SOLUTIONS TO EXERCISES (BY SIYE WU)

General Remarks

The classical moduli space

We recall that the quark superfield Q in an $N=1$ supersymmetric gauge theory is valued in a representation R of the gauge group G. The bosonic potential is $V = |D|^2$, where $D\colon R \to \mathfrak{g}^* \cong \mathfrak{g}$ is the moment map of the Hamiltonian G-action on R. Therefore the space $\mathcal{M}_{\mathrm{c}}$ of classical ground states (modulo gauge transformations), called the moduli space of classical vacua, is the symplectic quotient $D^{-1}(0)/G$.

The space $\mathcal{M}_{\mathrm{c}} = D^{-1}(0)/G$ is also the categorical quotient of R by $G^{\mathbb{C}}$.[1] That is to say, $\mathcal{M}_{\mathrm{c}}$ is a complex space whose coordinate ring is isomorphic to $\mathbb{C}[R]^G$. In fact, for any $Q \in R$, the orbit $G^{\mathbb{C}} \cdot Q$ is closed if and only if $G^{\mathbb{C}} \cdot Q \cap D^{-1}(0) \neq \emptyset$. If $Q \in D^{-1}(0)$, then $G^{\mathbb{C}} \cdot Q \cap D^{-1}(0) = G \cdot Q$. Let the semi-stable part of R be $R^{\mathrm{ss}} = \{Q \in R \,|\, G^{\mathbb{C}} \cdot Q \cap D^{-1}(0) \neq \emptyset\}$. Let $\pi\colon R^{\mathrm{ss}} \to R^{\mathrm{ss}}/G^{\mathbb{C}}$ be the quotient map. We have
1) the the inclusion $D^{-1}(0) \to R^{\mathrm{ss}}$ induces a homeomorphism $D^{-1}(0)/G \to R^{\mathrm{ss}}/G^{\mathbb{C}}$;
2) for any point $O \in R^{\mathrm{ss}}/G^{\mathbb{C}}$, $\pi^{-1}(O)$ contains exactly one closed $G^{\mathbb{C}}$-orbit in R;
3) for any open set $U \subset R^{\mathrm{ss}}/G^{\mathbb{C}}$, we have an isomorphism $\mathcal{O}(U) \cong \mathcal{O}(\pi^{-1}(U))^G$.
This justifies the identification of the structure sheaf $\mathcal{O}_{D^{-1}(0)/G}$ with $(\mathcal{O}_R)^G$. (However the invariants in $\mathbb{C}[R]^G$ do not distinguish the $G^{\mathbb{C}}$-orbits in R.)

In what follows we will consider the case $R = \mathrm{Hom}(F, C)$. Here C, called the space of colors, is an irreducible unitary representation of G. F, the space of flavors, is an Hermitian vector space on which G acts trivially. At the classical level, the global symmetry group is $U(F) \times U(1)_X$, where $U(1)_X$ is the rotation on the fermionic coordinates in the $N=1$ superspace. We write $U(F) = SU(F) \times_{Z(SU(F))} U(1)_A$, where $Z(SU(F))$ is the center of $SU(F)$ and $U(1)_A$ is the scalar multiplication of $U(1)$ on F.

Axial anomaly and the global symmetry

The classical action depends on the gauge group G, the matter representation R, the coupling constant g and the θ-term. In the quantum theory, the coupling constant

[1] G. Kempf, L. Ness, in: Lecture Notes of Math. 732, (1979) pp. 233-244; G. W. Schwarz, in: Progress in Math. 80, (1989) pp. 135-151; P. Heinzner, F. Loose, Geom. Funct. Anal. 4 (1994) 288-297

g depends on the renormalization scale μ: $g(\mu)$ runs according to $\mu\frac{dg}{d\mu}=\beta(g)$. Here $\beta(g)$ is the β-function; at one-loop it is given by

$$\beta(g)=-\frac{b_0}{(4\pi)^2}g^3, \tag{4.1}$$

where $b_0=\frac{3}{2}c_2(G)-\frac{1}{2}c_2(R)$. The quantum theory thus has a parameter of mass dimension 1: the (complex) scale Λ defined by

$$\Lambda^{b_0}=\mu^{b_0}e^{-\frac{8\pi^2}{g^2(\mu)}}e^{i\theta}=\mu^{b_0}e^{2\pi i\tau(\mu)}, \tag{4.2}$$

where θ is the theta angle and $\tau(\mu)=\frac{\theta}{2\pi}+\frac{4\pi i}{g^2(\mu)}$.

