The Dirac Index on Manifolds and Loop Spaces: Part I

Edward Witten*

December 10, 1997

I Introduction

One mathematical approach to the proof of the Hirzebruch index theorem is through the heat equation. One considers, for each $t > 0$, the operator $e^{-t\Delta}$ acting on the differential forms on a closed, oriented riemannian manifold Y. A direct spectral argument shows that the trace of this operator (or more precisely the difference of its trace on the self-dual forms and on the antiself-dual forms) is independent of t and limits as $t \mapsto \infty$ to the index of the manifold. On the other hand, the limit as $t \mapsto 0$ of this operator can be computed in terms of local invariants, curvature, of the manifold. One can even explicitly evaluate these local expressions, and in this manner recover the result that the signature of the manifold is given by L-polynomials of Hirzebruch. In a related context, that of equivariant signatures when there is a group action, one sees a localization taking place in the Atiyah-Hirzebruch formula for the equivariant signature in terms of local data (characteristic classes) at the fixed points. These localizations are reminiscent of localizations in quantum field theory, e.g., perturbation expansions around the classical solutions of Lagrange's equations, and raise the question of whether there are quantum field theory arguments establishing these and similar results.

The answer to this question is yes, and it is our purpose here to show how to establish results of a similar nature (for Dirac operators on a spin manifold) by doing a perturbation expansion of an appropriate path integral or Hamiltonian eigenstates. The explanation of the quantum field

*Lecture notes by John Morgan

theories that do indeed lead to index theorems for the Dirac operator on a spin manifold (an its equivariant analogue) are the first subject of these lectures. Once these classical cases are been covered, we turn to more sophisticated applications of the same techniques. This involves the index of Dirac operators on the loop spaces of spin manifolds. While the results on loop spaces were first discovered by the path integral approach that we take here, eventually, mathematical proofs were also found for these results [Bott-Taubes].

Before we get into the details of the computations, one general comment is in order. So far the perturbative expansions we have encountered have been around an isolated, non-degenerate critical point. That is to say the space of classical solutions (the space of minima of the Lagrangian) is a single point and the Hessian (the quadratic term in the Lagrangian) is positive definite at that point. The expansions that we encounter here will be of a more general type. The set of classical solutions will be a compact, finite dimensional smooth manifold (or super-manifold) and the quadratic term in the Lagrangian restricted to the normal bundle to the critical set is positive definite. The same techniques will work for oscillatory integrals as long as the Hessian in the normal directions is non-degenerate (the Chern-Simons functional in three dimensions gives examples of this type). Of course, one could meet situations where the critical set is singular and the Hessian has null-space which is not of constant rank. Such situations require techniques that are not discussed here.

In the situation where the critical set is a super-manifold M and at each point of the critical set the Hessian has null space equal to the tangent space of the critical set, the basic approach is to apply the usual perturbative expansion using Feynman diagrams and regularization in the normal direction where there is an isolated, non-degenerate critical point to obtain a result, and then integrate this result over the critical set. Of course, it may well happen that the individual expansions over the normal spaces have no intrinsic meaning and depend on choices but we expect the resulting integral over M to be well-defined. We will also be computing only the constant term in the expansions. These of course will depend only on the quadratic term in the Lagrangian and will be written in terms of the determinant of the propagator for the Bosons and Pfaffian of the propagator of the fermions integrated over the normal directions.

II The Dirac Operator on a Spin Manifold

In this section we shall give a path integral derivation of the index theorem for the Dirac operator on a spin manifold. Fix for this section Y a compact, oriented, $2n$-dimensional spin manifold.

II.1 The Lagranian Formulation

Let $\mathbf{R}^{1,1}$ denote the super Euclidean space with one even variable and one odd variable; i.e., $C^\infty(\mathbf{R}^{1,1}) = C^\infty(\mathbf{R})\otimes\wedge^*(\mathbf{R})$. We denote by $t, \theta \in C^\infty(\mathbf{R}^{1,1})$ the natural even and odd variables, respectively. We consider a quantum theory of maps

$$X\colon \mathbf{R}^{1,1} \to Y,$$

where Y is a compact spin manifold of dimension $2n$. The Lagrangian we take is

$$\mathcal{L} = -\int_{\mathbf{R}^{1,1}} dt d\theta^{-1} \frac{1}{2} g_{I,J}(X) \frac{dX^I}{dt} DX^J = -\int_{\mathbf{R}^{1,1}} \frac{1}{2}\langle \dot{X}, DX\rangle,$$

where as usual $g_{I,J}$ denotes the metric tensor on Y and D is the vector field

$$D = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial t}$$

on $\mathbf{R}^{1,1}$.

Let us recall some of the material from Problem 2 in Problem Set One. We can view X as a map x from $\mathbf{R}^1$ to Y together with a section ψ of the odd vector bundle $\Pi(x^*TY)$ over $\mathbf{R}^1$. We write $X = x(t) + \theta\psi(t)$ with this understanding as to the meanings of x and ψ. Then we can re-write the Lagrangian as

$$\mathcal{L} = -\int_{\mathbf{R}^{1,1}} \frac{1}{2}\langle \dot{x}(t) + \theta\nabla_t(\psi(t)), \psi(t) - \theta\dot{x}(t)\rangle dt d\theta^{-1}. \tag{1}$$

If one prefers an expression in local coordinates, doing the θ-integration yields:

$$\mathcal{L} = \frac{1}{2}\int_{\mathbf{R}^1} \langle \dot{x}(t), \dot{x}(t)\rangle - \langle \nabla_t\psi(t), \psi(t)\rangle dt.$$

The classical solutions to this Lagrangian form a supermanifold of maps $X\colon \mathbf{R}^{1,1} \to Y$ satisfying

$$\nabla_t DX = 0.$$

We can also write this equation as a pair of equations:

$$\nabla_t \psi = 0$$

$$F(\psi, \dot{x})(\psi) = \nabla_t(\dot{x}),$$

where F is the curvature of Y. Fixing a time t_0 and evaluating at this time identifies this supermanifold with space of "initial conditions" at time $t = t_0$. The initial conditions are simply a pair – a tangent vector $\dot{x}_0$ at some point x_0 of Y and an element ψ_0 in a copy of TY_{x_0} considered as an odd vector bundle. That is to say, the space of initial conditions is identified with the odd vector bundle $p^*\Pi(TY)$ over TY, where $p: TY \to Y$ is the natural projection mapping. The symplectic structure on this space of classical solutions is given by (see Problem 2 of Problem Set One)

$$\omega = \omega_{T^*Y} + \frac{1}{2}(\nabla\psi, \nabla\psi) - \frac{1}{2}\left(F(\psi), \psi\right),$$

where the first term is the canonical symplectic form on the cotangent bundle which is identified with the tangent bundle using the riemannian metric on Y.

II.2 The Quantization

We wish to apply canonical quantization to this super-manifold of classical solutions. Were the space of classical solutions simply TY with the usual symplectic structure, then canonical quantization would yield the Hilbert space $L^2(Y)$ of complex-valued L^2-functions on Y. (More precisely, we would get the space of L^2 half-densities.) The functions on Y act by multiplication and sections of TY (i.e., vector fields) act by differentiation. But in our case where the space of classical solutions is a super-manifold, we must mix this canonical quantization of TY with a quantization of the vector spaces which are the fibers of the odd bundle $p^*(\Pi TY) \to TY$.

