Lecture 3: Perturbative renormalization (continued)

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In this lecture we will discuss composite operators, their renormalization and operator product expansion (OPE).

3.1. Local functionals in a classical field theory.

Consider an n-dimensional classical field theory with spacetime V and a Lagrangian $\mathcal{L}$ of the form (2.3), with fields $\phi_1, ..., \phi_N$. Let X be the space of classical solutions for this theory. We want to consider functions on X called local functionals, which are defined as follows.

Let $x \in V$ be a point in the spacetime.

Definition *A local functional at x is a function of the fields and finitely many derivatives of the fields, evaluated at x.*

Example In the theory of a scalar bosonic field,

$$\phi^l(x), \phi'(x)^2, \phi^2(x)\phi'(x)^2$$

are local functionals, but $\phi(x) + \phi(2x)$ is not.

From the previous lecture it is clear how to define dimension of a local functional. We want to consider only homogeneous functionals of finite dimension and their finite linear combinations. Therefore, if a field has positive dimension (which is always the case if $n > 2$), we only consider polynomial functionals of this field. However, in 2-dimensional theories, where bosonic fields are dimensionless, it is reasonable and useful to consider more general functions of them (for example, the Lagrangians of Toda theories contain expressions of the form e^{ϕ}).

If $n > 2$, the space of all functionals of a given dimension is finite-dimensional, but if $n = 2$, it is infinite-dimensional.

Since elements of X satisfy the classical field equations, the same local functional can be written in different ways. For example, in ϕ^4-theory the classical field equation is

$$\Delta\phi = m^2\phi + \frac{g}{3!}\phi^3, \tag{3.1}$$

[1] Notes by Pasha Etingof and David Kazhdan, TeXnical editing Misha Verbitsky

so the left and the right hand sides of (3.1) evaluated at $x \in V$ represent the same local functionals of ϕ. Thus, local functionals are all possible differential expressions in ϕ modulo the classical field equations.

Remark 1. One should be careful to distinguish between two different notions of field dimension which arise in field theory. The first is "the engineering dimension" and says in what units the field is measured (if the units are cm^{-d} then the engineering dimension is d). The second is "the scaling dimension", which is the dimension we have been talking about. These two dimensions are not always the same. For example, in the theory of one field ϕ defined by (3.1) the engineering dimension of the field $m^2\phi$ is 3 (it is measured in cm^{-3}), while the scaling dimension is the same as that of ϕ, i.e. 1.

It is the basic principle of physics that all meaningful expressions and equations are homogeneous with respect to the engineering dimension. In particular, it is true for the field equations in field theory (e.g. (3.1)), which implies that engineering dimension defines a grading on the space of local functionals. On the other hand, whenever a field theory is not scale invariant, its field equations are not homogeneous with respect to the scaling dimension, so scaling dimension defines a filtration, rather than a grading, on the space of local functionals. This means, the space of all local functionals is a union of subspaces of functionals of dimension $\leq d$, and a functional is said to be of dimension d if it is of dimension $\leq d$ but not $\leq d-1$. On the other hand, if the theory happens to be scale invariant, scaling dimension does define a filtration on the space of local functionals.

In this lecture, we will not use engineering dimension, and the word "dimension" will always mean scaling dimension.

Remark 2. From a local functional $\mathcal{O}$ one can obtain more general functionals on X of the form $\int \mathcal{O}(\phi, x) d\mu(x)$, where $d\mu(x)$ is a (generalized) density on V. Using this operation, one can obtain from any local functional all derivatives of thus functional (it is enough to take for $d\mu(x)$ all possible densities supported at x). Therefore, without loss of information we could consider local functionals modulo the image of derivatives. However, for the purposes of this lecture this is not necessary.

3.2. Quantization of local functionals in a free theory

We have seen (see Bernstein's Lectures and Witten's problem sets) that the space X of classical solutions of a meaningful classical field theory always carries a natural closed 2-form Ω. If this form is nondegenerate, X is a symplectic manifold. In this case, suitable functions on X form a Poisson algebra.

In the quantum theory, the space X should be quantized, and the Poisson algebra of functions on X should become the algebra of operators (observables) in some Hilbert space of states. In particular, we should be able to assign an operator to every local functional.

If V is a Minkowski space, and the field theory satisfies the Wightman axioms (for example, the free theory), the Hilbert space of states $\mathcal{H}$ is constructed as described in Kazhdan's lectures. In this case, Wightman fields $\phi(x)$ are distributions on V with values in the space of operators on the subspace $\mathcal{D} \subset \mathcal{H}$ of muptiparticle states. This means, for any Schwarz function f on V there is an honest operator $\varphi(f)$ on $\mathcal{D}$.

However, we would like to have more general operators of the form $\phi^N(x)$, $(\nabla\phi)^2(x)$, etc. which correspond to local functionals in the classical theory. That is, for any Schwarz function f we want to have operators $\varphi^N(f)$, $(\nabla\varphi)^2(f)$, etc.

