The Dirac Index on Manifolds and Loop Spaces: Part II

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I Introduction

In this lecture we shall extend the discussion from the last lecture to the case of signature operators and Dirac operators on loop spaces.

II The Lagrangian formulation: σ-models in two dimensions

We shall consider the space (or rather the super-space) of maps of $\mathbf{R}^{2|2}$ to a finite dimensional, complete riemannian manifold M of even dimension. Let u, v be the coordinates on the underlying space $\mathbf{R}^2$ of $\mathbf{R}^{2|2}$ and let θ_+, θ_- be the odd coordinates on this super manifold. The metric on the underlying $\mathbf{R}^2$ is the flat Minkowski metric $ds^2 = du\ dv$. On $\mathbf{R}^{2|2}$ we have the vector fields D_+ and D_- given by

$$D_+ = \frac{\partial}{\partial\theta_+} - \theta_+\frac{\partial}{\partial u},$$

and

$$D_- = \frac{\partial}{\partial\theta_-} - \theta_-\frac{\partial}{\partial v}.$$

The Lagrangian that we consider is

$$\mathcal{L} = \frac{1}{2}\int_{\mathbf{R}^{2|2}} d^2\theta du dv\ g_{IJ} D_+X^I D_-X^J,$$

or written invariantly using the metric $\langle\ ,\ \rangle_M$ on TM

$$\mathcal{L} = \frac{1}{2}\int_{\mathbf{R}^{2|2}} d^2\theta du dv \langle D_+X, D_-X\rangle_M = \frac{1}{2}\int_{\mathbf{R}^2} du dv\, (D_-D_+\langle D_+X, D_-X\rangle_M)\,|_{\theta=0}.$$

Let us write the superfield out in coordinates as

$$X = x + \theta_+\psi_- + \theta_-\psi_+ + \theta_+\theta_-F$$

where $x = X|_{\theta=0}$, $\psi_\pm = D_\pm(X)|_{\theta=0}$, and $F = D_-D_+(X)|_{\theta=0}$. Integrating out the odd variables yields

$$\mathcal{L} = \int_{\mathbf{R}^2} du dv \frac{1}{2}\left(\langle\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}\rangle + \langle\psi_+, D_u\psi_+\rangle + \langle\psi_-, D_v\psi_-\rangle + \frac{1}{4}R_{IJKL}\psi_-^I\psi_-^J\psi_+^K\psi_+^L + \langle F, F\rangle\right).$$

Of course, when we consider the action since F appears purely quadratically and with no derivatives we can simply integrate out these degrees of freedom introducing the condition that $F = 0$ as one of the equations of motion. From now on we drop this term.

The presence of the indefinite form $\langle\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}\rangle$ indicates that neither u nor v can be used as a time coordinate on $\mathbf{R}^2$. We switch coordinates letting (time coordinate) $t = (u+v)/2$ and (space coordinate) σ be $(u-v)/2$ with the metric $ds^2 = dt^2 - d\sigma^2$. We also allow the possibility that the spatial direction σ has been made periodic so that it is a circle instead of a copy of $\mathbf{R}$. Under this change of coordinates we have

$$\frac{d}{du} = \frac{d}{dt} + \frac{d}{d\sigma}$$
$$\frac{d}{dv} = \frac{d}{dt} - \frac{d}{d\sigma}$$

and analogously for covariant derivatives.

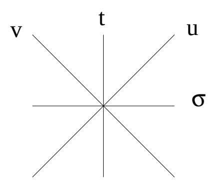

Figure 1.

Rewriting the Lagrangian in these variables yields

$$\mathcal{L} = \frac{1}{2}\int_{\mathbf{R}^2} dt d\sigma \Big(\langle \dot{x}, \dot{x}\rangle + \langle \psi_+, \dot{\psi}_+\rangle + \langle \psi_-, \dot{\psi}_-\rangle \\ -\langle \frac{\partial x}{\partial \sigma}, \frac{\partial x}{\partial \sigma}\rangle + \langle \psi_+, \frac{\partial \psi_+}{\partial \sigma}\rangle - \langle \psi_-, \frac{\partial \psi_-}{\partial \sigma}\rangle + \frac{1}{4} R_{IJKL}\psi_-^I\psi_-^J\psi_+^K\psi_+^L \Big).$$

The first three terms, those involving the time derivatives, are the kinetic energy terms. The other terms are potential terms.

As usual, we have symmetries of the Lagrangian associated with the right invariant vector fields

$$Q_+ = \frac{\partial}{\partial \theta^+} + \theta^+ \frac{\partial}{\partial u}$$
$$Q_- = \frac{\partial}{\partial \theta^-} + \theta^+ \frac{\partial}{\partial v}.$$