Recall that if $R=\mathrm{Hom}(F,C)$, then the classical theory has a global symmetry $U(F)\times U(1)_X$. Upon quantization, both $U(1)_A$ and $U(1)_X$ shift the theta angle θ, and hence Λ^{b_0}, whereas $SU(F)$ does not. Consequently neither of these $U(1)$ groups is a true symmetry of a fixed quantum theory; this is called the axial anomaly. Rather, the group $U(1)_A\times U(1)_X$ acts on a family of quantum theories. The $U(1)_A$ and $U(1)_X$ charges of Q, its fermionic component ψ_Q, the gluino λ and the scale Λ are summarized as follows:

(4.3)

	$U(1)_A$	$\times$	$U(1)_X$	$U(1)_R$
Q	-1		0	$\frac{c_2(R)-c_2(G)}{c_2(R)}$
ψ_Q	-1		-1	$-\frac{c_2(G)}{c_2(R)}$
λ	0		1	1
Λ^{b_0}	$c_2(R)$		$c_2(R)-c_2(G)$	0

Here and below, representations of $U(1)$ are labeled by their weights.

The anomaly-free subgroup of $U(1)_A\times U(1)_X$ is a product of a $U(1)$ subgroup, denoted by $U(1)_R$, and a finite group containing the center of $SU(F)$. We normalize the generator of the Lie algebra $\mathfrak{u}(1)_R$ so that it projects to that of $\mathfrak{u}(1)_X$. The weights of $U(1)_R$, which are also shown in the above table, are related to those of $U(1)_A$ and $U(1)_X$ by

$$R=X+\frac{c_2(G)-c_2(R)}{c_2(R)}A. \tag{4.4}$$

The true quantum symmetry is $SU(F)\times U(1)_R$ up to a discrete factor.

Global gauge anomalies

Consider a principal G-bundle P over the (Euclidean) space-time. Let $\mathcal{A}$ be the space of connections and $\mathcal{G}$, the group of gauge transformations. Let $\not{D}_R^+$ be the Dirac operator acting on Weyl spinors twisted by a representation R. A gauge theory with Weyl fermions has gauge anomalies if the determinant line bundle $\mathrm{Det}\ \not{D}_R^+$ over $\mathcal{A}/\mathcal{G}$ is not trivial. When the gauge group G is semi-simple, the curvature $\mathcal{F}$ of $\mathrm{Det}\ \not{D}_R^+$ is proportional to a multi-linear form d_R on the Lie algebra $\mathfrak{g}$ defined by

$$d_R(x,y,z)=\mathrm{tr}_R x\{y,z\} \qquad (x,y,z\in\mathfrak{g}). \tag{4.5}$$

Local gauge anomalies come from the non-vanishing of $\mathcal{F}$. For $G=Sp(n)$ or $G=SO(n)$, we have $d_R=0$ for any representation R of G. Consequently the bundle $\mathrm{Det}\ \not{D}_R^+$ is flat and there are no local gauge anomalies.

However if there is a non-contractible loop in $\mathcal{A}/\mathcal{G}$, the holonomy along it may be non-trivial even if $\mathcal{F}=0$. This is called global gauge anomaly. Since the space $\mathcal{A}$ is contractible, $\pi_1(\mathcal{A}/\mathcal{G})\cong\pi_0(\mathcal{G})$. When the (Euclidean) space-time is S^4, we have $\pi_0(\mathcal{G})\cong\pi_4(G)$. When G is a simple Lie group, $\pi_4(G)\neq 1$ only if only if the universal covering group is $Sp(n)$ for some $n\geq 1$, in which case $\pi_4(G)\cong\mathbb{Z}_2$. Thus the holonomy of Det $\not{D}^+_R$ for any R is ± 1, and consequently, the bundle (Det $\not{D}^+_R)^{\otimes 2}$ is always trivial. In particular, consider $R=\mathrm{Hom}(F,C)$, where G acts on F trivially. When $G=Sp(n)$ and $C\cong\mathbb{C}^{2n}$ is its defining representation, the bundle Det $\not{D}^+_R=$ (Det $\not{D}^+_C)^{\otimes\dim F}$ is trivial if $\dim F$ is even. This condition will guarantee the absence of global gauge anomalies. That the bundle Det $\not{D}^+_C$ itself is non-trivial is a deeper fact related to the Bott periodicity.

For $G=SO(n)$, the curvature $\mathcal{F}$ of Det $\not{D}^+_R$ vanishes for any representation R. If $n\geq 6$, since $\pi_4(G)=1$, the bundle Det $\not{D}^+_R$ is trivial for any representation R (including the defining representation). If G is simple and the representation R is non-trivial and of real type, then up to a finite subgroup of the center, G can be embedded into $SO(d)$, where $d=\dim R$. The G-bundle P induces a principal $SO(d)$-bundle P_0. Let $\mathcal{A}_0$ be the space of connections on P_0, and $\mathcal{G}_0$, the group of gauge transformations. Then there exist maps $\mathcal{A}\to\mathcal{A}_0$, $\mathcal{G}\to\mathcal{G}_0$ and $\mathcal{A}/\mathcal{G}\to\mathcal{A}_0/\mathcal{G}_0$. The determinant line bundle Det $\not{D}^+_R$ over $\mathcal{A}/\mathcal{G}$ is the pull-back of corresponding bundle over $\mathcal{A}_0/\mathcal{G}_0$. If $d\geq 6$, the latter is trivial. If $2\leq d\leq 5$, the case can be reduced to $d\geq 6$ since Det $\not{D}^+_{R\oplus R\oplus R}=$ (Det $\not{D}^+_R)^{\otimes 3}\cong$ Det $\not{D}^+_R$. Therefore Det $\not{D}^+_R$ is trivial for any real representation R of any compact Lie group G.

