law we get $\phi \to t^{\frac{n-2}{2}}\phi$, from the second $m \to tm$, and from the third $g \to t^{\frac{6-n}{2}}g$. In the future we will write these scaling laws like this:

$$[x] = -1, [d^n x] = -n, [\nabla] = 1, [\phi] = \frac{n-2}{2}, [m] = 1, [g] = \frac{6-n}{2}. \tag{2.2}$$

In other words, any homogeneous quantity a scales as $a \to t^{[a]}a$ under the dilation, and the number $[a]$ is called the dimension of a.

Remark. It is not difficult to define the notion of dimension completely formally (using the tensor calculus on the tangent bundle of V), but it is more illuminating to illustrate it on examples.

We see that the number 6 appears in the scaling law for g. More precisely, we see that the behavior of the ϕ^3 theory is determined by whether $[g]$ is positive, zero, or negative. Namely, we have three cases.

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1. $[g] > 0$ ($n < 6$): there is a finite number of superficially divergent graphs (by graphs we always mean connected graphs).

2. $[g] = 0$ ($n = 6$): the number of superficially divergent graphs is infinite, but $div(\Gamma)$ is bounded from above.

3. $[g] > 0$ ($n > 6$): there are infinitely many graphs with any number of external vertices and arbitrarily high $div(\Gamma)$.

Now we will explain why this method works, and prove that it can be used to compute the critical value of the dimension of the spacetime in a general field theory. In general, we can consider the following setup. Suppose we have a quantum field theory with fields $\phi_1, ..., \phi_N$, and the Lagrangian

$$\mathcal{L} = \int (\sum_i Q_i(\phi_i) + \sum_k g_k I_k(\phi_1, ..., \phi_N)) d^n x, \tag{2.3}$$

where Q_i is a free (quadratic) part, and I_k are interaction (coupling) terms (differential monomials in fields, cubic and higher). Each interaction term comes with a small parameter g_k, which is called the coupling constant.

Definition *An interaction I_k is called subcritical if $[g_k] > 0$, and critical if $[g_k] = 0$. A field theory is called superrenormalizable if all interaction terms in the Lagrangian are subcritical, and is called renormalizable, or critical, if all interaction terms are critical or subcritical, but not all of them are subcritical. Otherwise, the theory is called non-renormalizable.*

For example, the ϕ^3 theory is superrenormalizable for $n < 6$, renormalizable for $n = 6$, and non-renormalizable for $n > 6$.

We will assume that our theory is a perturbation of a free theory, (for the definition of a free theory, see Kazhdan's lectures). In a free theory, one can easily see that the dimension of bosonic fields is $\frac{n-2}{2}$, and of fermionic fields is $\frac{n-1}{2}$. Since dimensions of fields are determined from the quadratic part of the Lagrangian, these dimensions will be the same in the perturbed (classical) theory as well.

We will also assume that $n \geq 2$ (In the quantum mechanical case $n = 1$ renormalization theory is not necessary). Then $[\phi_i] \geq 0$.

Theorem 2.1 *(i) If a theory is superrenormalizable, there is a finite number of superficially divergent graphs in its Feynman diagram expansion.*

(ii) If a theory is renormalizable then the number of superficially divergent graphs is infinite, but $div(\Gamma)$ is bounded from above.

(iii). If a theory is non-renormalizable then there are infinitely many graphs with any number of external vertices and arbitrarily high $div(\Gamma)$.

Proof Let Γ be a graph in the Feynman diagram expansion of our theory. Types of internal vertices of such a graph correspond to interaction terms in the Lagrangian, and types of its external vertices and edges correspond to fields ϕ_i. Let e_i be the number of external vertices of type ϕ_i, and v_k be the number of internal vertices of type I_k. Then it is easy to show that

$$div(\Gamma) = n - \sum_i e_i[\phi_i] - \sum_k v_k[g_k]. \tag{2.4}$$

Statements (i)-(iii) of the theorem follows immediately from (2.4).

Definition *A theory of the form (2.3) is called classically scale invariant if the n-form under the integral in the Lagrangian is invariant under dilations.*

For example, the theory of a free scalar field is scale invariant iff it is massless.

It is clear that a massless theory is scale invariant if and only if it is purely critical, i.e. $[g_k] = 0$ for all k.

Remark. Scale invariant theories are always conformal. Indeed, the Lagrangian is always written naturally in terms of the metric on the spacetime, and depends only on the fields and their first derivatives. This implies that a scale invariant Lagrangian has to be invariant under conformal changes of metric.