Of course we have

$$Q_+^2 = \frac{\partial}{\partial u}$$
$$Q_-^2 = \frac{\partial}{\partial v}.$$

III Quantization

Now that we have a time direction t it makes sense to quantize in the Hamiltonian framework. In order to quantize we restrict to the slice $t = 0$. The fields $\psi_\pm$ should be viewed as spinors on the slice $t = 0$ with values in $x^*(TM)$, or equivalently as half-densities on the slice with values in $S_\pm \otimes x^*TM$. In the case that the spatial direction is a circle, we can view $\psi_\pm$ as half-density on S^1 with values in $T(\mathcal{L}M)$. Since we are taking periodic boundary values for the $\psi_\pm$ the spin bundles $S_\pm$ are trivial over the circle (i.e., give the non-bounding spin structure on S^1.) We can write

$$\int_{t=0} (\psi_+, \psi_+)$$

as a well-defined quadratic form on the space of sections on the slice. Quantizing this infinite dimensional quadratic space yields a spin space. Since we are in infinite dimensions now, the spin space depends not only on the underlying quadratic space but also on a polarization (i.e., a self-annihilating subspace in the complexification). The subspace we choose is given by the operator $D/D\sigma$ acting on the spinors. Since space is now a circle, this operator has discrete spectrum. For the spinors ψ_+ we use the polarization coming from the negative eigenspaces plus a polarization of the zero eigenspace. The choice of this latter polarization is irrelevant since it is finite dimensional. For the spinors ψ_- we choose the polarization consisting of the positive eigenspaces of $D/D\sigma$ plus a polarization for the zero eigenspace. The reason for these choices will be apparent presently. We call the resulting spin spaces for the two types of spinors S_R of right-moving spinors (the ψ_+) and S_L of left-moving spinors (the ψ_-). The Hilbert space after quantizing is the tensor product $S_R \otimes S_L$. This gives us a model for the differential forms on the loop space $\Omega^*(\mathcal{L}M)$. They are a type of middle dimensional forms in the sense that they are of neither finite dimension nor codimension.

The Hamiltonian operator is a perturbation of the operator given by the purely quadratic part of the kinetic plus potential energy. This operator derived from the quadratic energy terms has pieces

$$H_{\psi_+} = -\int_{S^1} (\psi_+, \frac{D}{D\sigma}\psi_+)$$

and

$$H_{\psi_-} = \int_{S^1} (\psi_-, \frac{D}{D\sigma}\psi_-).$$

Since we need these two operators to be bounded below, the choices we have made are forced upon us (up to finite dimensional change which does not affect the result spin space).

The supersymmetries $Q_\pm$ discussed above become operators after we quantize satisfying the same relations they did before quantization. In addition, the operators $Q_\pm$ commute with the Hamiltonian H, which is associated with the vector field

$$\frac{\partial}{\partial t} = \frac{1}{2}\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right).$$

Associated to the vector fields $Q_\pm$ are conserved quantities also denoted $Q_\pm$ given by

$$Q_+ = \int_{S^1} (\psi_+, D_u x)$$

$$Q_- = \int_{S^1} (\psi_-, D_v x).$$

Writing things out in local coordinates on M, we have

$$Q_+ = \int_{S^1} d\sigma g_{IJ} \psi_+^I(\sigma) \left(\frac{\partial x^J}{\partial t} + \frac{\partial x^J}{\partial \sigma}\right).$$

Under quantization $\partial x^I/\partial t$ becomes $-ig^{IJ}\delta/\delta x^J(t)$ so that

$$Q_+ = \int_{S^1} d\sigma \left(-i\psi_+^I \frac{\delta}{\delta x^I(t)} + (\psi, \frac{dx}{d\sigma})\right) .$$

The first term in this last formula for Q_+ is formally the Dirac operator for the free loop space $\mathcal{L}M$. The second term is needed to make the operator well-behaved quantum mechanically. Recall that in the last lecture (in the finite dimensional S^1-equivariant case) we had the operator $D + t\Gamma \cdot V$ where D was the Dirac operator and V was the Killing vector field of a circle action (t is a real number and Γ represents Clifford multiplication). In the case of the free loop space, there is a natural circle action which rotates the parametrization on the domain circle. Its Killing vector field V is $\frac{\partial}{\partial\sigma}$ so that the analogue of $t\Gamma \cdot V$ in this case is the second term in the last expression for Q_+. Notice that there is no parameter here playing the role of t in the finite dimensional discussion. If we tried to vary our operators replacing D/Du by $D/Dt + \epsilon D/D\sigma$, this would work fine as long as $\epsilon > 0$ (of course once $\epsilon \neq 1$ we would violate relativity, but that is not relevant here), but when $\epsilon < 0$ our Hamiltonian operators would no longer be bounded below. To achieve this we would have to pick the opposite polarizations for the spinors, so our operators and index theory would jump discontinuously in passing through $\epsilon = 0$. This is the basic reason that the vanishing results we will get for index theory on free loop space are weaker than the finite dimensional vanishing theorem. Half of the S^1-indices vanish in the loop space case rather than all of them as in the finite dimensional case. In the language of our discussion from last week, we are forced to keep $t > 0$ and hence can only show that half the indices vanish.

Of course, for a spin manifold M of finite, even dimension we have that the differential forms can be decomposed in terms of spinors (which we denote $S(M)$) by by $\Omega^*(M) = S(M) \otimes S(M)$. In general, in Riemannian geometry, there is a natural Dirac operator acting on sections of $S(M) \otimes E$ where E is any vector bundle with connection. By setting $E = S(M)$ and identifying $S(M) \otimes V$ with $S(M) \otimes S(M)$ in either of two ways (which differ by exchange of the factors), we have two Dirac operators acting on $\Omega^*(M)$. These coupled Dirac operators commute and are given by $D = d + d^*$, using the Dirac operator on the first factor, and $\tilde{D} = d - d^*$, using the Dirac operator on the second factor. If M has a circle action generated by a Killing vector field V, then we can form the operators

$$Q_+ = D + \Gamma \cdot V$$

$$Q_- = \tilde{D} - \tilde{\Gamma} \cdot V,$$

where Γ and $\tilde{\Gamma}$ represent Clifford multiplication in the first and second factor respectively. These operators commute and furnish finite-dimensional prototypes for the supersymmetry generators $Q_\pm$ in the loop space case.