Unfortunately, such operators are not automatically defined. For example, even $\varphi^2(f)$ does not, in general, make sense. Indeed, let $\delta_\varepsilon = e^{-|x|^2/\varepsilon}(2\pi/\varepsilon)^{n/2}$ be the smooth approximation to the δ-function, and let $\varphi\otimes\varphi$ be the operator-valued distribution on $V\times V$ given by $\varphi\otimes\varphi(f_1\otimes f_2) = \varphi(f_1)\varphi(f_2)$. The natural definition of $\varphi^2(f)$ would be that $\varphi^2(f)$ is the limit, as $\varepsilon \to 0$, of the operator $\varphi^2_\varepsilon(f) := \varphi\otimes\varphi(f(x)\delta_\varepsilon(x-y))$ (in the sense of convergence of matrix elements). However, it is easy to see that this limit does not usually exist, as Wightman functions usually have singularities on the diagonals.

One way to deal with this problem is to say that an "operator" A is just a collection of its matrix elements, i.e. a collection of correlation functions $\langle\phi(y_1)...\phi(y_l)|A|\phi(z_1)...\phi(z_r)\rangle$. If we accept this point of view, we might as well forget about the Hilbert space of states, i.e. perform a Wick rotation from Minkowski space to Euclidean space, and consider Schwinger functions instead of Wightman functions.

So from now on we will consider only the Euclidean situation, in which we will mean by an operator A a collection of functions

$$\langle\phi(y_1)...\phi(y_l)|A|\phi(z_1)...\phi(z_r)\rangle$$

with certain properties. Roughly, an operator is just a symbol which can be inserted in a correlation function.

Remark. Of course we should remember at all times that A is not really an operator and does not act in any Hilbert space. However, the information we get from studying A in the Euclidean situation can be used (after Wick rotation) for studying the Minkowski situation.

Consider now the problem of quantization of local functionals. By the definition, in order to quantize the local functional $\mathcal{O}(x)$, we should say what the functions $\langle\phi(y_1)...\phi(y_l)|\mathcal{O}(x)|\phi(z_1)...\phi(z_r)\rangle$ are. As our theory is Euclidean, the order of factors in the correlation function does not matter (any two distinct points are space-like separated). So, in order to define $\mathcal{O}(x)$, it is enough to define the correlation functions $\langle\phi(y_1)...\phi(y_r)\mathcal{O}(x)\rangle$ for all r.

Consider the theory of a scalar bosonic field in $n > 2$ dimensions, with the Lagrangian

$$\mathcal{L}(\phi) = \int (\frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + Q(\phi))d^n x, \tag{3.2}$$

We will give all definitions and constructions for this example. In other field theories, everything is done in a similar manner.

We first consider the case when the theory is free ($Q = 0$). In a free theory, quantization is done with the help of normal ordering, as follows. Consider a local functional $\mathcal{O}$ of degree J in ϕ. According to Feynman rules, in order to compute the correlation function $\langle\phi(y_1)...\phi(y_r)\mathcal{O}(x)\rangle$, we should consider all graphs with r external vertices and only one internal vertex v, which has J edges. We should put a certain function at the vertex v, and compute the term (amplitude) corresponding to this graph as usual in Feynman calculus. If $J > 1$, we will run into trouble: we will get some graphs with loops going from v to itself, and the integration over the loops is divergent. The easiest remedy is to ignore all such graphs. Then we obtain certain correlation functions, which define some operator. This operator is denoted by $:\mathcal{O}(x):$ and called the normal ordering of $\mathcal{O}(x)$. Apriori, $\mathcal{O}(x)$ does not make sense as an operator, while $:\mathcal{O}(x):$ does. We call operators of the form $:\mathcal{O}(x):$ composite operators. When no confusion is possible, we will drop the dots and write $\mathcal{O}(x)$.

Thus, we have assigned canonically to each differential polynomial $\mathcal{O}$ in ϕ an operator $:\mathcal{O}:$ in the free theory. However, recall that two different polynomials $\mathcal{O}_1, \mathcal{O}_2$ might define the same local functional. So does the map $\mathcal{O} \to :\mathcal{O}:$ actually define canonically a quantization of the local functional represented by $\mathcal{O}$? The answer turns out to be positive: whenever polynomials $\mathcal{O}_1$, $\mathcal{O}_2$ define the same local functional, the operators $:\mathcal{O}_1:, :\mathcal{O}_2:$ coincide. To check this is an easy exercise. For example, the polynomials $\mathcal{O}_1 = \phi\Delta\phi$, $\mathcal{O}_2 = m^2\phi^2$ define the same local functional, and the operators $:\phi\Delta\phi:, :m^2\phi^2:$ coincide.

3.3. Multiplication of composite operators.

It is easy to multiply composite operators supported at different points. That means, given two composite operators $\mathcal{O}$, $\mathcal{O}'$, and points $x, x' \in V$, $x \neq x'$, it is easy to define an operator $\mathcal{O}(x)\mathcal{O}'(x')$, using the Feynman diagram expansion. In this case we will have two special vertices v, v', and we should sum over all ways of connecting them which involve no loops from a vertex to itself. However, multiplication of two composite operators $\mathcal{O}, \mathcal{O}'$ at the same point x cannot be defined in the same way: on each edge connecting v with v', we will have to evaluate the Green's function at (x, x), where it is singular.

To avoid this problem, one can try to smear the operators, i.e. consider the operator $\mathcal{O}(f) = \int f(x)\mathcal{O}(x)d^n x$, where f is a compactly supported smooth function on V. Such smeared operators can sometimes be multiplied: for example, we can define the product $\phi(f_1)...\phi(f_s)$. (see Kazhdan lectures).