2.2. Critical dimensions of some field theories.

Now we will compute the critical dimension for several important field theories.

Example 1. Sigma-models.

Let M be a Riemannian manifold. The Lagrangian of the sigma-model on $\mathbb{R}^n$ with the target space M is

$$\mathcal{L}(\phi) = \int d^n x g_{ij}(\phi) \nabla\phi^i \cdot \nabla\phi^j. \tag{2.5}$$

Since this theory is conformal in two dimensions, 2 must be the critical dimension. Let us show by a direct computation that this theory is not renormalizable for $n > 2$.

Let ϕ_i be coordinates on M near some point. If the metric g_{ij} is constant in these coordinates, the theory is free. Consider a nonconstant metric of the form

$$g_{ij}(\phi) = \delta_{ij} + a_{ijk}\phi^k + r_{ijkl}\phi^k\phi^l + ... \tag{2.6}$$

(If we chose normal coordinates, we could get rid of a, but not of r, as r in normal coordinates is the Riemann curvature tensor). Substituting this into the Lagrangian, we find that $[r] = 2 - n$. This shows that sigma-model is not renormalizable beyond $n \geq 2$, unless the metric is flat.

Example 2. Gravity.

In the theory of gravity the spacetime is the space $\mathbb{R}^n$ with a metric of the form $g_{ij} = \delta_{ij} + h_{ij}\sqrt{G}$, where G is the Newton's constant (for us, it is just a formal parameter). The Lagrangian of the theory is

$$\mathcal{L}(g) = \frac{1}{G}\int R(g) d^n x, \tag{2.7}$$

where $R(g)$ is the scalar curvature of the metric. In terms of h, this Lagrangian can be rewritten as

$$\mathcal{L} = \int d^n x((\nabla h)^2 + \sqrt{G} h(\nabla h)^2 + ...), \tag{2.8}$$

so we get $[h] = \frac{n-2}{2}$, $[G] = 2 - n$. Thus, as in the previous example, the theory is non-renormalizable for $n > 2$ and critical for $n = 2$.

Remark. Here $h(\nabla h)^2$ stands for an expression which is linear in h and quadratic in ∇h. It is easy to compute what it is exactly, but it does not matter to us, since we are only interested in the dimension. So we will use such sloppy notation.

Example 3. Gauge theory.

In gauge theory fields are conections in a fixed principal G-bundle on the space $\mathbb{R}^n$, where G is a compact Lie group. The Lagrangian has the form

$$\mathcal{L}(\tilde{A}) = \frac{1}{e^2}\int \mathrm{Tr}(F_{\tilde{A}} \wedge *F_{\tilde{A}}), \tag{2.9}$$

where $F_{\tilde{A}}$ is the curvature of the connection $\tilde{A}$, and Tr is an invariant non-degenerate bilinear form on the Lie algebra $\mathfrak{g}$ of G.

In the computation of dimension, we will assume that our G-bundle is trivial, so a connection is represented by a 1-form A: $\tilde{A} = d + A$. Then $F = dA + A \wedge A$.

Consider the field $B = A/e$. In terms of B, the Lagrangian takes the form

$$\mathcal{L} = \int ((\nabla B)^2 + eB^2\nabla B + e^2 B^4) d^n x$$

We have $[B] = \frac{n-2}{2}$, so $[B^2\nabla B] = \frac{3n}{2} - 2$, and $[e] = \frac{4-n}{2}$. Thus, if the group G is noncommutative, the theory is superrenormalizable for $n < 4$, renormalizable for $n = 4$, and non-renormalizable for $n > 4$. In $n = 4$, the theory is conformal (If G is commutative, the theory is free).

Remark. Dimensions of fields in our computation agree with geometric dimensions. Indeed, in geometry, since $d \to td$ under the dilation $x \to t^{-1}x$, we have $[d] = 1$, so we must have $[A] = 1$. This coincides with our result: $[A] = [Be] = [B] + [e] = \frac{n-2}{2} + \frac{4-n}{2} = 1$.

Example 4. Gauge theory with a scalar bosonic field or with a fermionic field.