IV The Index of Q_+

Our goal is to calculate the index(Q_+) on the free loop space. Of course, this index must be interpreted properly since the free loop space is of infinite dimension. One thing we need to do in this infinite dimensional context is define the 'degree' of our 'forms' on the loop space, at least modulo two. How to do this was discussed at the end of the last lecture. Recall from that discussion that in our situation there is a group $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ with generators acting via

$$\psi_+ \mapsto -\psi_+; \quad \psi_- \mapsto \psi_-$$

and

$$\psi_+ \mapsto \psi_+; \quad \psi_- \mapsto -\psi_-.$$

As we saw, if M is spin then the full group of symmetries exists in the quantum theory; in any event, the diagonal copy of $\mathbf{Z}/2\mathbf{Z}$ always exists as a symmetry of the quantum theory. It turns out that using the diagonal $\mathbf{Z}/2\mathbf{Z}$ produces an index which plays the role of the Euler characteristic in finite dimensions. (After all in our finite dimensional analogy, the diagonal $\mathbf{Z}/2\mathbf{Z}$ acts with $(S^+ \otimes S^+) \oplus (S^- \otimes S^-)$ as the fixed space and $(S^+ \otimes S^-) \oplus (S^- \otimes S^+)$ as the minus one eigenspace. Under the isomorphism of $S \otimes S$ with $\Omega^*(M)$, this $\mathbf{Z}/2\mathbf{Z}$-grading corresponds to the degree modulo two. So it is not surprising that in the context of the loop space the index graded by the diagonal involution gives an Euler characteristic-type object.) But if we assume that M is spin, then we can take one of these involutions, say the involution ρ sending ψ_+ to $-\psi_+$. In the finite dimensional situation this involution is the Hodge $*$-operator, so it is reasonable to view the index computed using this involution as a generalization the usual signature of a finite dimensional manifold. This is the index that we shall study. But just using this involution is not enough. In order to get finite answers, we need to use the circle symmetry on the free loop space in addition to the involution. The way we set things up is to consider the index as a formal character of the circle. That is to say we consider

$$\text{index}\, Q_+ = \mathrm{Tr}_{\mathrm{Ker}^+(Q_+)} q^P - \mathrm{Tr}_{\mathrm{Ker}^-(Q_+)} q^P$$

where q is a formal variable, the operator P is given by

$$P = -\frac{\partial}{\partial\sigma},$$

and $\mathrm{Ker}^{\pm}(Q_+)$ refer to the decomposition of $\mathrm{Ker}(Q_+)$ into ± 1-eigenspaces under the involution ρ.

We will compute this index by quantizing the space of maps of a $2|2$ supermanifold whose underlying geometric manifold is $(S^1 \times \mathbf{R})$ into a compact even dimensional spin manifold M.

V The computation around the fixed points of the S^1-action.

As usual the computation localizes around the fixed points of the supersymmetry Q_+. Let us see what these are. The operator

$$Q_+^2 = H + P$$

is the Laplacian on forms plus $\mathcal{L}_V + |V|^2$, where V is the vector field generating the natural circle action on the free loop space. Thus, as in the last lecture we compute the index by localizing around the fixed points of V. The zeros of the vector field V acting on the loop space are easy to determine: they are simply the constant loops. Hence, they make a copy of the manifold M, embedded as a subspace of $\mathcal{L}M$. Thus, by an infinite dimensional analogue of the Atiyah-Bott fixed point theorem we expect the computation to yield an answer of the form

$$F(q) = \int_M \hat{A}(M)\mathrm{ch}(\mathcal{R})$$

where $\mathcal{R}$ is a bundle (or more precisely, a formal sum of bundles) that will be found presently by a computation on the normal bundle to M in the free loop space.

We made a similar computation in the last lecture for circle actions on finite dimensional manifolds. The method was to find the kernels of the Hamiltonian operators in the normal direction to the fixed point set. That is what we do here as well. Let us begin the discussion by parameterizing this normal bundle. We consider a constant loop at $x_0 \in M$. Then any nearby x can be written as

$$X(\sigma) = x_0 + \exp(\varphi(\sigma))$$

where $\varphi(\sigma) \in TM_{x_0}$ satisfies $\int_{S^1} d\sigma \varphi(\sigma) = 0$ and $\varphi(\sigma)$ satisfies the free wave equation:

$$\left(\frac{d^2}{dt^2} - \frac{d^2}{d\sigma^2}\right)\varphi = 0.$$

This means that we can decompose $\varphi(\sigma)$ uniquely as

$$\varphi(\sigma) = \varphi_+(\sigma) + \varphi_-(\sigma),$$

with φ_+ be left-moving and φ_- being right-moving. This decomposition is unique since we have required that the zero Fourier mode for φ vanishes.