However, most composite operators cannot be multiplied, even after smearing. Indeed, it can be deduced from reflection positivity that for any complex operator $\mathcal{O}$ $\langle \mathcal{O}(x)\mathcal{O}^*(x')\rangle \sim |x-x'|^{-2[\mathcal{O}]}$, $x \to x'$ where $[\mathcal{O}]$ is the dimension of $[\mathcal{O}]$, and $\mathcal{O}^*$ is the conjugate of $\mathcal{O}$. On the other hand, for a real operator $\mathcal{O}$, in order for the product $\mathcal{O}(f)\mathcal{O}(f')$ to be defined, it is necessary and sufficient that the functions $\langle \phi(y_1)...\phi(y_r)\mathcal{O}(x)\mathcal{O}(x')\rangle$ be distributions. The function $|x|^{-l}$ defines a distribution (without regularization) if and only if it is locally L^1, i.e. iff $l < n$. Thus, the product $\mathcal{O}(f)\mathcal{O}(f')$ is automatically defined iff $[\mathcal{O}] < n/2$.

Remark. If the Minkowski situation, the product $\mathcal{O}(f)\mathcal{O}(f')$ exists (in the sense of correlation functions) if and only if the "operator" $\mathcal{O}(x)$ is an operator-valued distribution, so that $\mathcal{O}(f)$ is actually an honest operator (i.e. matrix elements of $\mathcal{O}(f)$ are matrix elements of some operator on $\mathcal{D}$). Indeed, the "if" part is clear. The "only if" part: for any $f_1, ..., f_N$ we can regard the product $X = \mathcal{O}(f)\varphi(f_1)...\varphi(f_N)$ as a linear functional on $\mathcal{D}$. The fact that $\mathcal{O}(f)^2$ is defined implies that this functional has finite L^2-norm, as $||X||^2 = \langle \varphi(f_N)...\varphi(f_1)\mathcal{O}(f)^2\varphi(f_1)...\varphi(f_N)\rangle$. Therefore, X corresponds to a vector in $\mathcal{H}$, Q.E.D.

3.4. Operator product expansion (OPE) in the free theory

Composite operators in a reasonable field theory have an important algebraic property, which is called the operator product expansion (OPE). This is a property of algebraic nature, saying roughly that composite operators form an algebra of a certain kind. It does not follow obviously from Wightman axioms, but on the other hand is often satisfied when Wightman axioms fail. This property is actually useful in practice.

We will now define and compute the OPE for the free theory of a scalar bosonic field. The idea is to formalize the behavior of the product $\mathcal{O}(x)\mathcal{O}'(x')$ as $x \to x'$. In the classical theory, this is accomplished by the usual Taylor series. Indeed, we have

$$\mathcal{O}(x)\mathcal{O}'(x') = \mathcal{O}(x')\mathcal{O}'(x') + (x - x')_\mu \partial_\mu \mathcal{O}(x')\mathcal{O}'(x') + \ldots \tag{3.3}$$

Thus, for any two local functionals $\mathcal{O}, \mathcal{O}'$ the product $\mathcal{O}(x)\mathcal{O}'(x')$ decomposes in a series in products of powers of $(x - x')_\mu$ whose coefficients are other local functionals.

In the quantum theory, the product $: \mathcal{O}(x) :: \mathcal{O}'(x') :$, as we have seen, is singular at $x = x'$. Thus, we have to compute the asymptotic expansion of this product near $x = x'$.

We will assume that $\mathcal{O}, \mathcal{O}'$ are represented by some monomials in ϕ and its derivatives. According to Feynman rules, we should assign to both $\mathcal{O}, \mathcal{O}'$ a vertex, say v and v'. The numbers of edges at v, v' are J, J', where J, J' are the degrees of $\mathcal{O}, \mathcal{O}'$ as monomials of ϕ. At both v, v', we have to place symmetric distributions in J, J' variables corresponding to $\mathcal{O}, \mathcal{O}'$. Now we have to sum amplitudes over all ways of connecting some edges going from v with some edges going from v'. The amplitude of each particular graph is computed as follows: at each edge going from v to v' we put the function $D(y - z)$, the two-point Schwinger function of the free theory (the Green's function for the Helmholtz equation). On each remaining external edge we put $\phi(y)$. Then we compute formally the product of things on vertices and edges, and put it under the normal ordering sign. Since for each number K of connecting edges, we have only one possible graph, we should just compute the amplitude of this graph, and then multiply it by the number of ways to get this graph, i.e. by $J!J'!/K!(J - K)!(J' - K)!$.

This will give us a formula of the form

$$: \mathcal{O}(x) :: \mathcal{O}'(x') := \sum_{k=1}^{L} : \tilde{\mathcal{O}}_k(x)\tilde{\mathcal{O}}'_k(x') : E_k(x - x'), \tag{3.4}$$

where $: \tilde{\mathcal{O}}_k :, : \tilde{\mathcal{O}}'_k :$ are some local functionals, and $E_k(y)$ are polynomial differential operators in $D(y)$. Here $: \mathcal{O}(x)\mathcal{O}'(x') :$ is the amplitude of the graph in which the vertices v and v' are not connected.

Now we can use the Taylor formula (3.3) inside of the normal ordering, to get the following (already infinite) expansion:

$$: \mathcal{O}(x) :: \mathcal{O}(x') := \sum_s : \mathcal{O}_s(x') : D_s(x - x'), \tag{3.5}$$

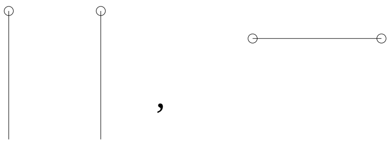
where $\mathcal{O}_s$ is local functional, and $D_s(y)$ is a product of the function $E_k(y)$ for some k and a polynomial in y.