Consider the setting of Example 3, and let V be a finite-dimensional representation of G. Let ϕ be a section of the corresponding vector bundle on $\mathbb{R}^n$ (associated with the above G-bundle). The Lagrangian of the gauge theory with ϕ is

$$\mathcal{L}'(\tilde{A}, \phi) = \mathcal{L}(\tilde{A}) + \int (\nabla_{\tilde{A}}\phi)^2 d^n x, \tag{2.10}$$

where $L(\tilde{A})$ is the Lagrangian (2.9). Writing $\tilde{A}$ in the form $\tilde{A} = d + A$, $A = eB$, we get

$$\mathcal{L}' = \int ((\nabla B)^2 + eB^2\nabla B + e^2B^4 + (\nabla\phi)^2 + 2e(\nabla\phi, B\phi) + e^2(B\phi)^2)d^n x$$

For this formula we get $[B] = [\phi] = \frac{n-2}{2}$, and (even when G is commutative!) $[e] = \frac{4-n}{2}$. Thus, the theory is superrenormalizable in $n < 4$, renormalizable in $n = 4$, and nonrenormalizable for $n > 4$ (regardless of the commutativity of G).

The same answer applies if we have a fermionic field. Let ψ be a section of $V \otimes S$, where S is the spin bundle over the spacetime. The Lagrangian of the gauge theory with ψ is

$$\mathcal{L}''(\tilde{A}, \psi) = \mathcal{L}(\tilde{A}) + \int (\psi, D_{\tilde{A}}\psi)d^n x, \tag{2.11}$$

where $D_{\tilde{A}}$ is (i times) the Dirac operator along the connection $\tilde{A}$. The critical dimension in this theory, as before, is 4, for any nontrivial compact group G.

Remark. If $G = U(1)$ and V is the standard 1-dimensional representation of G, this theory is the quantum electrodynamics (QED).

Example 5. Theory of a scalar bosonic field.

Consider the Lagrangian

$$\mathcal{L}(\phi) = \int (\frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + Q(\phi))d^n x,$$

where $Q(\phi) = \sum g_k\phi^k$. We already considered this type of Lagrangian in Lecture 1. In particular, the ϕ^3-theory is a special case of this situation, when Q is a cubic polynomial. We have $[g_k] = n - k\frac{n-2}{2}$. Thus, for $n = 2$ all terms in the Taylor expansion of $Q(\phi)$ are subcritical. For $n = 3$, the term ϕ^k is subcritical for $k < 6$, critical for $k = 6$, and non-renormalizable for $k > 6$. For $n = 4$, the term ϕ^k is subcritical for $k < 4$, critical for $k = 4$, and non-renormalizable for $k > 4$. For $n = 5, 6$, the terms ϕ^k, $k \geq 4$ are non-renormalizable, and the term ϕ^3 is subcritical for $n = 5$ and critical for $n = 6$.

In particular, the following theories are critical: the ϕ^3 theory in 6 dimensions, the ϕ^4-theory in 4 dimensions, and the ϕ^6 theory in 3 dimensions.

Example 6. Yukawa interaction.

Consider a theory with a scalar bosonic field ϕ and a fermionic field ψ. We will consider the Lagrangian

$$\mathcal{L}(\phi, \psi) = \int ((\nabla\phi)^2 + (\psi, D\psi) + g\phi[\psi, \psi])d^n x \tag{2.12}$$

(ψ takes values in a vector bundle W which is a direct sum of several copies of the spin bundle; (,),[,] are a symmetric and a skew-symmetric form on W which are invariant under gauge transformations). The cubic term $\phi[\psi, \psi]$ is called the Yukawa interaction. Let us compute the dimension of its coefficient g.

We have $[\phi] = \frac{n-2}{2}$, $[\psi] = \frac{n-1}{2}$, so $[g] = \frac{4-n}{2}$. So this theory is critical in dimension 4, superrenormalizable in dimension < 4, and non-renormalizable in dimension > 4.

We observe that in dimension 4 all interactions except ϕ^3, ϕ^4 and $\phi[\psi, \psi]$ are "bad" (non-renormalizable).

Example 7. Standard model.

Let us now try to write down the most general renormalizable theory that lives in our 4-dimensional physical world. According to the above examples, we cannot include gravity or Σ-model, but we can include connections, bosons with terms up to degree 4, and fermions with Yukawa interaction If we only take these fields, these are the only renormalizable terms we can write. Thus the most general Lagrangian we can write in dimension 4 giving a renormalizable theory is

$$\mathcal{L}(A, \phi, \psi) = \int (e^{-2}F_A^2 + (\psi, D\psi) + (\nabla\phi)^2 + g_1\phi^4 + g_2\phi\psi^2 + \text{lower terms})d^4x. \tag{2.13}$$

The Standard Model is a theory which belongs to this family, with the group G containing $SU(3) \times SU(2) \times U(1)$.