Let us consider a real, even dimensional super vector space V which is purely odd with respect to the $\mathbf{Z}/2\mathbf{Z}$-grading. Suppose further that V has a positive definite inner product and an orientation and a spin structure. We quantize this space by taking as the Hilbert superspace the complex spin bundle $S(V)$, with the $\mathbf{Z}/2\mathbf{Z}$-grading given by the usual decomposition of $S(V)$ into $S^+(V) \oplus S^-(V)$. The linear forms on V are quantized to operators on this space which act by Clifford multiplication:

$$V^* \otimes S(V) \to S(V).$$

Now we must mix together the two constructions. We have the odd vector bundle $p^*(\Pi TY)$ over the symplectic manifold TY. The Hilbert space is the space of $L^2(S(Y))$, the space of L^2-sections of the spin bundle $S(Y)$, again tensored by the half-densities on Y. The functions on Y act as usual by multiplication; the linear forms in TY act by covariant differentiation, and the linear forms on ΠTY act by Clifford multiplication. This space of sections of $S(Y)$ is a super Hilbert space – the $\mathbf{Z}/2\mathbf{Z}$-grading is induced with the usual decomposition of spinors in even dimensions $S(Y) = S^+(Y) \oplus S^-(Y)$.

The Hamiltonian operator for this quantization is given by quantizing the Hamiltonian function h associated with the flow $\partial/\partial t$ on the theory:

$$dh = -i_{\partial/\partial t}\omega.$$

In this case it is easy to see (cf, Problem 2 in Problem Set 1) that

$$h = \frac{1}{2}|\dot{x}|^2.$$

In quantizing this function there is in general an ambiguity in the resulting second order operator (adding zero and first order operators), but one representative for it is the Laplacian $\nabla^* \circ \nabla$ on spinor fields. In our situation the vector field

$$Q = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial t}$$

on $\mathbf{R}^{1,1}$ (whose square is $\partial/\partial t$) yields a vector field acting on the theory. The Hamiltonian function for this vector field, when quantized, yields a naturally defined first-order operator whose square is the Laplacian $\nabla^* \circ \nabla$ on spinor fields, up to lower order terms. This first-order operator is in fact the Dirac operator $\partial\!\!\!/$ on the spinor bundle. Thus, it is fairly natural to use the square of the Dirac operator on the spin bundle as the Hamiltonian operator. (Of course, all the various operators we could use will have the same symbol and hence give the same answers in the computations we do below.)

For any $\beta > 0$, let us denote by $[0,\beta]^{1,1}$ the pre-image in $\mathbf{R}^{1,1}$ under the natural projection mapping of the interval $[0,\beta] \subset \mathbf{R}$. Quantize as above. The Hamiltonian operator associated with infinitessimal time evolution is given by

$$H = \partial\!\!\!/^2$$

acting on the L^2-sections of spin bundle $S(Y)$ thought of as a super vector bundle over Y. Let $S^{1,1}$ denote the supercircle of length β obtained by dividing $\mathbf{R}^{1,1}$ by a translation of distance β. The partition function for the resulting supersymmetric sigma model of maps of $S^{1,1}$ into Y is the supertrace of the operator of time evolution from $t = 0$ to $t = \beta$; that is to say with

$$\mathrm{tr_s} rmexp(-\beta H) = \mathrm{tr_s}\left(\exp(-\beta \partial\!\!\!/^2)\right) \tag{2}$$

acting on the L^2-sections of the spin bundle $S(Y)$. Of course, by the usual spectral analysis (as alluded to int the introduction in the case of the heat equation proof of the HirzeCruch signature theorem), we see that the supertrace of this operator is independent of $\beta > 0$ and as $\beta \mapsto \infty$ it approaches

$$\dim\left(\mathrm{Ker}\ \partial\!\!\!/^2 | \Gamma(S^+(Y))\right) - \dim\left(\mathrm{Ker}\ \partial\!\!\!/^2 | \Gamma(S^-(Y))\right)$$

which is the index of

$$\partial\!\!\!/^+ : \Gamma(S^+(Y)) \to \Gamma(S^-(Y)).$$

Our strategy is to compute a perturbative expansion for this partition function near $\beta = 0$. In fact, as we mentioned above, we will only compute the constant term in this expansion.

II.3 The Path Integral

Now we wish to compute a power series expansion near $\beta = 0$ for the supertrace in Equation 2. Of course, as was explaned in Fadde'ev's lectures, the basic connection between the Lagrangian and Hamiltonian formulations is given by the Feynman-Kac formula. For the current application, the formula must be worked out in the context of spinor fields instead of scalar fields. This was done in detail in [Bismut]. In any event, the perturbative power series expansion that we shall compute by path integrals over Feynman diagrams is in fact a power series that is asymptotic to the above supertrace. We shall ignore all terms which are positive powers of β, thus computing the constant term in this power series. Since we know that the series is asymptotic to a constant function, this will suffice to compute the constant term in the series.

Recall that we are considering the space of maps of the supercircle $S^{1,1}$ of length β into Y. The super Lagrangian is

$$\mathcal{L}(X) = -\int_{\mathbf{S}^{1,1}} dt d\theta^{-1} \frac{1}{2} g_{I,J}(X) \frac{dX^I}{dt} DX^J,$$

The supermanifold of classical solutions is the supermanifold of maps of $S^{1,1} \to Y$, which are critical points for this Lagrangian. The underlying geometric manifold of this super-manifold is the space of closed geodesics in Y. The supermanifold M of critical points that minimize the Lagrangian is identified with $\Pi TY \to Y$, an odd vector bundle over the space of constants maps from S^1 to Y. We shall do our perturbation expansion around M. Other critical points give contributions which are exponentially small in β. (In fact, one can prove directly that the non-constant maps contribute zero to the final answer.)

By Equation 1 we can re-write this path integral as

$$\int_{x \in \mathrm{Maps}(S^1,Y); \psi \in \Gamma(x^* \Pi TY)} \mathcal{D}x \mathcal{D}\psi \exp\left(- \int_0^\beta dt \left[\frac{g_{I,J}}{2} \left(\frac{dx^I}{dt} \frac{dx^J}{dt} - \psi^I \frac{D\psi^J}{Dt} \right) \right] \right).$$

Rescale by setting $t' = \beta t$ and $\psi' = \frac{1}{\sqrt{\beta}} \psi$. Doing this and then rewriting the equation dropping the primes from the notation yields

$$\int_{x \in \mathrm{Maps}(S^1,Y); \psi \in \Pi x^* TY} \mathcal{D}x \mathcal{D}\psi \exp\left(- \frac{1}{\beta} \int_0^1 dt \left[\frac{g_{I,J}}{2} \left(\frac{dx^I}{dt} \frac{dx^J}{dt} - \psi^I \frac{D\psi^J}{Dt} \right) \right] \right). \tag{3}$$

This change of variables on the circle introduces a change in the measure of integration by a formally infinite constant. The regularization procedure will determine a finite value for this constant which depends only on the dimension of Y. In fact it is of the form C^{2n} for some universal constant. (Recall that the dimension of Y is $2n$.) We shall not try to explicitly evaluate this constant nor others like it which arise later in the argument, though it is possible to do so. Thus, our final answer must be taken up to a multiplicative constant of the form $C^{\dim Y}$.

As we have already remarked a minimum for the Lagrangian is a pair (x_0, ψ_0) where $x_0 \in Y$ is regarded as the constant map of S^1 to Y and $\psi_0(t)$ is a constant section of $\Pi T(Y)_{x_0}$. We expand to first order around this point by taking $x = x_0 + a$ where $a(t) \in TY_{x_0}$ satisfies

$$\int_0^1 a(t) = 0 \tag{4}$$

and $\psi = \psi_0 + \eta$ where $\eta(t) \in \Pi TY|_{x_0}$ satisfies

$$\int_0^1 \eta dt = 0. \tag{5}$$

These variations describe the normal bundle to M at (x_0, ψ_0). To project this parametrization of the normal bundle to a submanifold in the space of maps transverse to M we consider

$$x(s,t) = \exp_{x_0}(sa(t))$$

as a map of $S^1 \times (-\epsilon, \epsilon)$ into Y and

$$\psi(s,t) = P_{(s,t)}\left(\psi_0 + \eta(t)\right)$$

where $P_{s,t}$ denotes parallel translation in ΠTY from x_0 along the path $x(s', t)$ as s' runs from 0 to s. Thus, $\psi(s,t) \in \Pi TY_{x(s,t)}$, so that $x(s,t) + \theta\psi(s,t)$ is a map from $S^{1,1}$ to Y.