Since we are considering 'differential forms' on the loop space as being the tensor product of two spin bundles $S_R \otimes S_L$, we also have to consider the fermions

$$\psi_-(v), \psi_+(u).$$

If we want to consider the Dirac operator Q_+ that acts on spinors S_R, then the roles of S_R and S_L are not symmetrical. S_R is the bundle of "spinors on free loop space," while S_L is an auxiliary bundle on free loop space that just happens to be a spin bundle constructed with a different polarization. (In fact, at the end of the lecture we will replace S_L by a more general vector bundle on free loop space.) Let us then focus on the role of S_R, which is obtained by quantizing ψ_+. By quantizing the zero modes of ψ_+, we get the spinors on M (regarded as the fixed point subset of $\mathcal{L}M$), while quantization of the nonzero modes of ψ_+ gives the spinors on the normal bundle N to M. The tensor product is $S_R = S(M) \otimes S(N)$, where the two factors are $S(M)$, the spinors on M, obtained by quantizing the constant modes of ψ_+, and $S(N)$, the spinors of the normal bundle, obtained by quantizing the part of ψ_+ that is orthogonal to the zero modes. Thus $\psi_+ = \psi_{+,0} + \psi_{+,\perp}$, where $\psi_{+,0}$ is constant and

$$\int_{S^1} d\sigma \psi_{+,\perp}(u) = 0.$$

$S(N)$ is obtained by quantizing $\psi_{+,\perp}$.

First suppose that M is the flat manifold $\mathbf{R}^{2n}$. Then the theory would have no interaction terms and the action would be purely quadratic. The resulting Hilbert space would be the tensor product of spinor-valued L^2 functions on $\mathbf{R}^{2n}$ (obtained by quantizing x_0, $\dot{x}_0$, and $\psi_{+,0}$) with a tensor product Hilbert space

$$\chi = \chi^+ \otimes \chi^-,$$

where χ^+ and χ^- are Hilbert spaces of right-movers and left-movers, respectively. Concretely χ^+ comes from quantizing φ_+ and $\psi_{+,0}$, and χ^- comes from quantizing φ_-, ψ_-. (Because the Dirac operator whose index we will compute is Q_+ – a choice that breaks the symmetry between left and right – it is very helpful to separate out the zero modes from ψ_+, as we have done, but there is no need to similarly separate out the zero modes from ψ_-.) χ^+ is a Fock space since, for $M = \mathbf{R}^{2n}$, the theory is free. χ^- is almost a Fock space; it is actually the tensor product of a copy of $S(M)$ (obtained by quantizing the zero modes in ψ_-) with a Fock space obtained by quantizing the nonzero modes of ψ_- and φ_-. The Dirac operator Q_+ of the free model is the sum of an operator that acts in χ^+ and an operator (namely $(\psi_{+,0}\dot{x}_0)$, which is interpreted as the ordinary Dirac operator on $\mathbf{R}^{2n}$) that acts on the spinors on $\mathbf{R}^{2n}$.

Now suppose that M is not flat, but scale up the metric of M by a large factor so that the curvature is everywhere very small. Working near the constant loops in $\mathcal{L}M$ and using the fact that the flat model is everywhere a good approximation, we make a fiberwise version of the description in the last paragraph. The Hilbert space of the theory consists of spinor-valued L^2 sections of a certain bundle of Fock-like spaces over M. The bundle over M is a tensor product $\chi^+ \otimes \chi^-$ where χ^+ and χ^- are constructed exactly as in the flat model. (We call the factors Fock-like only because χ^- is not quite a Fock space.) One might think in terms of a projection from a small neighborhood of $M \subset \mathcal{L}M$ back to M. The idea is to compute the index of the Dirac operator by using this fibration; as usual the first step is to solve the Dirac equation along the fibers, which are copies of the normal bundle to M in $\mathcal{L}M$. We write $Q_{+,\perp}$ for the part of Q_+ that acts in χ.

The Dirac operator in the fibers is easily described. The operator $Q_{+,\perp}$ acts in χ^+ as the tensor algebra of its action on the space generated by $\varphi_+(n), \psi_+(n)$ for $n > 0$. (As usual we make Fourier expansions $\varphi_+ = \sum_n e^{in\sigma}\varphi_+(n)$, and likewise for other fields.) In each such space,

$Q_{+,\perp}$ has a one-dimensional kernel, so the kernel of $Q_{+,\perp}$ in the full Fock space χ^+ is the infinite tensor product L of the one-dimensional invariant subspaces. Hence its kernel in χ is identified with χ^- tensored with L:

$$\mathrm{Ker}(Q_{+,\perp}) = L \otimes \chi^- .$$

The involution that we are using acts nontrivially on χ^+ but trivially on L and trivially on χ^-. It then follows that the involution acts trivially on the kernel of $Q_{+,\perp}$. To understand better that kernel, let us examine χ^- in more detail. We have fields

$$\varphi_-(\sigma) = \sum_{n\neq 0} \varphi_n e^{in\sigma}$$

$$\psi_-(\sigma) = \sum_n \psi_n e^{in\sigma}.$$

These produce creation and annihilation operators in the Fock space χ^- which satisfy:

$$[\varphi_n^i, \varphi_m^j] = \frac{1}{m}\delta_{n+m}\delta^{ij}$$

$$\{\psi_n^i, \psi_m^j\} = \delta^{ij}\delta_{n+m},$$

where δ^{ij} is the metric in the tangent space to M at x_0. We quantize so that H is bounded below. In particular, the bosons can be quantized with a vacuum state $|\Omega\rangle$, such that

$$\varphi_n|\Omega\rangle = 0, \quad n > 0,$$

and hence the bosonic states in the Fock space are

$$\prod_{\ell=1}^{\infty} \varphi_{-\ell}^{\epsilon_\ell}|\Omega\rangle$$

of energy $\sum_\ell \ell\epsilon_\ell$.