We list a few consequences of this result. First, R and R' are both real or quarternionic representations of G, then the representation $R\otimes R'$ is real. So the bundle Det $\not{D}^+_{R\otimes R'}$ is trivial. Second, if $G=SO(n)$ for any $n\geq 3$, and $C\cong\mathbb{C}^n$, the defining representation, then the bundle Det $\not{D}^+_C$ is trivial. Finally, since the adjoint representation is always of real type, the bundle Det $\not{D}^+_{\mathrm{adj}}$ is trivial. Thus the gluino λ does not contribute to the global gauge anomaly.

SOLUTIONS TO EXERCISES (BY SIYE WU)
Problems 1–2

Solution to Problem 1

Let $C \cong \mathbb{C}^{2N_c}$ be an Hermitian vector space with a compatible symplectic form ω_c. C is the defining representation of the compact group $G = Sp(C) \cong Sp(N_c)$. We write the complexification $G^{\mathbb{C}} = Sp(C, \mathbb{C})$. The representation on C is quarternionic. We have $c_2(C) = 1$. The dual Coxeter number of G is $h = N_c + 1$. Therefore $c_2(G) = 2(N_c + 1)$. Let the flavor space F be of dimension $2N_f$. (It is even to ensure the absence of the global gauge anomaly.) Then $c_2(R) = 2N_f$ and $b_0 = 3(N_c + 1) - N_f$.

Classical theory

The moduli space of classical vacua is $\mathcal{M}_{\mathrm{c}} \cong \mathrm{Hom}(F, C)^{\mathrm{ss}}/Sp(C, \mathbb{C})$. We define gauge invariant quantities

$$M = Q^* \omega_c \in \wedge^2 F^*. \tag{5.1}$$

Clearly, M satisfies the constraint

$$\wedge^{N_c+1} M = 0. \tag{5.2}$$

It is a classical result of Weyl that the ring of invariants $\mathbb{C}[\mathrm{Hom}(F, C)]^G$ is generated by $M \in \wedge^2 F^*$ with the (only) relations (5.2).[1] So the moduli space $\mathcal{M}_{\mathrm{c}}$ is parametrized by M subject to the constraint (5.2).

Recall that C has a quarternionic structure. We have an orthogonal decomposition of the Lie algebra

$$\mathfrak{u}(C) = \mathfrak{u}(C, \mathbb{H}) \oplus i\{\mathbb{H}\text{-self-adjoint operators on } C\}. \tag{5.3}$$

Since $\mathfrak{u}(C, \mathbb{H}) = \mathfrak{sp}(C)$, the moment map is

$$D(Q) = \text{orthogonal projection of } iQQ^\dagger \text{ onto } \mathfrak{u}(C, \mathbb{H}). \tag{5.4}$$

[1] H. Weyl, *Classical groups*, §VI.1

So $D(Q) = 0$ if and only if $QQ^\dagger$ is $\mathbb{H}$-self-adjoint on C. In this case, there is an $\mathbb{H}$-unitary basis of C such that $QQ^\dagger$ has the form

$$QQ^\dagger \sim \begin{pmatrix} a_1^2 & & \\ & \ddots & \\ & & a_r^2 \\ & & & \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.5}$$

where $a_1, \ldots, a_r > 0$, $r = \frac{1}{2}\,\mathrm{rank}\, Q$. Hence there is a unitary basis in F such that Q is of the form

$$Q \sim \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r \\ & & & \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.6}$$

In the complex language, any $Q \in \mathrm{Hom}(F, C)^{\mathrm{ss}}$ can be put into the standard form (5.6) by a complexified gauge transformation and a global symmetry. The stabilizer of Q in $Sp(C, \mathbb{C})$ is isomorphic to $Sp(N_c - r, \mathbb{C})$ and that in $Sp(C)$ is $Sp(N_c - r)$. At a generic point in $\mathcal{M}_c$, the gauge group is broken to $Sp(N_c - N_f)$ if $N_c > N_f$; it is completely broken if $N_c \leq N_f$.