2.3. Perturbative renormalization of critical theories.

From now on we will consider only critical theories. As a model example we can consider ϕ^4-theory in 4 dimensions, which has one critical interaction ϕ^4, or its extension containing fermions, which has an additional critical interaction $\phi\psi^2$ (the Yukawa interaction).

Consider the Lagrangian of ϕ^4-theory:

$$\mathcal{L} = \int \left(\frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4 \right) d^n x. \tag{2.14}$$

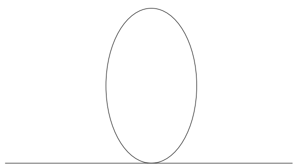
As before, we want to study the Schwinger function $\langle \phi(x_1)...\phi(x_N) \rangle$ (it is enough to consider only even N, as the Schwinger function vanishes for odd N). As usual, it is more convenient to consider its Fourier transform. This Fourier transform has the form $G_N(k_1, ..., k_N)\delta(k_1 + ... + k_N)$, where G_N is a function on the hyperplane $k_1 + ... + k_N = 0$.

Consider the Feynman diagram expansion of the function G_N. In ϕ^4 theory we have to sum over graphs whose internal vertices have 4 edges. As usual, we can restrict ourselves to connected graphs, as the sum over disconnected graphs expresses via Schwinger functions with fewer insertion points. In fact, as shown in Kazhdan's lectures, we can always restrict to 1-particle irreducible graphs (i.e. those which cannot be split by cutting one edge). So from now on we only consider connected, 1-particle irreducible graphs.

From formula (2.4) we get that the superficial divergence index of any graph equals $4 - E$, where E is the number of external edges. Thus, any graph with 2 external edges is quadratically divergent, any graph with 4 external edges is logarithmically divergent, and any graph with > 4 external edges is superficially convergent.

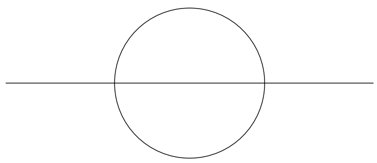
Now we will explain how to renormalize all N-point functions at all orders in g.

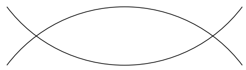
First of all, we can exclude graphs with 2 external edges which connect to the same internal vertex, e.g.



and all those which contain such a loop-like graph as a subgraph. Indeed, these graphs produce a constant function of k^2, so they can be removed by renormalization of mass. More precisely, there exists a function $P(\Lambda, g) = gP_1(\Lambda) + g^2P_2(\Lambda) + \ldots$, such that the sum of terms corresponding to all graphs for the theory with mass m^2, computed with the cutoff propagator (see lecture 1) equals the sum of terms corresponding only to graphs without loop-like subgraphs, but with mass $M(\Lambda)$ such that $M^2 = m^2 + P(\Lambda, g)$. So we can assume from the beginning that we have a theory with mass $M(\Lambda)$ and not worry about loop-like graphs.

Now we have no divergent graphs with one internal vertex, so we have no corrections to make in the first order in g. Let us look at the second order in g. In this case we have the following bad graphs:





Denote the first graph by Γ_2 and the second by Γ_4.

Thus, in order g^2 we have a problem in the 2-point and the 4-point functions. The problem in the 2-point function is created by Γ_2. The graph Γ_2 diverges quadratically. Therefore, the term corresponding to Γ_2 computed using the cutoff propagator is a function of k^2, Λ of the form $g^2\Sigma_2(k^2, \Lambda)$, where

(2.15)

$$\Sigma_2(k^2, \Lambda) = Ak^2 \ln(\Lambda/\mu) + B(\Lambda) + O(1), B(\Lambda) = B_0\Lambda^2(1 + o(1)), \Lambda \to +\infty$$

This asymptotics follows from the fact that the first derivative of (2.15) with respect to k^2 is logarithmically divergent, and the second one is convergent.

Another problem we have is in the 4-point function, created by the graph Γ_4. This graph is logarithmically divergent. This means, if we compute the term corresponding to this graph using the cutoff propagator, we will get a function of k_i, Λ of the form $g^2\Theta_2(k_1, k_2, k_3, \Lambda)$, where

$$\Theta_2(k_1, k_2, k_3, \Lambda) = C \ln(\Lambda/\mu) + O(1), \Lambda \to +\infty \tag{2.16}$$

This asymptotics follows the fact that the derivative of the integral corresponding to Γ_4 with respect to k_i is convergent.