For the fermions ψ_- the zero mode is allowed so that the quantization of ψ_0 gives a copy of the spin bundle S of the manifold M. In particular, there is not a unique vacuum since any state of S is of zero energy. We use $|s\rangle$ for such zero energy states. As before, we want $\psi_n|s\rangle = 0,\ n > 0$ and for any zero energy state $|s\rangle$. Thus, the states in the Fock space are

$$\prod_{n=1}^{\infty} \psi_{-n}^{\delta_n}|s\rangle$$

of energy $\sum_{n=1}^{\infty} n\delta_n$. Now we are in position to describe $\mathrm{Ker}(Q_+) = \chi^-$ as an S^1-module. Clearly, from the above description we see that it is given by

$$\chi^- = S \otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TM) \otimes_{n=0}^{\infty} \Lambda_{q^n}(TM).$$

Here we use the abbreviations

$$\mathrm{Sym}_{q^n}(TM) = \oplus_{m=0}^{\infty} q^{nm} \mathrm{Sym}^m(TM),$$

and similarly

$$\Lambda_{q^n}(TM) = \oplus_{m=0}^{\infty} q^{nm} \Lambda^m(TM).$$

Here Sym^m and Λ^m denote as usual the m^{th} symmetric or exterior powers. Finally, a factor q^n in front of a vector bundle is to indicate that the action of the circle on this bundle is by the n^{th} power of the standard representation.

After solving the Dirac equation along the fibers to get the kernel $L \otimes \chi^-$, the next step is to solve the Dirac equation along M with values in this kernel. Thus, formally at least, we have

$$F(q) = q^{-\dim M/24} \int_M \hat{A}(M) \mathrm{ch} \left(S \otimes_{n-1}^{\infty} \mathrm{Sym}_{q^n}(TM) \otimes_{m=1}^{\infty} \Lambda_{q^m}(TM) \right),$$

where $F(q)$ is Signature$(\mathcal{L}M)(q)$ as defined by the index of Q_+. Several comments are in order about this formula. First of all the power of q out front comes from the action of the circle on the fixed complex line L; we will explain later why this is the correct factor. Secondly, notice that only positive powers of q appear under the integral sign. But since there are positive powers which do appear in this formula, the signature is not constant as a representation of S^1, unlike the finite dimensional case. What replaces the constancy for the character is the fact that in this context the character is a holomorphic function of q, with positive powers only (multiplying a fixed prefactor). We shall see shortly that setting $q = e^{2\pi i\tau}$, this character extends to a holomorphic function of τ in the entire upper half-plane, a function that is modular for the subgroup $\Gamma_0(2)$ of $SL_2(\mathbf{Z})$.

VI Path Integral Approach

In the path integral approach to computing the index under discussion we consider, for real θ, the expression

$$\mathrm{Tr}\left(e^{-\beta H} e^{i\theta P} (-1)^{F_\rho}\right)$$

where ρ is the involution sending ψ_+ to $-\psi_+$ and ψ_- to ψ_-. Since $Q_+^2 = H+P = 0$ on $\mathrm{Ker}(Q_+)$, on this kernel $H = -P$. This trace is equal to

$$\mathrm{Tr}_{\mathrm{Ker}^+(Q_+)} e^{-(\beta+i\theta)H} - \mathrm{Tr}_{\mathrm{Ker}^-(Q_+)} e^{-(\beta+i\theta)H}.$$

This of course is simply the supertrace

$$F(q) = \mathrm{str}_{\mathrm{Ker}(Q_+)} q^H$$

for $q = e^{-(\beta+i\theta)}$. Notice that $|q| < 1$ since $\beta > 0$. The path integral has a propagator $e^{-\beta H}$ (so that time is Euclidean) for fields on a circle. The term $e^{i\theta P}$ says that before we glue the ends of the cylinder together we rotate by an angle θ. This yields a riemann surface diffeomorphic to T^2. As we have seen from the Hamiltonian discussion previously, there is a power series expression for this supertrace which is a holomorphic function of $q = \exp(2\pi i\tau)$ for τ in the upper half-plane. One can also prove directly using path integrals that this path integral is a holomorphic function of τ in the upper half-plane. (The key step in the proof is to write certain components of the stress tensor in the form $\{Q_+, \ldots\}$; one then uses this to prove that the $\overline{\tau}$ derivative of the index vanishes.)

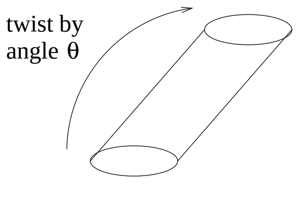

Figure 2.