Quantum theory

Let Λ be the (complex) scale of the quantum theory. Since Λ^{b_0} has $U(1)_A$-charge $c_2(R) = 2N_f$, it is appropriate to write

$$\Lambda^{3(N_c+1)-N_f} \in \wedge^{2N_f} F^*. \tag{5.7}$$

The anomaly-free subgroup in $U(1)_A \times U(1)_X$ is $U(1)_R \times \mathbb{Z}_{2N_f}$, where the R-charge can be computed by, following (4.4),

$$R = X + \frac{N_c + 1 - N_f}{N_f} A. \tag{5.8}$$

Since $\mathbb{Z}_{2N_f} = Z(SU(F))$, the global symmetry of the quantum theory is $SU(F) \times U(1)_R$. The superfields Q and M transform as

(5.9)

	$SU(F)$ $\times$	$U(1)_R$
Q	F^*	$1 - \frac{N_c+1}{N_f}$
M	$\wedge^2 F^*$	$2(1 - \frac{N_c+1}{N_f})$

The vacuum degeneracy in the classical moduli space $\mathcal{M}_c$ may be lifted by a dynamically generated superpotential W for the light fields M. W must be invariant under the gauge and global symmetries. The fields in M are already gauge invariant; the only combination invariant under $SU(F)$ is the Pfaffian $\mathrm{Pf}\, M$. Also, the superpotential is holomorphic on $\mathcal{M}_c$ hence in M. These restrictions imply that W is of the form

$$W \sim (\Lambda^{b_0})^x (\mathrm{Pf}\, M)^y. \tag{5.10}$$

The exponents x, y can be fixed as follows:
1) Since $\Lambda^{b_0}, \mathrm{Pf}\, M \in \wedge^{2N_f} F^*$ and W is a number, we have $x + y = 0$.
2) As a superpotential, W has $U(1)_X$ charge $2(N_c + 1 - N_f)x = 2$.
3) W has mass dimension $b_0 x + 2N_f y = 3$.

Notice that 1) and 2) together imply that W has $U(1)_R$-charge 2. The above three equations for two unknowns x and y are consistent and we get $x = -y = \frac{1}{N_c+1-N_f}$. Hence the superpotential is

$$W = A\left(\frac{\Lambda^{3(N_c+1)-N_f}}{\mathrm{Pf}\, M}\right)^{\frac{1}{N_c+1-N_f}}, \tag{5.11}$$

where A is a constant that depends only on N_c, N_f and the renormalization scheme. Notice that if $N_f \geq N_c + 1$, then the superpotential W in (5.11) does not exist because $\mathrm{Pf}\, M = 0$. In this case, the classical vacuum degeneracy is not lifted by quantum effects. When $N_f \leq N_c$, the decay of W at $M \to \infty$ is consistent with asymptotic freedom, which holds when $N_f < 3(N_c + 1)$.

Solution to Problem 2

Let $C \cong \mathbb{C}^{N_c}$ be a complex vector space with a real structure, i.e., C is the complexification of $C_{\mathbb{R}} \cong \mathbb{R}^{N_c}$. Suppose $C_{\mathbb{R}}$ is equipped with a metric g_c and a compatible volume form v_c, which provides a trivialization of $\det C_{\mathbb{R}}^*$. (In the symplectic case, v_c is fixed by the symplectic form ω_c.) C is the defining representation of the compact Lie group $G = SO(C) \cong SO(N_c)$. (The complexification is denoted by $G^{\mathbb{C}} = SO(C, \mathbb{C})$.) Let the flavor space F have dimension N_f. For $N_c \geq 5$, the dual Coxeter number $h = N_c - 2$ and $c_2(C) = 2$. So $c_2(G) = 2(N_c - 2)$, $c_2(R) = 2N_f$ and $b_0 = 3(N_c - 2) - N_f$. For $N_c = 4$, $G = (G_+ \times G_-)/\mathbb{Z}_2$, where $G_\pm \cong SU(2)$. For each factor $G_\pm$, we have $c_2(G_\pm) = 4$, $c_2(C) = 2$, $c_2(R) = 2N_f$. So $b_0 = 6 - N_f$ for the coupling constant of each $G_\pm$. These are formally the continuation of the cases $N_c \geq 5$. For $N_c = 3$, $G \cong SU(2)$ up to a $\mathbb{Z}_2$ factor. We have $c_2(G) = c_2(C) = 4$, $c_2(R) = 4N_f$ and $b_0 = 6 - 2N_f$.

Classical theory

The moduli space of classical vacua is $\mathcal{M}_c \cong \mathrm{Hom}(F, C)^{\mathrm{ss}}/SO(C, \mathbb{C})$. The gauge invariants are the mesons $M = Q^* g_c \in S^2 F^*$ and, if $\mathrm{rank}\, Q = N_c$, the baryons $B = Q^* v_c \in \wedge^{N_c} F^*$. They satisfy the constraints

$$\mathrm{rank}\, M \leq N_c, \qquad B \otimes B = \wedge^{N_c} M. \tag{5.12}$$

Here $\wedge^{N_c} M \in S^2(\wedge^{N_c} F^*)$ corresponds to the product of non-zero eigenvalues of M. The ring of invariants $\mathbb{C}[\mathrm{Hom}(F, C)]^G$ is generated by $M \in S^2 F^*$ and, if $N_f \geq N_c$, $B \in \wedge^{N_c} F^*$ with the relations (5.12).[2] So M and B subject to the constraints in (5.12) parametrize the moduli space $\mathcal{M}_c$.

