Appendix to Lecture 4: Remarks on renormalization and asymptotic freedom

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1. Ambiguity in operator products. In the last lecture we considered Lagrangians of the form

$$\mathcal{L} = \int d^4x(\frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{g}{4!}\phi^4 + \sum_i \varepsilon_i(x)\mathcal{O}_i(x)),$$

where $\mathcal{O}_i$ are local functionals. We saw that defining correlation functions of such Lagrangians to the first order in $\varepsilon_i(x)$ (i.e. $\varepsilon_i\varepsilon_j = 0$) is equivalent to defining composite operators corresponding to $\mathcal{O}_i$. Now let us work to the second order in ε_i ($\varepsilon_i\varepsilon_j\varepsilon_k = 0$). This corresponds to considering products of composite operators. Before we considered products of composite operators at non-coinciding points x, y, and saw that such a product $\mathcal{O}_i(x)\mathcal{O}_j(y)$ is defined automatically once we have defined $\mathcal{O}_i$ and $\mathcal{O}_j$. Now we will allow the points x and y to coincide. Then, as we know, the product is not automatically defined, and its definition requires additional renormalization. This means, we have to introduce counterterms in the Lagrangian, i.e. consider a new Lagrangian of the form

$$\mathcal{L}' = \mathcal{L} + \sum_{i,j,k} L_k\varepsilon_i(x)R_k\varepsilon_j(x)W_k(\Lambda, g)\mathcal{O}_k(x),$$

where L_k, R_k are differential operators (in x) with constant coefficients, and W_k are some functions which diverge as $\Lambda \to \infty$. The functions W_k are usually not uniquely determined and cannot be chosen canonically.

Let us consider an example. In one of the homework problems we computed the 1-loop correction to the 1-particle irreducible bosonic 2-point function Σ in QED. We discussed that this correction can be computed (in momentum space) as

$$\int e^{ikx}\langle J_\mu(x)J_\nu(0)\rangle dx,$$

where $J_\mu(x) = \bar\psi\gamma_\mu\psi$ is the operator of current, and the correlator in (1) is in the theory of a free fermion. Now, as follows from the above discussion,

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the product $J_\mu(x)J_\nu(x)$ is defined up to operators of lower order, and the expectation value $\langle J_\mu(x)J_\nu(0)\rangle$ is non-uniquely defined, which causes an ambiguity in the computation of (1). However, this non-uniqueness occurs only at $x=0$. In fact, its is easy to show that the function $\langle J_\mu(x)J_\nu(0)\rangle$ is well defined up to adding a multiple of $\delta(x)$. Therefore, the ambiguity in (1) is a constant (i.e. is independent of k).

In QED, to preserve gauge invariance, it is necessary to choose this constant in such a way that the condition $k^\mu\Sigma_{\mu\nu}(k)=0$ is satisfied. This gives a unique way to fix the constant.

2. Symmetry breaking. A symmetry that exists in a classical field theory may be lost in a particular renormalization scheme for the corresponding quantum theory. Of course it is possible that there exists a better scheme which preserves this symmetry, but it is also possible that there exists no such scheme, i.e. the symmetry is broken at the quantum level. For example, consider N free massless fermions $\psi_1,...\psi_N$, with Lagrangian $\sum(\bar\psi_i, D\psi_i)$, $\psi_i\in S_+$, $\Psi_-\in S_-$. In this theory we have a $U(N)$ symmetry. Let us add interactions in such a way that part of this symmetry is preserved. For example, add a gauge field A (i.e. regard the fermions together to make a section ψ of the vector bundle $S_+\otimes E$, where E is an N-dimensional Hermitian vector bundle over the spacetime), which takes values in the Lie algebra of a subgroup $H\subset U(N)$, and consider the Lagrangian $(\bar\psi, D_A\psi)$, where D_A is the Dirac operator along the connection A. The classical symmetry of this theory is the centralizer $Z(H)$ of H. In quantum theory, however, this symmetry may fail for topological reasons. In other words, a topological anomaly may appear.

3. An oversimplified version of experimental confirmation of asymptotic freedom.

Consider a field theory with electromagnetic and strong interactions, which contains electrons (which interact only electromagnetically), and quarks (which interact both electromagnetically and strongly). The Lagrangian of such a theory can be written as follows. The fields are:

(i) An $SU(3)$-connection A_g (the field of strong interactions),

(ii) A $U(1)$-connection A_e (the electromagnetic field),

(iii) Quarks q_i and an electron ε (which are fermions with values in $\mathbb{C}^3$ and $\mathbb{C}$ respectively).

The Lagrangian is

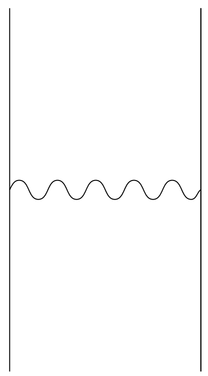
$$\int d^4x\Big(\sum_j \bar q_j(iD+A_g+A_e-m_i)q_j+\bar\varepsilon(iD+A_e-m)\varepsilon+\frac{1}{e^2}F_{A_e}^2+\frac{1}{g^2}F_{A_g}^2\Big).$$

Here e is the charge of the electron and g is the coupling of the strong interaction.