In this path integral description, the fields $\psi_\pm$ are sections over T^2 of the spin bundles $S_\pm$. Since there are four spin structures on T^2, there are 16 different possibilities for the pair of spin bundles $S^{\pm}$ and hence 16 different possibilities for the path integral

$$\int_{X:T^2\to M} \mathcal{D}X\mathcal{D}\psi_+\mathcal{D}\psi_- e^{-\mathcal{L}}.$$

We need to know which spin structures we should use. These are of course determined by the boundary conditions, or gluing conditions on the fields $\psi_\pm$. In the spatial direction for both ψ_+ and ψ_- we are using the ordinary periodicity and hence the spin structure in that direction is the non-bounding one for both types of spinors. In the time direction for ψ_+ we are computing the supertrace, this means that the spin structure for S^+ in that direction is the non- bounding one. In the time direction for ψ_- we are computing the usual trace and hence the spin structure for S^- in that direction is the bounding one. Letting $+$ denote the non-bounding spin structure on S^1 and $-$ the bounding one we see that the spin structures are given as pictured below:

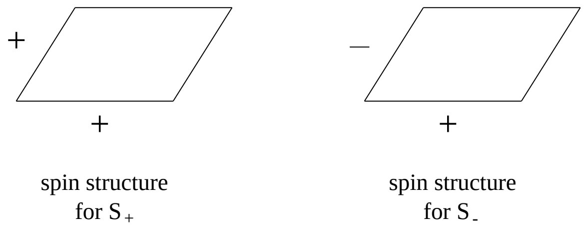

Figure 3.

Let us consider the action of $SL_2(\mathbf{Z})$ on these spin structures. First of all, the action of an element in $SL_2(\mathbf{Z})$ on a spin structure depends only on its reduction modulo two. The spin structure for S_+ is in fact invariant under the full group $SL_2(\mathbf{Z})$ whereas the subgroup leaving second one invariant is $\Gamma_0(2)$, which consists of matrices whose lower left entry is zero modulo two. This is a subgroup of index three in $SL_2(\mathbf{Z})$ reflecting the fact that the orbit of this spin structure under $SL_2(\mathbf{Z})$ has three elements. The fundamental domain for $\Gamma_0(2)$ acting on the upper half-plane is:

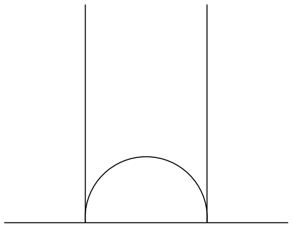

Figure 4. Fundamental Domain for $\Gamma_0(2)$

Notice that the quotient of the upper half-plane by $\Gamma_0(2)$ has three cusps. The one represented at ∞ in the upper half-space corresponds to $\beta \mapsto \infty$. Computing in this limit shows that the path integral reduces to the supertrace on $\mathrm{Ker}(Q_+)$ of q^H, as discussed above. The other cusps correspond to different computations. The cusps are interchanged by the action of $SL_2(\mathbf{Z})$ and hence computing in the limit at one of these means computing the same path integrals but with different spin structures.

As we have already mentioned, the action of $SL_2(\mathbf{Z})$ leaves invariant the spin bundle S_+, so that when we compute at the other cusps ψ_+ is still a section of the same spin bundle. What changes is the spin bundle S_-. Computing at one of the two other cusps means taking ψ_- as a section of

$$x^*TM \otimes \epsilon$$

where ϵ is a real line bundle on T^2 which is non-trivial over the the spatial circle. The two other cusps differ by whether ϵ is trivial in the time direction; we first take the case that it is.

The previous analysis can be repeated to determine the index that corresponds to a path integral with this spin structure. Namely, the Hilbert space of states χ for the theory is a tensor product $\chi = \chi_+ \otimes \chi_-$ of Fock spaces. Nothing has changed with χ_+, but let us examine χ_-. As before, we have, for example, the Fourier decomposition

$$\varphi_-(\sigma) = \sum_{n \neq 0} \varphi_n e^{in\sigma}.$$

The difference is that because of the change in the spin structure S_- around the spatial circle, the Fourier expansion for ψ_- is now

$$\psi_-(\sigma) = \sum_{m \in \mathbf{Z}+\frac{1}{2}} \psi_n e^{im\sigma}.$$

In particular, in the Fock space there is no copy of S, since there is no zero Fourier coefficient for ψ_-. (So things are now a bit simpler: $\chi^{\pm}$ are true Fock spaces, while before this was not quite so for χ^-.) Thus, the expansion at this cusp for the index is

$$F(q) = q^{-\dim M/24} \int_M \hat{A}(M) \mathrm{ch}\left(\otimes_{n=1}^{\infty} \mathrm{Sym}_{q^n}(TM) \otimes_{m=1/2,3/2,\ldots} \Lambda_{q^m}(TM)\right).$$

The third cusp is the image of the second cusp under the map $\tau \mapsto \tau + 1$. Thus, the expansion at the third cusp differs from this one simply by changing the sign of $q^{1/2}$ and thus, in the last formula, changing the sign of q^m for all m. (Recall that m is in $\mathbf{Z} + \frac{1}{2}$.)

Now there is one important point that we have not yet discussed. This is the source of the factor $q^{-\dim M/24}$ in front of these integral expressions. This comes from the S^1 action on

$$L = \mathrm{Ker}(Q_{+,\perp})$$

In fact, L is an infinite tensor product of one dimensional spaces which are obtained by quantizing $\varphi_+(n), \psi_+(n)$ for $n = 1, 2, 3, \ldots$. On the n^{th} such space, the circle acts with character

$$\frac{n}{2} \dim M.$$

Thus, on the tensor product the circle action contributes a multiplicative factor

$$q^{\frac{1}{2}\sum_{n=1}^{\infty} n \dim M}.$$

Of course, this factor is regularized in the renormalization procedure. It becomes

$$q^{\frac{1}{2}\zeta(-1)\dim M} = q^{-\dim M/24}.$$

As an alternative to the Hamiltonian approach that we have followed, the index of Q_+ can be evaluated by directly computing the path integral. One gets equivalent formulas, of course. Scaling up the metric of M reduces the evaluation of the path integral to one-loop computations. One requires computing Pfaffians for D_+ and D_- on spinors in a non-trivial but flat bundle over T^2 and computing the determinant of the Laplacian for the bosons. We will not explore this direction today.