Consider the orthogonal decomposition

$$\mathfrak{u}(C) = \mathfrak{so}(C) \oplus i\{\mathbb{R}\text{-self-adjoint operators on } C_{\mathbb{R}}\}. \tag{5.13}$$

Note that $\mathfrak{so}(C)$ is the set of skew-self-adjoint operators on $C_{\mathbb{R}}$. The moment map of the G-action on $\mathrm{Hom}(F, C)$ is

$$D(Q) = \text{orthogonal projection of } iQQ^\dagger \text{ onto } \mathfrak{so}(C). \tag{5.14}$$

So $D(Q) = 0$ if and only if $QQ^\dagger$ is $\mathbb{R}$-self-adjoint on $C_{\mathbb{R}}$. In this case, there is an orthogonal basis of C (i.e., up to a gauge transformation) such that $QQ^\dagger$ has the

[2] H. Weyl, *Classical groups*, §II.9

form

$$QQ^\dagger \sim \begin{pmatrix} a_1^2 & & \\ & \ddots & \\ & & a_r^2 \\ & & \end{pmatrix}, \tag{5.15}$$

where $a_1, \ldots, a_r > 0$ if $r = \operatorname{rank} Q < N_c$ and $a_1, \ldots, a_{N_c-1} > 0$, $a_{N_c} \neq 0$ if $r = N_c$. In the latter case, $B^{1,\ldots,N_c} = a_1 \ldots a_{N_c}$ is the only non-zero component of B. Hence up to gauge and global symmetries Q is of the form

$$Q \sim \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_r \\ & & \end{pmatrix}. \tag{5.16}$$

The stabilizer of Q in $SO(C, \mathbb{C})$ is isomorphic to $SO(N_c - r, \mathbb{C})$, and that in $SO(C_{\mathbb{R}})$, $SO(N_c - r)$. At a generic point in $\mathcal{M}_c$, the gauge group is broken to $SO(N_c - N_f)$ if $N_c \geq N_f + 2$; it is totally broken if $N_c < N_f + 2$.

Quantum theory

Let Λ be the (complex) scale of the quantum theory. We have

$$\Lambda^{3(N_c-2)-N_f} \in (\wedge^{N_f} F^*)^{\otimes 2} \tag{5.17}$$

if $N_c \geq 4$. (For $N_c = 4$, this holds for the scale of each $SU(2)$ theory.) If $N_c = 3$, then

$$\Lambda^{6-2N_f} \in (\wedge^{N_f} F^*)^{\otimes 4}. \tag{5.18}$$

The anomaly-free subgroup of $U(1)_A \times U(1)_X$ is $U(1)_R \times \mathbb{Z}_{2N_f}$ if $N_c \geq 4$ and $U(1)_R \times \mathbb{Z}_{4N_f}$ if $N_c = 3$. Here the R-charges are computed by

$$R = X + \frac{N_c - 2 - N_f}{N_f} A. \tag{5.19}$$

The $\mathbb{Z}_{N_f}$ subgroup in $\mathbb{Z}_{2N_f}$ or $\mathbb{Z}_{4N_f}$ is the center of $SU(F) \cong SU(N_f)$. Therefore the global symmetry of the quantum theory is $(SU(F) \rtimes \mathbb{Z}_2) \times U(1)_R$ if $N_c \geq 4$ and $(SU(F) \rtimes \mathbb{Z}_4) \times U(1)_R$ if $N_c = 3$. Here $SU(F) \rtimes \mathbb{Z}_n = \{U \in U(F) \,|\, U^n \in SU(F)\}$. For all $N_c \geq 3$, the superfields Q and M transform as

(5.20)

	$SU(F)$	$\times$	$U(1)_R$
Q	F^*		$1 - \frac{N_c - 2}{N_f}$
M	$S^2 F^*$		$2(1 - \frac{N_c - 2}{N_f})$

Invariance under the gauge and global symmetries and holomorphicity imply that the superpotential, if it exists, is of the form

$$W \sim (\Lambda^{b_0})^x (\det M)^y \tag{5.21}$$

for some exponents x, y. When $N_c \geq 4$, considerations exactly the same as in the symplectic case yield three equations $x + y = 0$, $2 = 2(N_c - N_f - 2)x$ and

$b_0 x + 2N_f y = 3$ for the two unknowns x and y. Again, they have a solution $x = -y = \frac{1}{N_c - 2 - N_f}$. Hence

$$W = A\left(\frac{\Lambda^{3(N_c-2)-N_f}}{\det M}\right)^{\frac{1}{N_c-2-N_f}}, \tag{5.22}$$

where A is a constant that depends only on N_c, N_f and the renormalization scheme. Clearly W does not exist if $N_f = N_c - 2$. If $N_f > N_c$, then $\operatorname{rank} M < N_f$ and $\det M = 0$. So W does not exist either. If $N_c - 2 < N_f \leq N_c$, $W \to \infty$ as $Q \to \infty$; this is in contradiction with asymptotic freedom ($b_0 > 0$ in this range). So the superpotential is not generated when $N_f \geq N_c - 2$. When $N_c = 4$, (5.22) is the superpotential generated by each of the $SU(2)$ factors. We limit the analysis to the case of equal scales for the two $SU(2)$ factors. When $N_c = 3$, the three equations $2x + y = 0$, $2 = (4 - 4N_f)x$ and $b_0 x + 2N_f y = 3$ are also consistent and have a solution $2x = -y = \frac{1}{1-N_f}$. So,

$$W = A\left(\frac{\Lambda^{3-N_f}}{\det M}\right)^{\frac{1}{1-N_f}}. \tag{5.23}$$