Since we are at a critical point, the first-order variation of the Lagranian with respect to a and η is zero. We are interested in the quadratic term, which is all that is needed to compute the constant term in the β-expansion of the path integral in Equation 3. To compute the quadratic term with respect to a and η in the Lagrangian we take the second derivative at a minimum (x_0, ψ_0). We find that the quadratic term is

$$I = -\frac{1}{2\beta}\int_0^1 dt \left[|\dot{a}|^2 + \langle \eta, \dot{\eta}\rangle - \frac{1}{2}\psi_0^I\psi_0^J R_{IJKL} a^K \dot{a}^L\right].$$

As noted above the higher order terms in a and η in the action will contribute to positive powers of β in the asymptotic expansion. In this expression the term $\dot{\eta}$ is the ordinary derivative of η with respect to t, which makes sense because for all t, the image $\eta(t)$ is contained in the vector space ΠTY_{x_0}. The term R_{IJKL} is the curvature of Y. Notice that in this integral there is no mixing in the quadratic terms between the a's and the η's. Thus, I is a sum $I_1 + I_2$ where

$$I_1(x,\psi) = -\frac{1}{2\beta}\int_0^1 |\dot{a}|^2 - \frac{1}{2}\langle a, \psi_0^I\psi_0^J R_{I,J}\dot{a}\rangle$$

and

$$I_2 = -\frac{1}{2\beta}\int_0^1 \langle \eta, \dot{\eta}\rangle.$$

Hence, the path integral is a product

$$\int_a \mathcal{D}a \, \exp\left(-I_1\right) \cdot \int_\eta \mathcal{D}\eta \, \exp\left(-I_2\right).$$

Also notice that the η integral

$$\int_\eta \mathcal{D}\eta \exp\left(-\frac{1}{2\beta}\langle\eta,\dot\eta\rangle\right)$$

is independent of x_0, a and ψ, since this integral depends only on TY_{x_0} with its inner product and the isomorphism class of this finite dimensional positive definite inner product space depends only on the dimension of Y. This path integral is purely quadratic and is identified with the Pfaffian of $\frac{d}{dt}$ acting on the Hilbert space of maps η. This is a constant which depends only on the dimension. In fact it is of the form C^{2n} for some universal constant C. Notice also that in the regularization that we are performing, these purely quadratic integrals are independent of β.

Thus, ignoring such multiplicative constants, to compute the constant term of the expansion in β, we must compute

$$\int_a \mathcal{D}a\ \exp\left(-\frac{1}{2}\int_0^1 |\dot a|^2 - \frac{1}{2}\langle a, \psi_0^I\psi_0^J R_{I,J}\dot a\rangle\right)$$

By integrating by parts from 0 to 1 we can re-write the integrand of the functional integral as the exponential of

$$-\frac{1}{2}\int_0^1 \langle D_B(a), a\rangle$$

where the propagator D_B depends on (x_0,ψ_0) and is given by

$$D_B(x_0,\psi_0) = -\frac{d^2}{dt^2} - \frac{1}{2}\psi_0^I\psi_0^J R_{I,J}\frac{d}{dt}.$$

Thus, modulo β, the path integral over the normal space to the super manifold M of minima at the point (x_0,ψ_0) of I_a gives

$$\frac{1}{\sqrt{\det_a\ D_B(x_0,\psi_0)}}$$

where the determinant is computed over the space of a for fixed (x_0,ψ_0) and is regularized using the ζ-function regularization. The final formula for the path integral is then

$$\int \mathcal{D}x\mathcal{D}\psi\ \exp(-I_a - I_\eta) \equiv C^{2n}\int_{\Pi TY}\frac{1}{\sqrt{\det\ D_B(x,\psi)}}\quad (\mathrm{mod}\ \beta). \tag{6}$$

Here C is a universal constant (explicitly calculable given enough patience).

It is convenient to realize that this integral over the supermanifold ΠTY can be re-interpreted as an ordinary integral of a top dimensional form over the underlying manifold Y. To see this note that, as explaned in Berstein's lectures, the C^∞-functions on ΠTY are simply the C^∞-sections of the bundle $\wedge^*(T^*Y)$ over Y with the usual $\mathbf{Z}/2\mathbf{Z}$-grading; that is to say the C^∞-functions on ΠTY are exactly the C^∞-differential forms on Y with the $\mathbf{Z}/2\mathbf{Z}$-grading being the degree modulo two. Let f be such a C^∞-function and let ω be the corresponding differential form on Y. We have an orientation on Y. For a function $f \in C^\infty(\Pi TY)$ the integral

$$\int_{\Pi TY} f$$

is simply the integral over the fundamental cycle of Y of the the corresponding form ω. The point is $\mathrm{Ber}(TY)$ and $\mathrm{Ber}(T^*Y)$ are naturally dual line bundles so that their product is canonically trivial.

Thus, we have the path integral expression for the index of the Dirac operator:

$$\mathrm{index}(\partial\!\!\!/^{+}) = (C')^n \int_Y \frac{1}{\sqrt{\det\, D_B(x,\psi)}},$$

where the determinant is computed using the ζ-function regularization, where C' is a universal constant, and, as discussed above, the ψ-dependence turns the integrand into a differential form (with compnents of various degrees).

II.4 Evaluation of the Determinant

Our next task is to compute this determinant as a differential form. Let us fix first a element $g \in so(TY_{x_0})$ and consider the operator

$$D_g = -\frac{d^2}{dt^2} - \frac{1}{2} g \frac{d}{dt}$$

acting on the space of maps $a\colon S^1 \to TY_{x_0}$ which satisfy the constraint given in Equationi 4. We can decompose TY_{x_0} into two-dimensional spaces invariant under g such that on the i^{th}-space g is given by the matrix

$$\begin{pmatrix} 0 & x_i \\ -x_i & 0 \end{pmatrix}$$

with respect to an orthonormal basis. The determinant of D_g will be the product over the blocks in this decomposition of the determinants associated to the individual two-dimensional pieces associated with each block. This allows us to reduce to the two-dimensional case. (This step is similar in spirit to the reduction step to riemann surfaces in some other proofs of the same result.) Fix a two-dimensional subspace V of TY_{x_0} on which g has the form

$$g = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}.$$

We use the metric and orientation of TY_{x_0} to induce a complex stucture on V. With respect to this structure g is simply multiplication by $-ix$. We can decompose the complexification of the space of maps of S^1 into V into two-dimensional spaces of the form $\exp(2\pi ikt)a$ for $a \in V$. Notice that by Condition 4, this decomposition is over $k \neq 0$. It is easy to see that the eigenvalues of D_g on the two-dimensional subspace indexed by k are

$$(2\pi k)^2 \pm (2\pi k)ix/2.$$

Thus,

$$\sqrt{\det D_g} = \prod_{\pm} \prod_{k\neq 0} \sqrt{((2\pi k)^2 \pm i(2\pi k)x/2)}.$$

Grouping together the four terms for each pair $k, -k$ we get

$$\sqrt{\det D_g} = \prod_{k=1}^{\infty} (2\pi k)^4 \left(1 + \left(\frac{x}{4\pi k}\right)^2\right).$$

As in Problem 2 of Problem Set Five, the ζ-function regularization sets the infinite factor

$$\prod_{k=1}^{\infty} (2\pi k)^4$$

equal to C^2 for some universal constant C. Thus, the result is that

$$\sqrt{\det D_g} = C^2 \prod_{k=1}^{\infty} \left(1 + \left(\frac{x}{4\pi k}\right)^2\right) = C^2 \left(\frac{x/2}{\sinh(x/2)}\right)^{-1}.$$