In order to renormalize the 2- and 4- point functions in order g^2, we choose renormalized functions $\Sigma_2^R(k^2)$, $\Theta_2^R(k_1, k_2, k_3)$. Here Σ_2^R is an arbitrary function of k^2 whose second derivative is given by the (convergent!) integral obtained by applying $\left(\frac{d}{dk^2}\right)^2$ to the quadratically divergent integral corresponding to Γ_2. Analogously, Θ_2^R is an arbitrary function of k_1, k_2, k_3 whose derivatives are given by the convergent integrals obtained by differentiating the logarithmically divergent integral corresponding to Γ_4. It is clear that the function Σ_2^R is defined uniquely up to addition of a function of k^2 of the form $ak^2 + b$, and the function Θ_2^R is defined uniquely up to addition of a constant.

Now we will make second order corrections to the coefficients of $(\nabla\phi)^2$, ϕ^2 and ϕ^4 in the Lagrangian. Namely, we will consider a new Lagrangian of the form

(2.17)

$$\mathcal{L} = \int \left(\frac{1}{2}(\nabla\phi)^2 + \frac{M^2}{2}\phi^2 + \frac{g}{4!}\phi^4 + \frac{\alpha(\Lambda)}{2}(\nabla\phi)^2 + \frac{\beta(\Lambda)}{2}\phi^2 + \frac{\gamma(\Lambda)}{4!}\phi^4 \right) d^n x,$$

Let $\Sigma_2'(k^2, \Lambda, \alpha, \beta, \gamma)$, $\Theta_2'(k_1, k_2, k_3, \Lambda, \alpha, \beta, \gamma)$ be the functions Σ_2, Θ_2 for this Lagrangian and the cutoff propagator. We will choose the functions α, β, γ in such a way that

$$\begin{aligned} \lim_{\Lambda\to\infty} \Sigma_2'(k^2, \Lambda, m, \alpha(\Lambda), \beta(\Lambda), \gamma(\Lambda)) &= \Sigma_2^R(k^2), \\ \lim_{\Lambda\to\infty} \Theta_2'(k_1, k_2, k_3, \Lambda, m, \alpha(\Lambda), \beta(\Lambda), \gamma(\Lambda)) &= \Theta_2^R(k_1, k_2, k_3), \end{aligned} \tag{2.18}$$

As we are working modulo g^3, we can choose α, β, γ in the form $\alpha = g^2\alpha_2$, $\beta = g^2\beta_2$, $\gamma = g^2\gamma_2$, where $\alpha_2, \beta_2, \gamma_2$ are independent of g. It is easy to check that in order for (2.18) to hold, the functions $\alpha_2, \beta_2, \gamma_2$ should have the following asymptotics:

(2.19)

$$\alpha_2 = g^2(A\ln(\Lambda/m) + D_1) + o(1), \beta_2 = g^2(-B(\Lambda) + D_2) + o(1),$$
$$\gamma_2 = g^2(C\ln(\Lambda/m) + D) + o(1), \Lambda \to +\infty,$$

where D_1, D_2 depend on the choice of Σ_2^R, and D depends on the choice of Θ_2^R. Of course, there are many ways to choose such functions, but they are unique up to adding terms $o(1)$, $\Lambda \to \infty$.

Thus, we have renormalized the graphs Γ_2, Γ_4. This removes divergence in all correlation functions modulo g^3. Thus, all correlation functions of our theory are now defined modulo g^3.

Now we proceed inductively in the order of g. Suppose we have removed divergences and defined all correlation functions modulo g^K. Consider the $2N$-point function (for the deformed Lagrangian and the cutoff propagator) modulo g^{K+1}:

$$F_{2N} = \sum_{j=0}^{K-1} g^j F_{2N,j}^R + g^K F_{2N,K} \tag{2.20}$$

(the superscript R means that the corresponding coefficient has already been renormalized). The term $F_{2N,K}$ is represented by the sum over all graphs with K internal vertices. This sum has superficial divergence index $4-2N$. Therefore, the second derivative of F_2 by k^2, the first partial derivatives of F_4, and F_{2N}, $N \geq 3$, are superficially convergent. The crucial fact for renormalization theory, which follows from the "Strong Weinberg Theorem" (see Lecture 1), is

Proposition 2.2 *There exists finite limits, as $\Lambda \to \infty$, of the functions $F_{2N,K}(k_1, ..., k_{2N-1}, \Lambda)$, $\nabla_k F_{4,K}(k_1, k_2, k_3, \Lambda)$, and $\left(\frac{d}{dk^2}\right)^2 F_{2,K}(k^2, \Lambda)$ $N \geq 3$.*