Similarly, W does not exist if $N_f = 1$ or $N_f > 3$. When $N_f = 2$, the form of W is not consistent with asymptotic freedom ($b_0 = 2 > 0$ in this case). When $N_f = 3$, we have $b_0 = 0$, but the two-loop β-function is positive. The theory is infrared free and does not have a superpotential. In summary, no superpotential can be generated when $N_c = 3$.

SOLUTIONS TO EXERCISES (BY SIYE WU)

Problems 3–4

Solution to Problem 3

$N_f = N_c$: instanton generated superpotential

Recall that when $N_f \leq N_c$, it is possible to have a dynamically generated superpotential (5.11). This superpotential is generated by instantons when $N_f = N_c$, because the following three requirements are independently met:
1) The result of any calculation in instanton perturbation theory is proportional to $e^{2\pi i\tau(\mu)} \sim \Lambda^{b_0}$. The superpotential W in (5.11) has the same power of Λ when $N_f = N_c$.
2) An instanton vertex absorbs $2(N_c+1)$ of the λ and $2N_f$ of the ψ_Q. The usual vertex is given by the term $\lambda\psi_Q Q$. When $N_f = N_c$, it is possible to form a diagram with 2 external lines of ψ_Q.
3) The $Sp(N_c)$ gauge symmetry is completely broken when the renormalization scale μ is small compared to $\langle Q\rangle$. This means the coupling constant stays small for all μ. Hence the instanton calculation is trustworthy.
Thus we expect $A \neq 0$ when $N_f = N_c$.

We study how the constant A depends on N_c and N_f using deformations on the moduli space. Suppose Q acquires an expectation value $\langle Q\rangle \in \mathrm{Hom}(F, C)$ with $\frac{1}{2}\operatorname{rank}\langle Q\rangle = r$. Then $C = \hat{C} \oplus \hat{C}^\perp$, $F = \hat{F} \oplus \hat{F}^\perp$, where $\hat{C} = (\mathrm{im}\langle Q\rangle)^\perp$, $\hat{F} = \ker\langle Q\rangle$ have dimensions $2(N_c - r)$, $2(N_f - r)$, respectively. The expectation value $\langle Q\rangle$ breaks the gauge group to $Sp(\hat{C})$. Let $\langle M^\perp\rangle \in \wedge^2 \hat{F}^{\perp *}$ be the restriction of $\langle M\rangle$ to $\hat{F}^\perp$; $\langle M^\perp\rangle$ is non-degenerate. Because of the superpotential (5.11), the excitations of M in the $\hat{F}^\perp$ directions become massive. In the low energy theory, M is of the form

$$M = \begin{pmatrix} \hat{M} & \\ & \langle M^\perp\rangle \end{pmatrix}, \tag{6.1}$$

where $\langle M^\perp\rangle$ is fixed and $\hat{M} \in \wedge^2 \hat{F}^2$ contains the light modes. The space of light flavors is $\hat{F}$.

When the renormalization scale $\mu >> (\mathrm{Pf}\,(\langle M^\perp\rangle/2))^{1/2r}$, the coupling constant $g(\mu)$ runs according to

$$(\Lambda/\mu)^{3(N_c+1)-N_f} = e^{-\frac{8\pi^2}{g^2(\mu)}}. \tag{6.2}$$

When $\mu << (\mathrm{Pf}\,(\langle M^\perp\rangle/2))^{1/2r}$, the low energy theory has $N_c - r$ colors and $N_f - r$ flavors. Let $\hat{\Lambda}$ be its scale. Then

$$(\hat{\Lambda}/\mu)^{3(N_c-r+1)-(N_f-r)} = e^{-\frac{8\pi^2}{g^2(\mu)}}. \tag{6.3}$$

We use the $\overline{\mathrm{DR}}$ subtraction scheme. Matching the coupling constant $g(\mu)$ in (6.2) and (6.3) at $\mu = (\mathrm{Pf}\,(\langle M^\perp\rangle/2))^{1/2r}$, we get

$$\Lambda^{3(N_c+1)-N_f} = \mathrm{Pf}\,(\langle M^\perp\rangle/2)\,\hat{\Lambda}^{3(N_c-r+1)-(N_f-r)}. \tag{6.4}$$

This relation is valid for any N_c, N_f and $r \leq N_c, N_f$. Using (6.4), the superpotential of the low energy theory has the same form (5.11) but with a different coefficient $\hat{A}$ related to A by

$$A = 2^{\frac{r}{N_c+1-N_f}}\,\hat{A}. \tag{6.5}$$

When $N_f = N_c$, if we choose $r = N_c - 1$, then $\frac{1}{2}\dim\hat{C} = \frac{1}{2}\dim\hat{F} = 1$. Due to the accidental isomorphism $Sp(1) \cong SU(2)$ and their defining representations, the low energy theory is identical to the $SU(2)$ supersymmetric gauge theory with a single flavor of quark. We know from the $SU(2)$ theory that $\hat{A} = 1$ under the $\overline{\mathrm{DR}}$ scheme. So $A = 2^{N_c-1}$ when $N_f = N_c$.