Definition. Expansion (3.5) is called the operator product expansion (OPE).

Now recall that the Green's function $D(x)$ has the following behavior at $x = 0$: $D(x) \sim |x|^{2-n}$. Therefore, all functions $d_k(x)$ have the property $|d_k(x)| \le C_k |x|^{r_k}$ for small x, where the numbers r_k are almost all positive. This proves an important property of the OPE: only finitely many terms are singular. However, these singular terms carry the most interesting information in many situations.

Remark. We see that the operator product expansion (3.5) is in fact the quantum analogue of the Taylor expansion (3.3). Indeed, it is an expansion of the form similar to (3.3), but also involving finitely many singular terms at $x = x'$.

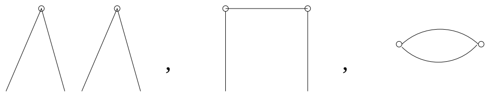
Let us now consider examples of OPE (we will always write the finite formula (3.4)). Let us first compute $\phi(x)\phi(x')$. Using the above rules, we get two graphs:



which yields the formula

$$\phi(x)\phi(x') =: \phi(x)\phi(x') : + D(x - x'). \tag{3.6}$$

Now we compute $: \phi^2(x)\phi^2(x') :$. This product gives us three graphs:



so we get the formula

$$: \phi^2(x) :: \phi^2(x') :=: \phi^2(x)\phi^2(x') : +4 : \phi(x)\phi(x') : D(x - x') + 2D^2(x - x'). \tag{3.7}$$

In general, it is easy to prove that

$$: \phi^J(x) :: \phi^{J'}(x') := \sum_{K=0}^{\min(J,J')} \frac{J!J'!}{K!(J-K)!(J'-K)!} \times \\ \times D(x - x')^K : \phi^{J-K}(x)\phi^{J'-K'}(x') : \tag{3.8}$$

The last formula can be written more nicely using generating functions:

$$: e^{\alpha\phi(x)} :: e^{\beta\phi(x')} := e^{\alpha\beta D(x-x')} : e^{\alpha\phi(x)} e^{\beta\phi(x')} :, \tag{3.9}$$

where α, β are constants.

If we take α, β to be any differential operators on V with constant coefficients, this formula remains valid. (α acts on x, β on x'). In this form, formula (3.7) represents the most general OPE for the free theory of a scalar bosonic field.

Remark. Dimensions of fields in quantum theory may differ from the dimensions of their classical analogues. For example, in a (2-dimensional) classical field theory, if a field ϕ is 0-dimensional, so is $f(\phi)$, where $f : \mathbb{R} \to \mathbb{R}$ is any function. However, in quantum theory a renormalized composite operator $f(\phi)$ may acquire a nontrivial (anomalous) dimension. For example, in the free 2-dimensional theory of a massless scalar bosonic field, $D(y) = -\ln |y|$, so we have from (3.9):

$$\langle : e^{i\alpha\phi(x)} :: e^{-i\alpha\phi(y)} :\rangle = |x - y|^{-\alpha^2}.$$

But we know that $\mathcal{O}(x)\mathcal{O}^*(y)\rangle \sim |x - y|^{-2[\mathcal{O}]}$. Thus, the scaling dimension of $e^{i\alpha\phi}$ is $d(\alpha) = \alpha^2/2$. Observe that the function d is not linear in α, so for quantum dimensions, to the contrary with the classical dimensions, $[: \mathcal{O}_1\mathcal{O}_2 :] \neq [: \mathcal{O}_1 :] + [: \mathcal{O}_2 :]$.

The dimension function $d(\alpha) = \alpha^2/2$ appears in the theory of vertex operator algebras, namely, in the Frenkel-Kac vertex operator construction.

In an interacting theory, even polynomial fields can have different dimensions quantum-mechanically than they do classically. This is essential, however, mostly in the non-perturbative setting. In the perturbative setting we can count dimensions as in the classical theory, since dimensional anomalies are infinitesimally small.

3.5. Normal ordering and renormalization.

Now we will reformulate the definition of normal ordering in the free theory in terms of renormalization theory. This reformulation will be crucial in understanding what is the analogue of normal ordering in interacting theories.

At the beginning of this lecture, we defined normal ordering of a local functional by formally throwing away graphs with self-loops, which produced divergence. Another way of dealing with self-loops is renormalizing them, as we did in two previous lectures. Suppose we want to renormalize all local functionals of dimension $\leq d$. For this purpose we replace the standard propagator $\frac{1}{k^2+m^2}$ with a cutoff propagator $\frac{1}{k^2+m^2}\left(\frac{\Lambda^2}{k^2+\Lambda^2}\right)^l$, where

l is sufficiently big. Now all the loop integrals converge, and for any local functional $\mathcal{O}$ of degree $\leq d$ we can consider the cutoff correlation function $\langle\phi(y_1)...\phi(y_r)\mathcal{O}(x)\rangle_\Lambda$. If we take $\Lambda \to \infty$, we will of course find that the limit does not exist. However, it is not diffucult to prove the following.