Now suppose that we scatter two electrons against each other with momenta p_1, p_2, and measure the amplitude of the event that after scattering they will have momenta q_1, q_2. As we know, this amplitude is defined by the 4-point function $\Gamma_4(p_1, p_2, q_1, q_2)$. Let us try to compute this function and thus predict the result of measurement.

First of all, we can use the fact that e^2 is small. This means, we can trust the perturbative expansion in powers of e.

To order e^2, we can assume that the electrons, during scattering, exchange only one photon, which does not interact while it moves from one electron to the other. This corresponds to the following Feynman diagram:



Thus we have

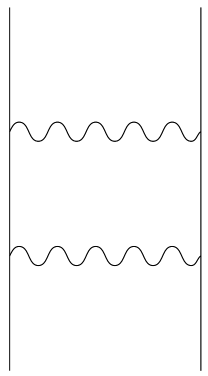
$$\Gamma_4(p_1, p_2, q_1, q_2) = e^2(G_2((p_1 - q_1)^2) + G_2((p_1 - q_2)^2)),$$

where $G_2(p^2)$ is the free photon propagator. Here G_2 is regarded as an operator from $S \otimes S$ to $S \otimes S$, where S is the space of spinors.

Remark. In principle, we should include the diagrams where one of the electrons exchanges a photon with itself, but we will not consider them, regarding them as absorbed in the electron propagator.

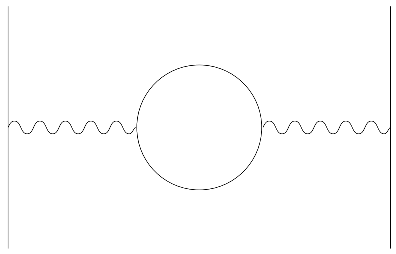
To order e^4, we have two possiblities.

1) The electrons could exchange two non-interacting photons. The amplitude of the corresponding 1-loop diagram

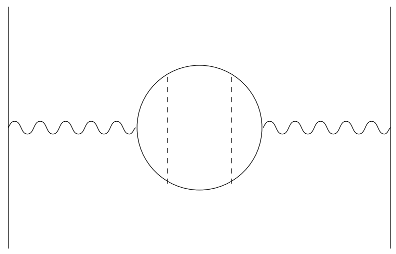


can be computed within the framework of QED.

2) The electrons could exchange only one photon, but on its way it could split in an electron and positron, or in a quark and an antiquark. The first splitting scenario

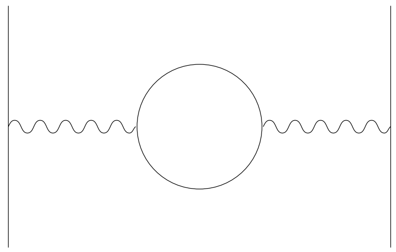


is harmless, since it gives only one 1-loop diagram with no strong interactions, and we can compute the amplitude of this diagram as in QED. However, the second scenario (with quarks) really gives us trouble. Indeed, the coupling constant g of the strong interaction is not small, so we cannot trust the perturbation expansion in g and thus have to take into account infinitely many Feynman diagrams with any number of loops:



(here dotted lines denote the field of strong interaction).

However, we know that the theory of strongly interacting quarks is asymptotically free (when the number of quarks is not too large). Thus, we should expect that the perturbative expansion in the effective coupling $g_{eff} \sim g(\ln p^2)^{-1/2}$ should be valid at high momenta. This would mean that at high momenta we can restrict to the 1-loop diagrams, which involve no strong interactions. Since we have one such diagram for each type of quark,



the total amplitude of these diagrams is $\sum e_i^2 \Sigma_i(p)$, where e_i are charges of quarks, and Σ_i are amplitudes of the corresponding diagrams for a particle with charge 1. Of course, the functions $\Sigma_i(p)$ depend on the (unmeasurable!) masses m_i of quarks, but at high momenta masses are irrelevant, and all functions Σ_i are approximately equal to each other. Thus, the amplitude is $\Sigma(p)(\sum e_i^2)$, where $\Sigma(p)$ is a universal function computed from QED (as in one of the homework exercises).

It follows from asymptotic freedom that the (relative) error of this computation is of order $1/\ln p^2$. If we compute the two-loop correction (still working to order e^4), we will get an additional term of the order $1/\ln p^2$, and the error will be of order $1/\ln^2 p^2$. More generally, if we take into account N-loop diagrams, the error will be of order $1/\ln^N p^2$. Thus, we get an asymptotic series, with very slowly decaying terms, but at very high p one can hope that it gives a reasonably good approximation to the 4-point scattering amplitude. This approximation (to the 0-th order) could in principle be checked experimentally, and can be regarded as a confirmation of asymptotic freedom.

Remark. In practice, asymptotic freedom was checked experimentally in a different way, but the ideology is similar to the one described above.

Correction to the text of lecture 3 (by Pavel Etingof)

Unfortunately, in Section 3.3 of Lecture 3 there is a wrong statement (noticed by D.Freed). Namely, the statement "$\langle ...\mathcal{O}(x)\mathcal{O}(x')\rangle \sim |x-x'|^{-[\mathcal{O}]-[\mathcal{O}']}$" is incorrect. For example, it fails if $\mathcal{O}' = 1$ and $\mathcal{O}$ is a nontrivial operator. This statement is true, however, if $\mathcal{O}' = \mathcal{O}$, which is enough to make the point which was being made in the text.