$N_f < N_c$ and the pure gauge theory

To derive the superpotential for smaller values of N_f but with a fixed N_c, we need to apply a gauge invariant mass deformation. We perturb the theory by adding at the tree level a superpotential $W_{\mathrm{tree}} = \langle M, m\rangle$, where $m \in \wedge^2 F$. Let $r = \frac{1}{2}\,\mathrm{rank}\,m$. We write $F = \hat{F} \oplus \hat{F}^\perp$ such that $\frac{1}{2}\dim\hat{F} = N_f - r$ and the projection $m^\perp \in \wedge^2\hat{F}^\perp$ of m is non-degenerate. The gauge group of the low energy theory is still $Sp(C)$, but the flavor space is $\hat{F}$. Let $\hat{\Lambda}$ be the scale. When $\mu >> (\mathrm{Pf}\,m^\perp)^{1/r}$, $g(\mu)$ runs according to (6.2). However when $\mu << (\mathrm{Pf}\,m^\perp)^{1/r}$, there are $N_f - r$ flavors of light quarks, and we get

$$(\hat{\Lambda}/\mu)^{3(N_c+1)-(N_f-r)} = e^{-\frac{8\pi^2}{g^2(\mu)}}. \tag{6.6}$$

Matching the coupling constant $g(\mu)$ in (6.2) and (6.6) at $\mu = (\mathrm{Pf}\,m^\perp)^{1/r}$, we get

$$\hat{\Lambda}^{3(N_c+1)-(N_f-r)} = \mathrm{Pf}\,m^\perp\,\Lambda^{3(N_c+1)-N_f}. \tag{6.7}$$

This relation is valid for any N_c, N_f and $r \leq N_f$.

If $N_f \leq N_c$ in the ultraviolet theory, the total superpotential after the mass deformation is

$$W = A\left(\frac{\Lambda^{3(N_c+1)-N_f}}{\mathrm{Pf}\,M}\right)^{\frac{1}{N_c+1-N_f}} + \langle M, m\rangle. \tag{6.8}$$

We eliminate the heavy fields in M by minimizing bosonic potential. At the extremum of the superpotential (6.8), M is of the block-diagonal form (6.1), where

(6.9)

$$\langle M^\perp\rangle = \left(\frac{A}{N_c+1-N_f}\right)^{\frac{N_c+1-N_f}{N_c+1-(N_f-r)}}\left(\frac{\mathrm{Pf}\,m^\perp\Lambda^{3(N_c+1)-N_f}}{\mathrm{Pf}\,\hat{M}}\right)^{\frac{1}{N_c+1-(N_f-r)}}(m^\perp)^{-1}.$$

Here $(m^\perp)^{-1} \in \wedge^2\hat{F}^{\perp *}$ is induced by $m^\perp \in \wedge^2\hat{F}^\perp$. We compute the superpotential of the light mesons $\hat{M}$ by substitute in (6.8) the solution of $\langle M^\perp\rangle$. Using

$\langle (m^\perp)^{-1}, m^\perp\rangle = r$, $d\,\mathrm{Pf}\,M = \mathrm{Pf}\,M\langle dM, M^{-1}\rangle$ and (6.7), we deduce that the superpotential has precisely the same form as (5.11), with the coefficient

$$\hat{A} = (N_c + 1 - (N_f - r))\left(\frac{A}{N_c + 1 - N_f}\right)^{\frac{N_c+1-N_f}{N_c+1-(N_f-r)}}. \tag{6.10}$$

If we have $N_f = N_c$ to begin with, then $A = 2^{N_c-1}$ and $\hat{A} = (N_c + 1 - (N_f - r))2^{\frac{N_c-1}{N_c+1-(N_f-r)}}$. Equivalently, for a theory with $N_f \le N_c$, the coefficient A in (5.11) is given by

$$A = (N_c + 1 - N_f)2^{\frac{N_c-1}{N_c+1-N_f}}. \tag{6.11}$$

In all cases of $N_f \le N_c$, the superpotential is non-zero. When $1 \le N_f \le N_c$, the classical vacuum degeneracy is lifted by W and the quantum theory has no vacuum.