This computation was done under the assumption that TY was two-dimensional. In the general case when TY is a sum of two-dimensional spaces of the above form we find

$$\sqrt{\mathrm{detD_g}} = C^{2n} \prod_{i=1}^{n} \left(\frac{x_i/2}{\sinh(x_i/2)}\right)^{-1}.$$

This power series can be re-written using the elementary symmetric functions of the x_i. When we do so by the definition of $\hat{A}$ we obtain

$$(\det D_g)^{-1/2} = C^{2n} \hat{A}(g).$$

Now let us return to $\det D_B^{-1/2}$ as a function on ΠTY, or as we have seen as a differential form on Y. Consider the function $\sum_{I,J} R_{I,J}(x_0)\psi_0^I\psi_0^J$ on ΠTY with values in $\mathrm{ad}(TY)$. As indicated above this function is identified with a two-form on Y with values in $\mathrm{ad}(TY)$. Not suprisingly under this identification the sum becomes the curvature two-form $R(Y)$ of Y. Thus,

$$\det(D_B)^{-1/2} = C^{2n}\hat{A}(R(Y)).$$

We have now established that

$$\text{index }(\partial\!\!\!/^{+}) = C^{2n}\int_Y \hat{A}(R(Y))$$

where C is a universal constant. Of course, the usual formula of Atiyah-Singer is

$$\text{index } \partial\!\!\!/^{+}) = \int_Y \hat{A}(R(Y)).$$

In this derivation we have been careless with universal constants depending only on dimension (infinite but regularized in some manner). To evaluate the constant C one can either follow through the regularization procedure or simply evaluate it on one example where the index is non-trivial.

III The Case of a Circle Action

In this section Y is a compact $2n$-dimensional spin manifold with an isometry $g\colon Y \to Y$. We suppose that this automorphism lifts to the bundle of spinors. We define the g-index of the Dirac operator to be

$$\mathrm{Tr}_{\mathcal{H}_+} g - \mathrm{Tr}_{\mathcal{H}_-} g$$

where $\mathcal{H}_\pm$ are the plus and minus harmonic spinors. Equivalently, we can compute this as

$$\mathrm{tr_s}\left(g\ \exp(-\beta H)\right),$$

where H is the square of the Dirac operator. As before, there is a path integral computation of a power series expansion in β aysmptotic to this supertrace at $\beta = 0$. The constant term in this series computes this g-index. In fact, this procedure works for any isometry g which lifts to the spin bundle, (see the first problem in the final exam for the Fall Semester). But here we shall suppose that g lies in a connected, compact group of isometries, which is equivalent to assuming that it lies in a circle of symmetries.

So we consider an action $S^1 \times Y \to Y$ by isometries. At the expense of doubling the circle, this action automatically lifts to the bundle of spinors. We take a Killing vector field V for this action (the vector field generating the action), and denote by $\mathcal{L}_V$ the Lie derivative of this vector field on the spinors. We consider

$$F(\theta) = \mathrm{tr_s}\left(\exp(i\theta\mathcal{L}_V)\exp(-\beta H)\right) = \sum_{n\in\mathbf{Z}} \exp(in\theta)\mathrm{ind}_n \partial\!\!\!/^{+},$$

where $\mathrm{ind}_n \partial\!\!\!/^{+}$ is the index of the Dirac operator from subspace $\mathcal{H}_n^+$ of plus spinors on which the character of S^1 is the n^{th}-power to the corresponding subspace $\mathcal{H}_n^-$ of the minus spinors. In what follows we denote $\mathrm{ind}_n \partial\!\!\!/^{+}$ by a_n.

Our goal is to prove the theorem of Atiyah-Hirzebruch [AH];

Theorem III.0.1 *If $S^1 \times Y \to Y$ is a non-trivial isometric circle action on a spin manifold Y then* $\mathrm{index}_n(\partial\!\!\!/)^{+} = a_n = 0$ *for all n.*

III.1 Proof of Atiyah-Hirzebruch Theorem using the Hamiltonian Formalism

Instead of computing this via path integrals as we did in the case of trivial action above, we introduce a Hamiltonian approach. We perturb the Dirac operator to an operator depending on a real parameter t defined by

$$\partial\!\!\!/_t(\psi) = \partial\!\!\!/(\psi) + tV \cdot \psi$$

on spinors. (Here, $V \cdot \psi$ is Clifford multiplication.) It is a formally self-adjoint operator. The Hamiltonian is

$$H_t = (\partial\!\!\!/_t)^2 = (\partial\!\!\!/)^2 + t^2|V|^2 + t\left(2\nabla_V + \mathrm{div}(V) + dV^* \cdot \psi\right),$$

where V^* is the one-form dual under the metric to V and the last term is Clifford multiplication by dV^*. Once again this is a formally self-adjoint operator. Since the circle action perserves the metric, $\mathrm{div}(V) = 0$. Thus, we have

$$H_t = (\partial\!\!\!/_t)^2 = (\partial\!\!\!/)^2 + t^2|V|^2 + t\left(2\nabla_V + dV^* \cdot \psi\right).$$

We also have a formula for the Lie derivative $\mathcal{L}_V$ as follows

$$\mathcal{L}_V = \nabla_V + \frac{1}{4} dV^* \cdot (\ ),$$

so that we can re-write the Hamiltonian as

$$H_t = (\partial\!\!\!/_t)^2 = (\partial\!\!\!/)^2 + t^2|V|^2 + t\left(2\mathcal{L}_V + \frac{1}{2} dV^* \cdot (\ )\right).$$

Since $\mathcal{L}_V$ generates a circle action on the spin bundle, its eigenvalues are integers. We denote by $\mathcal{H}_n$ the eigenspace for $\mathcal{L}_V$ of eigenvalue n in the Hilbert space of all L^2-sections of the spin bundle. The vector field V generates a symmetry of H and in fact of H_t for every $t \in \mathbf{R}$. Hence, $\mathcal{L}_V$ commutes with H_t. The index a_n we are trying compute is simply the index of the restriction of the operator defined by H (or H_t) from the invariant space $\mathcal{H}_n^+$ to the invariant subspace $\mathcal{H}_n^-$; or put another way, using self-adjointness of H_t it is the dimension of the kernel of H_t on $\mathcal{H}_n^+$ minus the dimension of the kernel of H_t on $\mathcal{H}_n^-$. We shall prove the theorem by showing that if $tn \geq 0$ and $|t| >> 1$ then H_t is positive on $\mathcal{H}_n$ and in particular has no kernel on this subspace. It follows immediately that $a_n = 0$.