VII Bundles with whose signature or Dirac operator has constant character for circle actions

Let us examine the formulas we get for the index of Q_+ at the various cusps. For one cusp we get a formula of the form

$$F(q) = \int_M \hat{A}(M)\mathrm{ch}\left(S \otimes (\oplus_{n=1}^{\infty} q^n R_n)\right).$$

At the other cusps we get a formula of the form

$$F(q) = \int_M \hat{A}(M)\mathrm{ch}(\oplus_{m \in \mathbf{Z}/2} q^m \tilde{R}_m).$$

Both the bundles R_n and $\tilde{R}_m$ are SO-bundles associated to the tangent bundle of M by certain representations which we can make explicit. We claim that these bundles have a special property. Suppose that M has an $U(1)$-symmetry $G: S^1 \times M \to M$, with generating vector field K. Then

$$F(q,\alpha) = \mathrm{Tr}_{\mathrm{Ker}^+(Q_+)} q^H e^{i\alpha K} - \mathrm{Tr}_{\mathrm{Ker}^-(Q_+)} q^H e^{i\alpha K}$$

is independent of α. Of course, this is really a statement about the signature operator coupled with the finite dimensional vector bundles R_m (or the Dirac operator with $\tilde{R}_m$) on a finite dimensional spin manifold with circle action. The statement is that for the special bundles arising from consideration of the loop space index, the indices of these operators, viewed as characters on the circle, are constant.

To establish this result, let us consider the path integrals where instead of using maps of $T^2 \to M$ we consider sections of a bundle over an M-bundle over T^2 where as we pass from time $t = 0$ to $t = \beta$ we identify M with itself by twisting by $\exp(\alpha K)$. (Of course, we could do the same sort of twisting in the other direction as well. That we will do presently.) We expand the resulting index as

$$F(q,\alpha) = q^{-\dim M/24} \sum_{k,m} q^k e^{im\alpha} c_{k,m}$$

for constants $c_{k,m} \in \mathbf{Z}$. We have already seen that

$$c_{k,m} = 0 \quad \text{for } k < 0.$$

Next we claim that:

$$c_{k,m} = c_{k+m,m}.$$

If we can establish this claim, then together with the first fact, it immediately follows that $c_{k,m} = 0$ for $m \neq 0$, which is exactly the constancy of $F(q,\alpha)$ with respect to α.

It remains to establish the above claim. To do this, let us introduce another $\gamma \in S^1$, and let us consider the Dirac operator Q_+ on a twisted loop space $\mathcal{L}_\gamma M$, the M bundle over the spatial circle obtained by gluing the ends together using the automorphism $\exp(\gamma K)$. The σ-model makes sense as a function of γ, and $F(q,\alpha)$, being an index, is independent of the continuous parameter γ. On $\mathcal{L}_\gamma M$, we have

$$e^{2\pi \frac{d}{d\theta}} = e^{\gamma K}.$$

Let us define subspaces $\mathcal{H}_{\gamma,n,m}$ in the Hilbert space of states by the conditions

$$\mathcal{H}_{\gamma,n,m} = \left\{ \psi | K\psi = \psi \text{ and } P\psi = \left(n - \frac{\gamma m}{2\pi}\right)\psi \right\}.$$

The dimension of the space $\mathcal{H}_{\mathrm{Id},m,n}$ is the index $c_{n,m}$. Once again, we know that the dimension of these spaces is independent of γ (the Fredholm property of the operators). As we let γ go from 0 to 2π this has the effect of shifting the $c_{n,m}$ to $c_{n+m,m}$. This, then, establishes the claim and hence the constancy statement.

In this way we have found an infinite sequences of bundles $\tilde{R}_m$ (or R_n) associated to the tangent bundle of a compact even dimensional spin manifold M with the following property. If M has an isometric circle action, then the index of the Dirac (or signature) operator coupled to these bundles is a constant character of S^1. The first few of the $\tilde{R}_m$'s are:

$$\begin{aligned}
\tilde{R}_0 &= \mathcal{O};\ \text{ the trivial bundle} \\
\tilde{R}_{1/2} &= TM \\
\tilde{R}_1 &= TM \oplus \wedge^2 TM.
\end{aligned}$$

By computing relevant bordism groups Landweber-Stong showed that for general spin manifolds the $\tilde{R}_m$ are the only bundles naturally associated to the tangent bundle with this constancy property.

VIII Generalization to Vector Bundles over the Loop Space

There is an analogue of this theory where one considers spinors on the loop space with values in certain vector bundles $\mathcal{V}$ over the loop space. The relevant $\mathcal{V}$'s are constructed from ordinary finite-rank vector bundles $V \to M$. Physically, this analogue is very important because it arises in heterotic string theory. Mathematically, one finds that the index of the Dirac operator with values in certain bundles made from V (rather than from the tangent bundle of M) is a constant character of the circle if M has a circle action obeying certain conditions. Let us sketch briefly the relevant version of the nonlinear sigma model and its mathematical application.