Let A_d be the space of local functionals of dimension $\leq d$.

Proposition 3.1. There exists a Λ-dependent linear map $R_\Lambda : A_d \to A_d$, strictly triangular with respect to the filtration of A_d by dimension, such that for any local functional $\mathcal{O} \in A_d$ there exists a limit

$$\lim_{\Lambda\to\infty} \langle\phi(y_1)...\phi(y_r)R_\Lambda\mathcal{O}(x)\rangle_\Lambda, \tag{3.10}$$

Thus, Proposition 3.1 allows us to assign to every local functional $\mathcal{O}$ an operator $\tilde{\mathcal{O}}$, which is, by definition, the operator whose matrix elements are the limits of the corresponding matrix elements of $R_\Lambda\mathcal{O}$. However, given $\mathcal{O}$, the operator $\tilde{\mathcal{O}}$ is defined apriori noncanonically, as the map R_Λ is not unique: it is defined up to left multiplication with a Λ-independent strictly triangular map.

That is, $\tilde{\mathcal{O}}$ is defined uniquely up to adding composite operators of lower dimension. This shows that the space of composite operators is naturally a filtered object (by dimension), and not a graded object.

Remark. Of course, if the theory is not classically scale invariant, we saw that there is no grading on functionals already at the classical level. The statement here is that even for a classically scale-invariant theory, where the space of classical functionals automatically has a grading by scaling dimension, the grading is usually lost in the process of quantization. An exception is a free scale-invariant theory, where there is a canonical quantization by normal ordering, and therefore the grading survives quantization.

Thus, in general we may be able to quantize naturally the space of local functionals of dimension $\leq d$, but not every functional separately.

3.6. Composite operators in an interacting critical theory.

Now consider an interacting renormalizable field theory. As a model example we consider the Lagrangian (3.2) and take $Q(\phi) = \frac{g}{4!}\phi^4$. Let $\mathcal{O}$ be any local functional represented by a monomial. To quantize $\mathcal{O}$, we proceed as in the free case, but we will formulate everything in a slightly different language.

We want to consider correlation functions $\langle\phi(y_1)...\phi(y_r)\mathcal{O}(z)\rangle$. These correlation functions can be viewed as the ε-coefficient in the usual Schwinger functions for a perturbed Lagrangian, of the form

$$\mathcal{L}_\varepsilon(\phi) = \mathcal{L}(\phi) + \varepsilon\mathcal{O}(\phi, z), \tag{3.11}$$

where ε is a formal variable such that $\varepsilon^2 = 0$. In the language of Feynman diagrams, this means that we are introducing an additional vertex v corresponding to $\mathcal{O}$, and summing over all graphs which contain exactly one such vertex (with no self-loops at this vertex) and are otherwise as usual.

Remark. One should remember that there is no momentum conservation at the vertex v.

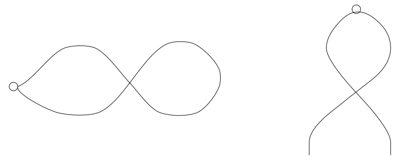
In general, such an alteration will worsen the divergence properties of the Feynman graphs. More precisely, now the superficial divergence of a graph with E external edges is given by $div(\Gamma) = [\mathcal{O}] - E$. However, we can renormalize these divergences, using the cutoff propagator considered in the previous section. Then, analogously to Proposition 3.1, one can prove the following.

Proposition 3.2. There exists a Λ-dependent linear map $R_\Lambda : A_d \to A_d$, triangular (in general, not strictly) with respect to the filtration of A_d by dimension, such that for any local functional $\mathcal{O} \in A_d$ there exists a limit

$$\lim_{\Lambda\to\infty} \langle \phi(y_1)...\phi(y_r) R_\Lambda \mathcal{O}(x)\rangle_\Lambda, \tag{3.12}$$

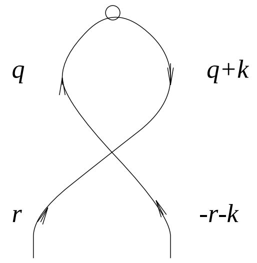
As in the free theory, this Proposition allows to quantize the space of local functionals of dimension $\leq d$, but in general there is no canonical quantum analogue for each classical local functional. This non-uniqueness is not only due to the non-uniqueness of renormalization, but also due to the non-uniqueness of representation of a given functional by a differential polynomial.

Example. Let $\mathcal{O} = \phi^2/2$. Let us compute the renormalization of $\mathcal{O}$ of order g. The only divergent graphs we have in this order are



Let us call the first graph by Γ_0 and the second by Γ_2. The graph Γ_0 is quadratically divergent. If we replace the usual propagator with the cutoff propagator, the integral will converge to a Λ-dependent constant of the form gC_Λ, where C_Λ, which grows quadratically in Λ.

Now consider the graph Γ_2. As usual, it is more convenient to work in the momentum space, i.e. consider the Fourier transform of the term corresponding to this graph. Let k_1, k_2, k be the corresponding momentum variables. Then the Fourier transform of the term corresponding to Γ_2 is of the form $F_2\delta(k_1+k_2+k)$, where F_2 is a function on the plane $k_1+k_2+k=0$. Set $k_1=r$, then $k_2=-r-k$ (on this plane). Thus, $F_2=F_2(r,k)$, and its order g correction is the amplitude of the Feynman diagram



This amplitude is given by the integral

$$I(k)=-\frac{g}{2}\int\frac{d^4q}{(2\pi)^4}\frac{1}{(q^2+m^2)((q+k)^2+m^2)}. \tag{3.13}$$

This integral is logarithmically divergent. If we replace the usual propagator with the cutoff propagator, the integral will converge to a function $I_\Lambda(k)$, which has the asymptotic behavior

$$I_\Lambda(k)=-\frac{gA}{2}\ln(\Lambda/\mu)+O(1),\Lambda\to\infty$$

where A is a constant.