For large $|t|$ the spectrum of H_t restricted to the subspace $\mathcal{H}_n$ can be calculated in an asymptotic expansion in powers of $1/|t|$ by expanding near the minima of the potential. These minima are the zeros of V. We shall work first in the case when the S^1-action has isolated fixed points. We consider the more general case briefly at the end of this section. Let us consider an isolated fixed point $\alpha \in Y$. We decompose the linearization of the S^1-action at this fixed point into a sum of two-planes:

$$TY_\alpha = \sum_j W_j$$

where the restriction of V on W_j is given in orthonormal coordinates (x_{2j-1}, x_{2j}) by

$$k_j \left(x_{2j-1} \frac{\partial}{\partial x_{2j}} - x_{2j} \frac{\partial}{\partial x_{2j-1}} \right), k_j \in \mathbf{Z}^+.$$

We use the orientation induced on W_j by these coordinates. The condititіon that the S^1-action lifts to the spin bundle implies that

$$\sum_j k_j \equiv 0 \pmod{2}. \tag{7}$$

To leading approximation in $1/|t|$ the restriction of H_t near α is given by

$$H_t^{\mathrm{loc}}(\alpha) = -\sum_{j=1}^{2n} \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^{n} t^2 k_j^2 (|x_{2j-1}|^2 + |x_{2j}|^2) - t \sum_{j=1}^{n} k_j \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2t\mathcal{L}_V .$$

Ignoring the last term and splitting along the two-dimensional subspaces for the monent we consider the operator

$$H_t^{\mathrm{loc},j}(\alpha) = -\frac{\partial^2}{\partial x_{2j-1}^2} - \frac{\partial^2}{\partial x_{2j}^2} + t^2 k_j^2 (|x_{2j-1}|^2 + |x_{2j}|^2) - t k_j \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1 \le j \le n. \tag{8}$$

With these definitions we have

$$H_t^{\mathrm{loc}}(\alpha) = \sum_{j=1}^{n} H_t^{\mathrm{loc},j}(\alpha) + 2t\mathcal{L}_V .$$

Let us consider the spectrum of the harmonic oscillator which is the sum of all but the last term in the expression for $H_t^{\mathrm{loc},j}(\alpha)$:

$$-\frac{\partial^2}{\partial x_{2j-1}^2} - \frac{\partial^2}{\partial x_{2j}^2} + t^2 k_j^2 (|x_{2j-1}|^2 + |x_{2j}|^2).$$

Its ground state Ω has eigenvalue $2|t||k_j|$, and its spectrum is

$$\{(4\ell + 2)|t||k_j\} \;\; \ell = 0, 1, 2, \cdots$$

with each eigenvalue being of multiplicity two.

The last term in Expansion 8 commutes with the harmonic ocsillator and has eigenvalues $\pm t k_j$. Thus, it splits each of the two-dimensional eigenspaces into two one-dimensional spaces with distinct eigenvalues. Thus, the lowest eigenvalue for $H_t^{\mathrm{loc},j}(\alpha)$ is $|t||k_j|$, and its spectrum is

$$\{(2r + 1)|t||k_j|\}, \;\; r = 0, 1, 2, \cdots .$$

Though we do not need it now, it will be important later to understand how $\mathcal{L}_V$ acts on these eigenstates. A direct computation in these local coordinates shows that the eigenstate of $H_t^{\mathrm{loc},j}(\alpha)$ with eigenvalue $(2r + 1)k_j|t|$ is an eigenstate for $\mathcal{L}_V$ with eigenvalue $(r + 1/2)k_j$. Also notice that if $n > 0$ then $\mathrm{Ker} H_t \cap \mathcal{H}_n$ consists of plus spinors and if $n < 0$ it consists of minus spinors.

Summing up over the two-planes and using the definition of $\mathcal{H}_n$, we see that the lowest eigenvalue for $H_t^{\mathrm{loc}}(\alpha)$ on $\mathcal{H}_n$ is

$$\sum_j |t||k_j| + 2tn.$$

As $|t| \mapsto \infty$, any element of norm one in the kernel of the global Hamiltonian operator must localize around the fixed points. In fact it must converge at each fixed point an element in the kernel of $H_t^{\mathrm{loc}}(\alpha)$. But we have already seen that for $tn \geq 0$ the local operators are positive and have no kernel. It follows that if $t >> 0$ and $tn \geq 0$ there is no kernel for the restriction of H_t to $\mathcal{H}_n$.

This proves the Atiyah-Hirzebruch theorem stated above at least in the case when the circle action has isolated fixed points. The case of non-isolated fixed points is briefly discussed at the end of this section.

Notice that in proving this result for all n we had to use both $t > 0$ and $t < 0$. When we get to the loop space setting in the next lecture, we will not be able to take $t = 0$ and hence we can not pass from $t > 0$ to $t < 0$. The consequence of this is that we will only establish that half the S^1-indices vanish.

III.2 A local integral for the index of the Dirac operator

Even though we have established the Atiyah-Hirzebruch theorem that the a_n vanish, it is still interesting to give a different computation of the a_n in terms of local data around the fixed points of the circle action. This computation is particularly relevant when we get to the case of twisted Dirac operators where the equivariant indices do not all vanish.

The function $F(\theta) = \sum_n a_n e^{in\theta}$ is computed as supertrace of the circle action on the kernel of the operator H_t for any t. The convergence of the operators H_t to the sum of the local operators H_t^{loc} for large $|t|$ implies that in fact we can do the computation by using the kernels of the local operators at the fixed points. Let us fix $t < 0$ and compute the kernel of $H_t^{\mathrm{loc}}(\alpha)$. Note that the argument in the last subsection did not show that this kernel vanished, only that the restriction of H_t to the $\mathcal{H}_n$ with $tn \geq 0$ had no kernel.

As we have already seen, the spectrum of the operator $H_t^{\mathrm{loc},j}(\alpha)$ is $\{(2r+1)k_j||t|\}$, $r = 0, 1, \cdots$, with the eigenspaces of dimension one, generated by λ_r, say. Furthermore, the vector field V is the infinitessimal generator of an S^1-action on the state $/lambda_r$ is given by

$$e^{(r+1/2)k_j\theta}.$$

Thus, V is the infinitessimal generator of an S^1-action whose character on the sum of the states given by

$$e^{k_j i\theta/2}+e^{3k_j i\theta/2}+\cdots=\frac{e^{k_j i\theta/2}}{1-e^{-i\theta k_j}}=\frac{i}{2}\frac{1}{\sin(k_j\theta/2)}. \tag{9}$$

Nex we consider the sum of all possible tensor products over j of eigenspaces of the $H_t^{\mathrm{loc},j}$: The vector field V is the infinitessimal generator of an S^1-action on this sum of tensor products whose character is given by

$$F_\alpha(\theta)=\prod_{j=1}^{n}\frac{i}{2}\frac{1}{\sin(k_j i\theta/2)}. \tag{10}$$

Notice also that since $t<0$, the computation above of the action of $\mathcal{L}_V$ on the eigenstates of the $H_t^{\mathrm{loc},j}$ immediately implies taht all these tensor product states are in the kernel of our local operator $H_t^{\mathrm{loc}}(\alpha)$ and are plus spinors. It is clear that this accounts for the entire kernel of the local operator. Notice that even though the k_j are not necessarily even, all the exponents that appear in the above product expansion are of the form

$$((2r_1+1)k_1+\cdots+(2r_d+1)k_d)\,/theta/2,$$

and by the spin condition (Equation 7) are integral multiples of θ. Hence, $F_/alpha(\theta)$ is a character of an action of the original S^1, not some covering of it. It is the character of the action of the circle on the kernel of $H_t^{\mathrm{loc}}(\alpha)$. Thus, $F_\alpha(\theta)$ is the local contribution at α of the S^1-equivariant index.

Summing up over the various isolated fixed points α gives

$$F(\theta)=\mathrm{tr_s}\left(e^{i\theta\mathcal{L}_V}e^{-\beta H}\right) \quad = \quad \left(\frac{i}{2}\right)^n\sum_\alpha(-1)^{\sigma_\alpha}\prod_{j=1}^{n}\frac{1}{\sin(k_{\alpha,j}\theta/2)}.$$

where α ranges over the fixed points of the circle action (still assumed to be isolated), and σ_α compares the orientation on TY_α coming from the ambient orientation of Y with the orientation induced from the direct sum of the orientations on the various two-planes $V_{\alpha,j}$. (The $k_{\alpha,j}$, $j=1,\ldots n$ are the rotation numbers of the circle action at the fixed point α.)

Of course, in this case we have shown that this sum is in fact zero. This is a non-trivial statement about the nature of the S^1-action at the fixed points. (Recall that we are assuming in the discussion so far that the S^1-action has only isolated fixed points.)