We use "chiral" superfields; we set θ^- and ψ_- to zero and consider maps

$$X\colon \mathbf{R}^{2|1} \to M.$$

The Lagrangian we consider is

$$\int d\theta_+ du dv \left((D_v X, D_{\theta_+} X\right) = \int du dv \left(g_{IJ} \partial_u x^I \partial_v x^J + \psi_+ D_v \psi_+\right).$$

To get something interesting we add another superfield

$$\Lambda \in \Gamma\left(\mathbf{R}^{2|1}, S_- \otimes x^*(V)\right),$$

where $V \to M$ is a real $SO(k)$-bundle with a connection. We express Λ in components as $\Lambda = \lambda + \theta_+ G$. We form a new Lagrangian

$$\int d\theta_+ du dv \left((D_v X, D_{\theta_+} X) + (\Lambda, D_{\theta_+} \Lambda)\right)$$

which after performing the theta integral and integrating out the auxiliary field G becomes schematically

$$\int du dv \left(g_{IJ} \partial_u X^I \partial_v x^J + \psi_+ D_v \psi_+ + \lambda D_u \lambda + F \psi_+ \psi_+ \lambda\lambda\right).$$

Here F is the curvature of the connection on V.

In this theory we have the operator Q_+ as before, but no longer do we have Q_-. When we go to evaluate path integrals we find that for a fixed choice of the Riemann surface Σ and the map $X : \Sigma \to M$, the fermion path integral takes values in a product of determinant line bundles $\det D_+(\lambda) \cdot \det D_-(\lambda)$. In order for the path integral to be a real number, and not just a section of a line bundle, we need to trivialize this product of determinant line bundles. The first question then is a topological one: when is this product of determinant line bundles topologically trivial? The answer depends a bit on what choices of spin structures one wishes to use. Assuming that we want to be able to make independent choices of spin structure for both ψ_+ and λ (this is what one needs to do in the heterotic string theory, and it leads to the most interesting mathematical results), one needs the theory to be invariant under the full $\mathbf{Z}_2 \times \mathbf{Z}_2$ of independent sign changes for ψ_+ and λ. For this, we need M to be a spin manifold and V a spin bundle. The triviality of the product of determinant line bundles for any Riemann surface and any spin structures gives further

$$\frac{p_1(TM)}{2} = \frac{p_1(V)}{2}. \qquad (*)$$

(Here p_1 is the first Pontryagin class, which is naturally divisible by two for spin manifolds and spin bundles.)

The next question is to find a flat connection on this product of determinant line bundles. Mathematically, there is a standard definition of a connection on the determinant line bundle. Though it is not flat, it can be modified to be flat if one is given a 3-form H on M such that

$$dH = \frac{\mathrm{tr} R \wedge R - \mathrm{tr} F \wedge F}{8\pi^2}.$$

Here, R is the curvature of the riemannian manifold M. The point of the formula is that dH is the four-form that appears in the standard mathematical formula for the curvature of the usual connection. The equality of $p_1(TM)$ and $p_1(V)$ ensures that such H's exist. If one is given such an H, then one modifies the usual connection on the determinant line bundle as follows. The Dirac operators whose product of determinants we wish to trivialize are defined when one is given a choice of a Riemann surface Σ and a map $X : \Sigma \to M$. A one-parameter family of such data give a three-manifold $W = \Sigma \times [0,1]$ and a map $X_W : W \to M$. We multiply the usual parallel transport from 0 to 1 in the determinant line by the phase factor

$$\exp(i \int_W X_W^*(H)).$$

The usual local formula for the curvature of the determinant line bundle ensures that the modified connection is flat; its global holonomy is also trivial if the formula $(*)$ holds integrally and not just at the level of differential forms.

This construction is related to some very important physics. Physically, H is the curvature of the two-form field that is usually called B, and the inclusion of the H-dependent phase factor is part of the Green-Schwarz anomaly cancellation mechanism.

To give a mathematical application, we suppose that M and V have S^1-actions and we ask, as before, if the index is a constant character for the S^1. Here a new question arises: the action of S^1 induces a symmetry of the Lagrangian, but is it a symmetry of the quantum field theory? For this, we want not just that the product of determinant lines considered before should be trivial, but that if $X : \Sigma \to M$ is an S^1-invariant map, the S^1 action on the fiber of the product of determinant lines over X should be trivial. Via some further index theory, one can show that the condition for this is that $(*)$ must hold in the S^1-equivariant cohomology, not just in ordinary integral cohomology:

$$\left(\frac{(p_1)_{S^1}(V)}{2} - \frac{(p_1)_{S^1}(TM)}{2}\right)^{S^1} = 0. \quad (**)$$

(Here, $(p_1)_{S^1}$ refers to the Pontryjagin class in S^1-equivariant cohomology.) Once we have this, the arguments as in the previous case apply to show that the Dirac index on M with values in $\tilde{R}_m(V)$ or in $S(V) \otimes R_n(V)$ is constant as a character of the circle.

One can actually make a somewhat similar statement if the left hand side of $(**)$ is not zero but is the pullback of a class in $H^4(BG)$. For if this is so, then by adding or in the sense of K theory subtracting from V a trivial bundle with nontrivial S^1 action, one can reduce to the case that $(**)$ is satisfied.