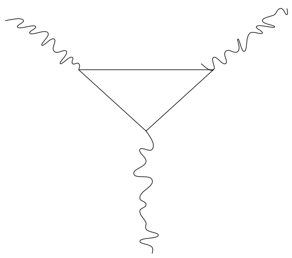
Remark. For ϕ^4 theory, this proposition holds for the term corresponding to each particular graph, but in general (for example, for theories with gauge fields) this is not the case: the sum over all graphs may have a meaning while each individual graph does not. However, an analogue of Proposition 2.2 (for the sum over all graphs) holds in any renormalizable theory.

Proposition 2.2 allows us to fulfil the induction step. It shows that the function F_2 (in the limit) is defined up to adding ak^2+b, the function F_4 is defined up to adding a constant, and F_{2N}, $N \geq 3$, is defined uniquely. So one can choose renormalized functions $F^R_{2N,K}$ and make corrections in the Lagrangian, $\alpha \to \alpha + g^K \alpha_K$, $\beta \to \beta + g^K \beta_K$, $\gamma \to \gamma + g^K \gamma_K$, to compensate the divergence in $F_{2N,K}$ and obtain $F^R_{2N,K}$ instead of it. This procedure is completely analogous to the one for order g^2. In this way we will complete the renormalization in order K.

Remark 1. At every step of our renormalization procedure we had to choose 3 constants of integration. This may create an impression that we get a family of theories parametrized by 3 infinite sequences of constants. However, it is easy to see that in fact we get a family of theories parametrized by only 3 constants. This means that any 4 invariants attached to the theory (for example, values of the 2-point function at 4 points in spacetime) are linked by a universal functional relation.

Remark 2. Even if in the original theory certain critical or subcritical interactions were not present, they may appear in the process of renormalization. In general, renormalization brings in all missing critical and subcritical terms, unless there is a symmetry which prevents it from doing so. Let us demonstrate it by a few examples.

Example 1. In the process of renormalization of the Lagrangian (2.12) in 4 dimensions we will be forced to introduce the subcritical term ϕ^3 and the critical term ϕ^4, in order to remove logarithmic divergence in the graphs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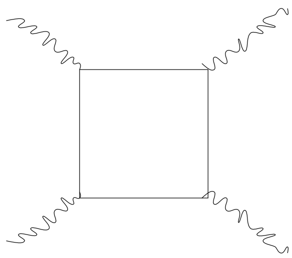
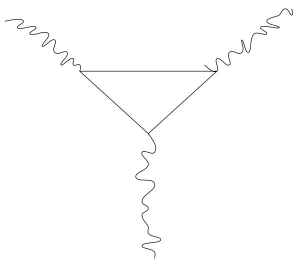


where wavy lines correspond to bosons and straight ones to fermions. However, the critical term $\phi^2\nabla\phi$ will not appear, since there is no Poincare invariant expression of this form. In terms of graphs, this means that the graph



whose superficial divergence is linear, in fact diverges only logarithmically, because of cancellations in the integrand caused by Poincare symmetry; so there is no linear divergence to compensate and hence no need for $\phi^2\nabla\phi$ to appear.

Example 2. In ϕ^4 theory in 4 dimensions, the subcritical term ϕ^3 does not appear in renormalization, since it is not preserved by the symmetry $\phi \to -\phi$ of the original theory. In the language of graphs, this is clear: there is no graphs with 3 external edges, so there is no divergence to compensate by ϕ^3.

In a general field theory, every type (in terms of external edges) of a superficially divergent graph corresponds to a number of critical and subcritical terms in the Lagrangian, which should be renormalized in order compensate the divergence in the corresponding graph. More precisely, divergent terms which are quadratic in k correspond to terms in the Lagrangian which have two derivatives by x, linear terms in k correspond to terms with one derivative, and constant divergencies correspond to terms without derivative.

Remark. It follows from formula (2.4) that in a renormalizable theory, all divergences in $N \geq 2$-point correlation function are no worse than quadratic. So the coefficient of k^2 in the 2-point function if divergent at most logarithmically. Therefore, if the quadratic forms Q_i have to be renormalized (like in ϕ^4 theory), they will be multiplied by coefficients which depend on the cutoff parameter Λ at worst logarithmically. This shows that the dimension of Q_i and hence of ϕ_i survives renormalization to all finite orders in the asymptotic expansion.