III.3 Dirac Operators with Values in a Bundle

If we pass from the case considered above of the pure Dirac operator to the case of a Dirac operator twisted by a bundle with a connection, then the vanishing theorem of Atiyah-Hirzebruch does not generalize. Nevertheless, the second argument computing the equivariant index in terms of local data at the fixed points does generalize. It produces the Atiyah-Bott formula [AB].

Let $\xi \to Y$ be a vector bundle over a compact, even dimensional spin manifold. We consider the Dirac operator twisted by this bundle. The local analysis around the fixed points α involves the two-plane bundles $W_{\alpha,j}$ at α as well as the restriction $\xi|_\alpha$. The formula is (assuming that the S^1-action has only isolated fixed points)

$$F_\xi(\theta) = \mathrm{ind}_{S^1}(\partial\!\!\!/ \otimes \xi) = \sum_\alpha (-1)^{\sigma_\alpha} \prod_j \prod_{j=1}^{n} \frac{1}{\sin(k_{\alpha,j}\theta/2)} \mathrm{Tr}_{\xi|\alpha}\left(e^{i\theta\mathcal{L}_V}\right).$$

If ξ is a trivial bundle then we are back to the case of the pure Dirac operator and we know that the S^1-index $F_\xi(\theta)$ is identically 0. There is an important generalization of this result for a certain class of spin bundles $\xi \to Y$. Let us consider the case when $\xi = S(Y)$, the spin bundle of Y. While it is no longer true that $F_\xi(\theta) = 0$, it is true is that $F_\xi(\theta)$ is a constant function. The reason is that $S(Y) \otimes S(Y)$ can be identified with the differential forms and hence the kernel spaces are identified with the harmonic forms. Of course, $\mathcal{L}_V$ acts trivially on these spaces. One could ask a more general question here: Let $P_Y \to Y$ be the principal $\mathrm{Spin}(2n)$-bundle of Y and let $\mathcal{R}$ be (finite dimensional) representation of $\mathrm{Spin}(2n)$. We define

$$\xi = P \times_{\mathrm{Spin}(2n)} \mathcal{R}$$

and consider the character-valued index function $F_\xi(\theta)$ as above. The question is: For which W constructed in this manner is the function $F_\xi(\theta)$ equal to a constant? We shall see in the next lecture that there is an infinite series of such representations $\mathcal{R}$ with this property. By constructing enough examples Landweber and Stong [LS] showed that the infinite series that we construct includes all the representations with this property.

III.4 The Case of Non-isolated Fixed Points

Let us finish this section with a brief discussion of the case when the S^1-action does not have isolated fixed points. Suppose that the fixed point set is $\coprod_r M_r$, with the M_r being connected. Again we consider the deformation

$$\partial\!\!\!/_t(\ ) = \partial\!\!\!/(\ ) + tV\cdot(\ ).$$

Exactly as before, taking $|t| \mapsto \infty$ localizes the low eigenstates of H_t around the fixed components M_r of the circle action. The local Hamiltonian H_t^{loc,M_r} at the fixed component M_r has harmonic ocsillators in the normal direction, analogous to the one encountered when the fixed points were isolated, and is the square of the Dirac operator along M_r. The decomposition of TY_α into two-dimensional spaces $V_{\alpha,j}$, which occurred in the case of isolated fixed points, is replaced in this case by a decomposition of the normal bundle V_r to M_r as a direct sum of sub-bundles $V_{r,j} \rightarrow M_r$ of dimension $2d_j$, say, on which the circle action is given by the character which is raising to the k_j^{th}-power. We orient the $V_{r,j}$ so that $k_j > 0$, hence inducing an orientation on V_r and consequently on M_r. (If M_r is a point, then this orientation differs from the natural one on the point by the sign $(-1)^{\sigma_\alpha}$ encountered before.) The argument given above generalizes easily to this case to show that for $tn \geq 0$ there is no kernel for H_t^{loc,M_r} on the subspace $\mathcal{H}_n$. Thus, we see in this case as well, that H_t has no kernel on $\mathcal{H}_n$ for $|t| >> 1$ and $tn \geq 0$, establishing the Atiyah-Hirzebruch theorem in the case of general circle actions.

One can also generalize the second argument to produce a formula in terms of fixed point data for the equivariant index. Once again it suffices to consider $|t| >> 1$ and to find the kernel of H_t^{loc,M_r}. Let us suppose first, for the purposes of illustration, that the component M_r is of codimension two and is a spin submanifold. Let k be the character of the circle action in the normal direction. Of course, our spin hypotheses imply that $k \equiv 0 \pmod 2$. For $t < 0$ we find, exactly as before, one state in the kernel of the normal operator restricted to $\mathcal{H}_n$ for $n = (2s+1)k/2$, $s = 0, 1, 2, \ldots$. Each of these states varies with the point of M_r and hence forms a complex bundle over M_r. We then have line bundles $L_1, L_3, \ldots$ with L_{2s+1} having S^1-character $(2s+1)k/2$. Let us relate the L_{2s+1} to the normal bundle V_r (which is a complex line bundle). Since the eigenstate φ_s with character $(2s+1)k/2$ is homogeneous of degree $2s+1$ in the spin bundle associated to V_r, it follows that $L_{2s+1} = \sqrt{V_r}^{\otimes(2s+1)}$, the $2s+1$ tensor power of the square root of the normal bundle of M_r. (The fact that M_r is spin implies that V_r has a square root and even picks out one, so that these bundles are well-defined.) Thus, the spaces of vacua in the normal direction fit together to form a sum of S^1-equivariant complex line bundles. The local contribution around this component of the fixed point set is given by a sum like the one given by Equation 9 with $ik\theta/2$ replaced by $(x+ik\theta)/2$ where $x = c_1(V_r)$. Of course, along the fixed point set M_r, the operator H_t^{loc,M_r} is the square of the Dirac operator. The usual index theorem for the Dirac operator then computes the local contribution at this component M_r to the equivariant index to be

$$\int_{M_r} \hat{A}(M_r)\frac{1}{2}\frac{1}{\sinh((x+ik\theta)/2)}.$$

If M_r is codimension-two but not spin, then two things breakdown in the above argument: There are no spin bundles on M_r and hence no Dirac operator on M_r, and we cannot form the square root line bundle to V_r. Nevertheless, the description of the kernel states in the normal direction is still valid locally. What happens is that the obstruction to defining the spin bundles and Dirac operator globally on M_r is exactly cancelled by the obstruction to forming the square root line bundle to V_r. Thus, the local descriptions fit together to define an operator which is locally described as a Dirac operator with values in a line bundle but is globally twisted. The index theorem still applies in this twisted situation and gives the same formula. (Though of course, $x/2$ is only a rational cohomology class.)