This shows that if instead of $\phi^2/2$ we use the renormalized functional

$$\left(\frac{\phi^2}{2}\right)_\Lambda:=\frac{\phi^2}{2}(1+\frac{gA}{2}\ln(\Lambda/\mu))-gC_\Lambda, \tag{3.14}$$

the matrix elements will have a finite limit modulo g^2. Observe that the constant μ can be chosen arbitrarily, so the renormalization is not canonical already in this case.

3.7. Stability of the classical field equations under quantization

In a free theory, we know that the classical field equations are also satisfied quantum-mechanically. This means, if the classical equations are $P\phi=0$, where P is a linear differential operator, then the correlation functions in the quantum theory satisfy the equation $P_x\langle\phi(y_1)...\phi(y_r)\phi(x)\rangle=0$ outside of the diagonals $x=y_i$. For example, the Green's function $D(x-y)=\langle\phi(x)\phi(y)\rangle$ is the fundamental solution of the equation $Pf=0$.

In an interacting theory, there are some problems. Namely, since the classical field equations for an interacting theory are nonlinear (e.g. (3.1)), they do not make sense quantum-mechanically in the setting of Wightman axioms. However, in the OPE setting they make sense in a suitable interpretation, and one can show that they are satisfied.

Remark. We will say that the equation $F(\phi) = 0$ is satisfied in quantum theory (where F is a renormalized local functional) if the function $\langle \phi(y_1)...\phi(y_r)F(\phi(x))\rangle$ vanishes outside of the diagonals $x = y_i$.

We will show that the classical field equations are satisfied in ϕ^4 theory. Consider the space of classical local functionals of dimension ≤ 3, which are Poincare-invariant and odd under the symmetry $\phi \to -\phi$. This space is 2-dimensional: it has 3 generators ϕ^3, $\Delta\phi$, and ϕ, but they are linearly dependent, since they satisfy relation (3.1).

Now consider the quantum theory. Consider composite operators of dimension ≤ 3, with the same invariance properties as above. In this space we have renormalized operators $\Delta\phi$, ϕ, ϕ^3. Our goal is to show that, like in the classical theory, they are linearly dependent. This is equivalent to the statement that ϕ satisfies the field equation $C\phi^3 = A\Delta\phi + B\phi$.

We will first prove the validity of the field equation for the theory with the cutoff propagator, since in this theory we do not have divergence problems. In the cutoff theory, the classical field equation is

$$P_\Lambda \phi = \frac{g}{3!}\phi^3, P_\Lambda = (\Delta - m^2)(1 - \frac{\Delta}{\Lambda^2})^l. \tag{3.15}$$

Proposition 3.3 Equation (3.15) holds in the quantum theory with the cutoff propagator, where ϕ and ϕ^3 are regarded as composite operators.

Proof. Consider the correlation function $\langle \phi(y_1)...\phi(y_r)P_\Lambda\phi(x)\rangle$. Consider the graph decomposition of this function. We should consider all possible graphs with r external edges, any number of 4-valent internal vertices, and a special vertex v with 1 outgoing edge. These graphs can be of two kinds: 1) graphs in which v connects to an external vertex; 2) graphs in which v connects to an internal vertex. The sum over graphs of the first type is the corresponding correlation function for the free theory, so it is supported on the union of diagonals $x = y_i$. Thus, outside of the diagonals we have to sum only over graphs of the second kind. But graphs of the second kind with N external edges are in 1-1 correspondence with usual Feynman graphs (without v) with $N + 3$ external edges (this correspondence is obtained by biting off the vertex v and the vertex it connects to). This shows that the sum over graphs of the second kind is just $\frac{g}{3!}\langle \phi(y_1)...\phi(y_r)\phi^3(x)\rangle$. Thus, the field equation is satisfied, Q.E.D.

Proposition 3.3 shows that for any Λ, the composite operator $\frac{g}{3!}\phi^3$ is linearly dependent of composite operators which are linear in ϕ. This property has to be preserved in the limit $\Lambda \to \infty$. This shows that in the renormalized ϕ^4 theory we have an equation $C\phi^3 = P\phi$, where P is a linear differential operator with constant coefficients, and C is a constant. Since the r.h.s. of this equation can only have terms of dimension ≤ 3, the operator P has to be of the form $A\Delta + B$. Thus in the quantum theory we have the equation

$$C\phi^3 = A\Delta\phi + B\phi \tag{3.16}$$

Remark. Of course, the constants A, B, C are not uniquely defined, as they depend on the choice of the renormalization. The statement is only that for any choice of renormalization, some nontrivial equation of the form (3.16) holds. In other words, the statement is that the space of Poincare invariant, odd composite operators of dimension ≤ 3 is 2-dimensional, i.e. has the same dimension as the corresponding space of classical local functionals.

In general, one can consider the space of local functionals of dimension $\leq N$. Denote its dimension by $d(N)$. One can prove the following proposition.