More generally, when the normal bundle V_r to M_r splits as a sum of complex line bundles invariant under the S^1-action, the space of vacua is a sum of terms which are tensor products of the above type, and the normal contribution is a product of terms of the above type:

$$\prod_j \frac{e^{(x_j+ik_j\theta)/2}}{1-e^{x_j+ik_j\theta}} = \prod_j \frac{1}{2\sinh((x_j+ik_j\theta)/2)},$$

where x_j is the first Chern class of the j^{th}-line bundle and k_j is the weight of the S^1-action on this line bundle. In the more general case when V_r does not split as a sum of complex line bundles invariant under the S^1-action case we get the expression for normal contribution is obtained using the splitting principal. Suppose that the normal bundle V_r to M_r splits as $\oplus_j V_{r,j}$ as described before. Recall that $d_j = \dim_{\mathbf{C}}(V_{r,j})$. The resulting S^1-equivariant Chern character is

$$\hat{A}_\theta(V_r) = 2^{-\dim(V_r)/2}\prod_j\prod_{\ell=1}^{d_j}\frac{1}{\sinh((x_{\ell,j}+ik_j\theta)/2)}$$

with the $x_{\ell,j}$ being the formal roots whose elementary symmetric functions are the Chern classes of $V_{r,j}$. (Of course, as always, this expression must be re-written in terms of the elementary symmetric functions in order to find the expression in terms of the Chern classes of $V_{r,j}$.) This describes the bundle of vacua over M_r for the operator H_t^{loc,M_r}. Of course, so far we have been describing this operator only in the normal directions to M_r. Along M_r it is the square of the usual Dirac operator. Thus, the index of H_t^{loc,M_r} is simply that of the Dirac operator with values in the bundle of vacua in the normal direction. (As before, if M_r is not a spin submanifold, then neither the spin bundles along M_r nor the spin bundle in the normal direction can be independently defined, but the index formula works exactly as if they could. The usual index theorem now evaluates the S^1-index of this operator to be

$$F_{M_r}(\theta) = \int_{M_r}\hat{A}(M_r)\hat{A}_\theta(V_r).$$

Adding up the above expression for the local S^1-indices over all the components of the fixed point set yields the following formula for the S^1-index of H:

$$F(\theta) = \sum_r\int_{M_r}\hat{A}(M_r)\hat{A}_\theta(V_r). \tag{11}$$

This formula together with the Atityah-Hirzebruch theorem that the equivariant signature vanishes can then be viewed as giving non-trivial information about the nature of the fixed points submanifolds of the S^1-action.

More generally, if we are considering a Dirac operator coupled to a connection on an S^1-vector bundle $\xi \to Y$ then we replace $\mathcal{V}_r$ in Equation 11 by

$$(\oplus_j(V_{r,j})\otimes\left(\oplus_b e^{i\theta b}\xi_{r,b}\right)$$

where

$$\xi|M_r = \oplus_b\xi_{r,b}$$

and the character of the circle on $\xi_{r,b}$ is the b^{th}-power. With this substitution for $\mathcal{V}_r$, the same equation gives the index of the coupled Dirac operator as a function on S^1.

IV σ-models in $1+1$ dimensions

Let us study in a rough qualitative way the case of maps of a two-dimensional supermanifold into a compact manifold Y of dimension $2n$. We shall consider several contexts, but always with an action given by the same formula. First, let us consider the super-manifold

$$\mathrm{Maps}(\mathbf{R}^{2,2}, Y),$$

with the action given by

$$\mathcal{L} = \int_{\mathbf{R}^{2,2}} dudv(d\theta_+)^{-1}(d\theta_-)^{-1} g_{I,J} D_+ X^I D_- X^J$$

where the even coordinates on $\mathbf{R}^{2,2}$ are (u, v), the odd coordinates are θ_+, θ_-, and

$$D_+ = \frac{\partial}{\partial\theta_+} - \theta_+ \frac{\partial}{\partial u}$$

$$D_- = \frac{\partial}{\partial\theta_-} - \theta_- \frac{\partial}{\partial v}$$

are vector fields on $\mathbf{R}^{2,2}$. Notice that a map of $X: \mathbf{R}^{2,2} \to Y$ can be written as

$$x + \psi_+\theta_+ + \psi_-\theta_- + F\theta_+\theta_-$$

wher x is a map $x: \mathbf{R}^2 \to Y$, ψ_+, ψ_- are sections of the pullback $x^*\Pi TY$ of the tangent bundle of Y, considered as an odd bundle and a section F of x^*TY considered as an even bundle. For any superriemann surface Σ (supermanifold of type $(2,2)$) we have this action, but invariantly the $\psi_\pm$ are sections of $S^\pm(\Sigma) \otimes x^*TY$. With this identification, we can re-write the action as

$$\mathcal{L} = \int_{\mathbf{R}^2} dudv \left[g_{I,J}\partial_u X^I \partial_v X^J + i\psi_+\partial_v\psi_+ + i\psi_-\partial_u\psi_- + R_{I,J,K,L}\psi_+^I\psi_+^J\psi_-^K\psi_-^L + g_{I,J}F^I F^J \right].$$

Since the field F enters the Lagrangian purely quadratically and only algebraically (no derivatives of F appear), it is easy to integrate out the F dependence. When we do this gaussian integration in F we are left with the constraint $F = 0$ and the same action (without the $|F|^2$-term to integrate over the space of fields $x, \psi_\pm$.

We shall study to some degree this Lagranian formulation of this supersymmetric σ-model, but we shall also need a Hamiltonian formulation. In the Hamiltonian formulation we study a quantum theory of maps of the super cylinder $\mathbf{R}^1/S \times \mathbf{R}^{1,2}$ into Y where S is translation by a fixed amount. Equivalently, we consider the space of maps of $\mathbf{R}^{1,2}$ into the loop space $\mathcal{L}Y$ of Y. Letting t be the geometric variable in $\mathbf{R}^{1,2}$ and letting s be the loop variable in $\mathcal{L}Y$, we identify $t = u + v$ and $s = u - v$. Under this identification the action above becomes a flow equation in this context: it is not the usual analogue of the geodesic flow equation for $\mathbf{R}^{1,2}$ in $\mathcal{L}Y$ since there are extra potential terms. Nevertheless, the space of classical solutions is given by the initial conditions at $t = 0$, where as before $F = 0$ is a constraint. That is to say, the space of classical solutions is identified with the odd vector bundle $\Pi T\mathcal{L}Y \oplus \Pi T\mathcal{L}Y$ over $T\mathcal{L}Y$ which is two copies of the tangent bundle to the loop space. Quantizing this bundle yields the super-Hilbert space of sections of $\mathrm{Sym}^*\Pi T(\mathcal{L}Y)$ which should be viewed as the differential forms on $\mathcal{L}Y$.

There is one important feature we wish to discuss. There is a $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$-symmetry of the Lagrangian given by

$$\sigma \cdot \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \mapsto \begin{pmatrix} -\psi_+ \\ \psi_- \end{pmatrix}$$

and

$$\tau \cdot \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \mapsto \begin{pmatrix} \psi_+ \\ -\psi_- \end{pmatrix}.$$

We wish to discuss whether these symmetries lift to the quantum field theory.

Before addressing this problem directly, let us consider the analogous question for the space of maps of $\mathbf{R}^{1,1}$ into a finite, even dimensional spin manifold Z. Quantizing the odd vector bundle ΠTZ over an even dimensional spin manifold Z yields the Hilbert space of L^2-sections of the spin bundle $S(Z)$, and the field ψ (analogous to say ψ_+ above) acts by Clifford multiplication. The analogue of the symmetry σ above is then implemented on this Hilbert space by the involution ϵ whose ± 1-eigenspaces are the sections of $S^{\pm}(Z)$. By this we mean

$$\mathrm{Cl}(-\psi_+) = \epsilon \circ \mathrm{Cl}(\psi_+) \circ \epsilon,$$

where $\mathrm{Cl}(\chi)$ denotes Clifford multiplication by χ. Now in the case of maps of $\mathbf{R}^{1,2} \to Z$ one can implement both of the symmetries σ and τ using the two copies of the spin bundle associated to the fields $\psi_\pm$. Thus, it is clear that in this finite dimensional analogue the symmetries σ and τ lift to the quantum level when Z is an even dimensional spin manifold.

Let consider the case of mapping $\mathbf{R}^{1,2}$ into an finite, even dimensional manifold Z, no longer assumed to be spin. Even though the manifold is not spin, as we indicated above, we can quantize the direct sum of the two copies of ΠTZ resulting in the differential forms on Z with their mod 2 grading. (Of course, if Z is spin we got the space of L^2-sections of the tensor product of $S(Z) \otimes S(Z)$, but this is naturally identified with the space of L^2 differential forms on Z.) With this quantization, it turns out that, provided that Z is oriented, it is still the case that the symmetries σ and τ lift to the realization. They are given by

$$(-1)^{\deg} = \sigma\tau$$

and

$$* = \sigma,$$

where $*$ is the Hodge $*$-operator multiplied by a factor of $i^{\deg}$ in order to make it an involution. Thus, in this finite dimensional analogue, the $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$-symmetry group lifts to the quantum theory of maps of $\mathbf{R}^{1,2}$ into Z as long as Z is orientable.

We claim that the answer in the infinite dimensional case is the same, with a fairly natural definition of the notion of $\mathcal{L}Y$ being orientable. We shall examine this question via path integrals considering the theory of all maps of super riemann surfaces Σ (supermanifolds of type $(2,2)$) into Y. Since σ and τ are symmetries of the Lagrangian, the question of whether they lift is the same as asking whether the symmetries are symmetries of the formal measure of integration $\mathcal{D}X\mathcal{D}\psi_+\mathcal{D}\psi_-$. It turns out that just as in the finite dimensional case $\sigma\tau$ always lifts to a symmetry of the quantum theory. Let us consider σ. First, we claim that σ is a symmetry of the measure of integration if and only if $\dim \mathrm{Ker}\mathcal{D}_+ \equiv 0 \pmod 2$ for all maps $x\colon \Sigma \to Y$ for any riemann surface Σ. The reason is that σ clearly preserves the $\mathcal{D}X\mathcal{D}_-$ and σ will be a symmetry of $\mathcal{D}_+$ provided that the space of sections of the pullback bundle $S^+(\Sigma) \otimes x^*TY$ is even dimensional. The idea is that σ is acting by -1 on these fields and thus is preserving the orthogonal form. The issue is whether formally σ preserves the orientation. (In finite dimensions -1 is a symmetry of the integration measure if and only if it is orientation-preserving which is equivalent to the statement that the space is even-dimenional.) Of course, in our context the space is infinite dimensiona, so in order to determine its dimension we have to regularize it. The way this is done is the following. We have a skew-symmetric pairing on this bundle of sections given by

$$\langle \chi_1, \chi_2 \rangle = \int_\Sigma \langle \chi_1, \mathcal{D}_+(\chi_2) \rangle.$$

Here $\mathcal{D}_+$ is the dirac operator on $S^+(\Sigma)$ coupled with covariant differentiation on the pullback x^*TY. This pairing has a finite dimensional null space which is exactly the kernel of the Dirac operator $\mathcal{D}_+$. Given a skew-pairing A on a finite dimensional vector space V we know that $\dim(V) \equiv \dim(\mathrm{Ker}(A)) \pmod 2$. In the infinite dimensional theory that we are studying here, this is true for the pairing given above. Thus, we see that for the dimension of the space of sections of $S^+(\Sigma) \otimes x^*(TY)$ modulo two we use the dimension of the kernel of $\mathcal{D}_+$. Since TY real this coupled Dirac operator is skew-adjoint. Its spectrum is purely imaginary and symmetric about zero. Thus, its algebraic spectral flow is zero, and the dimension of its kernel modulo two is a topological invariant (unchanged under deformation of the operators through ellipic skew-adjoint operators). In fact, there is a mod two index theorem for the dimension of the kernel. Since TY is even dimensional, the mod two index theorem identifies the dimension of the kernel with $w_2(x^*TY)$. Thus, we see that σ will lift to an involution on the quantum field theory for all maps of all superriemann surfaces Σ into Y if and only if $w_2(TY) = 0$.

At least in the case when $\pi_1(Y) = \{1\}$, it is natural to view the condition that $w_2(Y) = 0$ as the statement that the loop space $\mathcal{L}(Y)$ is orientable. (Of course, in analogy with finite dimensions, we do not expect to implement the Hodge star operator unless the manifold in question is orientable. Or put another way, we are attemping to use this theory of maps of $\mathbf{R}^{1,2}$ into $\mathcal{L}Y$ to compute equivariant signatures of the tautological circle action on $\mathcal{L}Y$. In order for these signatures to be defined, we must assume that $\mathcal{L}Y$ is orientable.) Let us explain how $w_2(Y) = 0$ should be viewed as equivalent to the orientability of $\mathcal{L}Y$. The tangent space of $\mathcal{L}(Y)$ at a map $x\colon S^1 \to Y$ is the space of sections of $x^*(TY)$. This vector bundle has a natural skew pairing on it given by

$$\omega_x(\alpha, \beta) = \langle \alpha, \beta \rangle = \int_0^1 (\alpha, \nabla_\theta \beta) d\theta$$

where the inner product on the right-hand-side is induced from the metric on Y. It is clear that the radical of this pairing is the space of covariantly constant sections of $x^*(TY)$. In particular, the radical of this pairing is always finite-dimensional. As before with the coupled Dirac operator, it is easy to see that the spectrum of the skew-adjoint operator associated to this pairing is discrete with all eigenvalues of finite multiplicity. Furthermore, the eigenvalues are purely imaginary and symmetric with repect to the origin. Given a closed path x_t in $\mathcal{L}(Y)$, then a local orientation of $\mathcal{L}(Y)$ at the initial point x_0 will be preserved around the loop if and only if the spectral flow of the family T_{x_t} from negative to positive is even. (By skew symmetry the full spectral flow is algebraically zero: here we are looking at half of it – the half from negative to positive.) This spectral flow modulo two is easily identified with $w_2(Y)$ evaluated on map of the torus $S^1 \times S^1$ into Y given by $(t, s) \mapsto x_t(s)$. Thus, provided that Y is simply connected, we have that $w_2(Y) = 0$ if and only if $\mathcal{L}Y$ is orientable if and only if the kernel of $\mathcal{D}_+$ is even dimensional for every map of a superriemann surface into Y.

If $w_2(Y) \neq 0$ this doesn't mean that we can not perform path integrals for maps X with $X^* w_2(Y) \neq 0$. It simply means that the results of such path integrals will be odd under σ, as the measure is. To get something non-zero we need an operator which is a product of local operators $\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)$, $x_i \in \Sigma$, which is $\sigma\tau$ invariant but σ anti-invariant. Then we can calculate the expectation value

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle.$$

The symmetry σ tells us that we get zero contribution at the minima (constant maps) and the non-zero contributions come only from maps $x\colon \Sigma \to Y$ with the property that $x^*(w_2(Y)) \neq 0$. Fix a homotopy class of maps with this property. Of course, since the maps must be homotopically non-trivial, we see that

$$\int_\Sigma |dx|^2$$

is bounded away from zero. Furthermore, the maps minimizing this integral are exactly the harmonic maps in the given homotopy class. That is to say we need to find minimal area surfaces $S \subset Y$ and then take holomorphic maps $\Sigma \to S \subset Y$. We consider $\Sigma = \mathbf{C}P^1$, or better delete a point and consider $\Sigma = \mathbf{R}^2$. Thus we are considering maps $X\colon \mathbf{R}^2 \to Y$ which converge at infinity to a constant map. The expectation value

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle$$

will be of the order $\exp(-tM)$ where M is the energy of the harmonic map. (Notice that we have not normalized this expectation value by dividing by the partition function Z. If we were to do that we must take the partition function which is the sum over all maps $x: \Sigma \to Y$ not just those that pullback w_2 non-trivially. The leading order term in this partition function will come from the constant maps.) The critical points of the Lagrangian will not be isolated: Even with the boundary condition at infinity of $\mathbf{R}^2$ there are four obvious parameters for such maps: rotation, translation and scaling of $\mathbf{R}^2$. There may well be others in addition to these universal ones. Thus, we will be computing an expression of the form

$$\int_{\mathcal{M}} (\text{perturbation expansion})$$

where $\mathcal{M}$ is the instanton moduli space, i.e., the space of harmonic maps in the given homotopy class. This time the propagator is more complicated and the linearization around the instanton moduli space gives only an asymptotic expansion.