Proposition 3.4 The space of composite operators of dimension $\leq N$ is also equal to $d(N)$.

Thus, all differential equations which are satisfied classically, have quantum analogues.

3.8. Operator product expansion in an interacting theory.

In a free theory, once we defined composite operators, it was no problem to define the product of several of them supported at different points. The same is true in the interacting theory. Namely, if $\mathcal{O}_1, ..., \mathcal{O}_N$ are renormalized composite operators, and $x_1, ..., x_N \in V$ distinct points, then the correlation function $\langle\phi(y_1)...\phi(y_r)\Theta(x_1)...\Theta(x_N)\rangle$ is just the coefficient of $\varepsilon_1...\varepsilon_N$ in the usual Schwinger function $\langle\phi(y_1)...\phi(y_r)\rangle$ for the Lagrangian

$$\mathcal{L}_{\varepsilon_1,...,\varepsilon_N}(\phi) = \mathcal{L}(\phi) + \sum_{i=1}^{N} \varepsilon_i \mathcal{O}_i(\phi, x_i), \tag{3.17}$$

where $\varepsilon_i^2 = 0$. This Schwinger function is, by definition, the limit of the corresponding function computed with the cutoff propagator. It is easy to see, by looking at Feynman diagrams, that this limit is always finite.

Remark. Roughly speaking, this means that once we renormalized each of the operators $\mathcal{O}_i$, we automatically renormalized the product

$$\mathcal{O}_1(x_1)...\mathcal{O}_N(x_N), \quad x_i \neq x_j.$$

Thus, now we are in as good shape as we were in the free theory before we defined OPE. It turns out, we can define OPE in the interacting theory as well. Namely, for any two composite operators $\mathcal{O}, \mathcal{O}'$ and for any $r \in \mathbb{R}$ there exists a finite collection of composite operators $\mathcal{O}_1, ..., \mathcal{O}_l$ such that

$$\mathcal{O}(x)\mathcal{O}(x') = \sum_{s=1}^{l} \mathcal{O}_s(x') D_s(x - x') + O(|x - x'|^r).$$

We will show how to partially compute the OPE on examples.

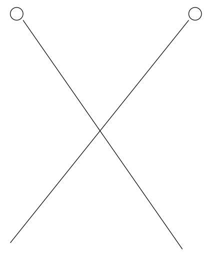
First we consider the product $\phi(x)\phi(x')$, and compute its asymptotics as $x \to x'$. In this case the expansion is of the form:

$$\phi(x)\phi(x') = \phi_R^2(x') f(x - x') + h(x - x') + \text{ regular part },\tag{3.18}$$

where ϕ_R^2 is the renormalized operator ϕ^2 defined by (3.17). (recall that ϕ_R^2 is defined non-uniquely). Let us find functions f, h.

The function $h(z)$ can be found by looking at the 0-point function for the operator $\phi(x)\phi(x')$. Indeed, since $\langle \phi_R^2(x) \rangle = 0$ by the definition, we get $h(x - x') = \langle \phi(x)\phi(x') \rangle$. If we compute the answer modulo g^2, we get no corrections to the free theory answer, so $h(z) = D(z)$ (see lecture 2).

Now let us find $f(z)$ (we remember that it is defined up to scaling $f \to a(g)f$, where $a = 1 \bmod g$). This function is found from the 2-point function of $\phi(x)\phi(x')$. We have $f(z) = 1 + gf_1(z) + O(g^2)$. The only graph that contributes to $f_1(z)$ is



The amplitude of this graph (in the position space) is

$$\begin{aligned} & g \int D(x - z) D(x' - z) D(y_1 - z) D(y_2 - z) dz \\ & = -\frac{gA}{2} \ln |\frac{x - x'}{\mu}| D(y_1 - x') D(y_2 - x') \\ & + \text{ regular part} \end{aligned}\tag{3.19}$$

(A is the same constant as in (3.17)). This shows that $f_1(z) = -\frac{gA}{2} \ln |\frac{x-x'}{\mu}|$, and thus

$$\begin{aligned} \phi(x)\phi(x') &= \phi_R^2(x')(1 - \frac{gA}{2} \ln |\frac{x - x'}{\mu}|) + D(x - x') \\ & + \text{ regular part } + O(g^2). \end{aligned}\tag{3.20}$$

Now consider a more complicated operator product, for example

$$\phi_R^2(x)\phi_R^2(x').$$

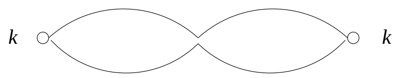
In this case the expansion is of the form

$$\begin{aligned}\phi_R^2(x)\phi_R^2(x') = {} & \phi_R^4(x')f_1(x-x') + (\nabla\phi_R^2)(x')f_2(x-x') \\ & + \phi_R^2(x')f_3(x-x') + f_4(x-x') \\ & + \text{ regular part },\end{aligned} \tag{3.21}$$

where the subscript R means "renormalized composite operator". Of course the functions $f_1, ..., f_4$ will depend on the choice of the renormalization, but they are well defined up to an upper triangular linear transformation.

For the sake of brevity we will only compute the expansion modulo $O(|x-x'|^{-1})$. Thus, the functions f_1, f_2 can be ignored, and we only have to compute f_3, f_4 modulo $O(|x-x'|^{-1})$.

The function f_4 is defined canonically up to multiplication by a scalar. As before, this function is computed using the 0-point function: $f_4(x-x') = \langle\phi_R^2(x)\phi_R^2(x')\rangle$. The only graph that contributes to the 1-st order in g of f_4 is



This shows that

$$f_4(z) = 2D(z)^2 + g\widehat{S(k)} + \text{ regular part } + O(g^2),$$

where $S(k)$ is the amplitude (in the momentum space) of the above graph, and hat denotes the Fourier transform. The function $S(k)$ equals $T(k)^2$, where $T(k)$ is the renormalized integral

$$T(k) = C\int_R \frac{dq}{(q^2+m^2)((k-q)^2+m^2)},$$

Remark. Since $T(k)$ is defined up to adding a constant, $f_4(z)$ is defined up to adding a multiple of $g\widehat{T(k)}$. Since $\widehat{T(k)}$ is proportional to $D^2(z)$ when $z \neq 0$, adding such multiple is equivalent to multiplying f_4 by $1 + cg$. This freedom is natural, as f_4 is defined up to multiplying by $1 + cg$.

Now let us try to compute $f_3(z)$, which is also defined canonically up to a scalar. For this purpose we should consider the 2-point function of

$$\phi_R^2(x)\phi_R^2(x'),$$

i.e.

$$\langle \phi(y_1)\phi(y_2)\phi_R^2(x)\phi_R^2(x')\rangle.$$

We will work with the Fourier transform of this function, which we denote by $F_2(p_1, p_2, q, q')$ (here p_1, p_2, q, q' are the dual variables to y_1, y_2, x, x').

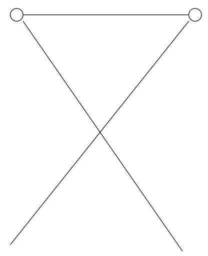
We will try to compute the function $f_3(z)$ by looking at the asymptotic expansion of $F_2(p_1, p_2, q, r-q)$ as $|q| \to \infty$, and using the following fact from calculus:

Claim Let f be an L^1-function on an n-dimensional Euclidean space V whose Fourier transform $\hat{f}$ satisfies the inequality $|\hat{f}(q)| \leq C|q|^{-n-N-\varepsilon}$, $\varepsilon > 0$. Then $f \in C^n(V)$.

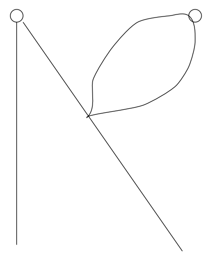
By doing so we will be able to find $f_3(z)$ modulo C^∞-functions, which is all we want at this point.

We have $f_3(z) = 4D(z) + gf_3^1(z)$. Looking at the Feynman diagram expansion of the 2-point function in the first order of g, we see terms of two kinds: 1) terms where the special vertices v, v' are directly connected with an edge; 2) terms where v, v' are not connected.

The only graph of the first kind that contributes to f_3^1 is



and the only graph of the second kind contributing to f_3^1 is



(this graph occurs 4 times, with 4 different labelings of vertices). It is easy to check directly that both graphs contribute to $\hat{f}_3(q)$ a term of the form $Cq^{-2}\ln|q/\mu| + O(q^{-4+\varepsilon})$, where C is a constant. Thus, $f_3^1(z)$ is the Fourier transform of the function $C \ln |q/\mu|/q^2$ modulo $O(|x-x'|)^{-1}$ and $O(g^2)$. As before, the constant μ depends on the choice of renormalization; changing of this constant is equivalent to adding a multilple of $D(z)$ to $f_3^1(z)$, which is the same as multiplying $f_3(z)$ by a scalar of the form $1+cg$.

Remark: Operator product expansion in conformal field theory.

In a 2-dimensional conformal field theory, the OPE of composite operators has especially simple form. In this case, $V=\mathbb{C}$, and the space of functions on $V\setminus 0$ has a "bigrading": the function $z^a\bar{z}^b$ has bidegree (a,b) $(a-b\in\mathbb{Z})$. The space of composite operators also has a bigrading: to any homogeneous operator $\mathcal{O}$ one assigns two numbers – the holomorphic dimension $d(\mathcal{O})$ and antiholomorphic dimension $\bar{d}(\mathcal{O}')$ $(d-\bar{d}\in\mathbb{Z})$. Therefore, if $\mathcal{O}(z)\mathcal{O}'(z')=\sum_k D_k(z-z')\mathcal{O}_k(z')$, then $D_k(z-z')$ has degree $(d_k,\bar{d}_k)$, where $d_k=d(\mathcal{O})+d(\mathcal{O}')-d(\mathcal{O}_k)$, $\bar{d}_k=\bar{d}(\mathcal{O})+\bar{d}(\mathcal{O}')-\bar{d}(\mathcal{O}_k)$. This implies that $D_k(z)=C_k z^{d_k}\bar{z}^{\bar{d}_k}$. Also, the action of the Virasoro algebra allows to reduce the problem of computing OPE of arbitrary operators to the problem of computing OPE for primary fields only, i.e. for fields which are highest weight vectors for the Virasoro algebra. In a rational conformal field theory, one has a chiral algebra of symmetries (for example, an affine Lie algebra) which is so big that there are only finitely many fields which are primary with respect to this algebra. This fact allows to treat a rational 2-dimensional conformal field theory in a purely algebraic setting, and reduce many of its problems to problems in algebraic geometry.