In the pure $Sp(C) \cong Sp(N_c)$ gauge theory, $N_f = 0$, and the superpotential

$$W = (N_c + 1)2^{\frac{N_c-1}{N_c+1}}\Lambda^3 = (N_c + 1)\mu^3 2^{\frac{N_c-1}{N_c+1}} e^{\frac{2\pi i}{N_c+1}\tau}. \tag{6.12}$$

This is $(N_c + 1)$-valued, corresponding to the fact that there are $N_c + 1$ vacua in quantum theory. In particular, when $N_c = 1$, (6.12) agrees with the result for $SU(2)$ gauge theory. Notice also that the superpotential W is a constant: although non-zero, it does not contribute to the effective Lagrangian. Nevertheless, the result (6.12) implies gaugino condensation. We consider an extended theory in which the parameter τ becomes a superfield and assume that supersymmetry is unbroken after quantization. The partition function

$$Z(\tau) = \int DA\, D\lambda\, e^{iS(A,\lambda,\tau)} = e^{i\int d^4x d^2\theta\, W(\tau)} \tag{6.13}$$

is holomorphic in τ and $W(\tau)$ has the same form as (6.12). Since the Lagrangian density of the pure gauge theory is

$$\mathcal{L} = \frac{1}{4\pi i}\int d^2\theta\ \tau\ \mathrm{tr} W_\alpha W^\alpha + \text{c.c.}, \tag{6.14}$$

where $W_\alpha = \lambda_\alpha + \cdots$ is the superfield strength, we get

$$\begin{aligned} \langle\lambda\lambda\rangle &= 4\pi\frac{\delta}{\delta F_\tau}\log Z(\tau) = 4\pi\frac{\partial}{\partial F_\tau}\int d^2\theta\, W(\tau) \\ &= 4\pi\frac{\partial}{\partial\tau}W(\tau) = 2^{4-\frac{2}{N_c+1}}\pi^2 i\Lambda^3 = 2^{4-\frac{2}{N_c+1}}\pi^2 i\mu^3 e^{\frac{2\pi i}{N_c+1}\tau}. \end{aligned} \tag{6.15}$$

Again, this is $(N_c + 1)$-valued.

$N_f = N_c + 1$

In the above mass deformation of theories with $N_f \le N_c$, if $r = N_f$, then $\hat{F}^\perp = F$, $m^\perp = m$ and $M^\perp = M$. The extremum of the total superpotential (6.8) is reached at

$$\langle M\rangle = 2^{\frac{N_c-1}{N_c+1}}(\Lambda^{3(N_c+1)-N_f}\,\mathrm{Pf}\,m)^{\frac{1}{N_c+1}}\, m^{-1}. \tag{6.16}$$

Just as in the supersymmetric $SU(N_c)$ gauge theories, the above result is valid in the range $N_f \ge N_c + 1$. For $N_f = N_c + 1$, the solution (6.16) satisfies

$$\mathrm{Pf}\,M = 2^{N_c-1}\Lambda^{2(N_c+1)}. \tag{6.17}$$

This is independent of m, and shall remain valid in the absence of the mass term. We claim that in the low energies the quantum theory is described by M subject to the constraint (6.17). To check that the theory reduces to $N_f = N_c$ via mass deformation, we add a mass term $\langle M, m\rangle$ with $\frac{1}{2}\operatorname{rank} m = 1$. Then $\frac{1}{2}\dim \hat{F} = N_c$. Because of the constraint (6.17), we have

$$\langle M, m\rangle = \frac{2^{N_c-1} m^{\perp} \Lambda^{2(N_c+1)}}{\operatorname{Pf}\hat{M}} = 2^{N_c-1}\frac{\hat{\Lambda}^{2N_c+3}}{\operatorname{Pf}\hat{M}}, \tag{6.18}$$

where the last equality follows from (6.7) when $N_f = N_c + 1$, $r = 1$. This is the superpotential (5.11) with the correct coefficient (6.11) when $N_f = N_c$.

The quantum moduli space $\mathcal{M}_{\mathrm{q}}$ is smooth. There are no additional massless fields other than M subject to the constraint (6.17). The absence of massless gluons is known as the confinement of color charges. We check the consistency of these postulates using 't Hooft's anomaly matching conditions. Each point $M_0 \in \wedge^2 F^*$ on $\mathcal{M}_{\mathrm{q}}$ is a symplectic form on F. Let $Sp(F, \mathbb{C})$ be the (complex) symplectic group preserving M_0. The most symmetric points on $\mathcal{M}_{\mathrm{q}}$ are those compatible with the Hermitian structure on F. There the global symmetry $SU(F) \times U(1)_R$ is broken to $Sp(F) \times U(1)_R$, where the (compact) symplectic group $Sp(F) = Sp(F, \mathbb{C}) \cap U(F)$ depends on the choice of M_0. Notice that when $N_f = N_c + 1$, $U(1)_X = U(1)_R$ is anomaly free. In the ultraviolet theory, the massless fermions are ψ_Q and λ. In the infrared theory, they are the fermionic partners ψ_M of M in $T_{M_0}\mathcal{M}_{\mathrm{q}}$. Under the unbroken global symmetry, these light fermions transform according to

(6.19)

		$Sp(F)$	$\times$	$U(1)_R$
ultraviolet	ψ_Q	F^*		$-\frac{N_c+1}{N_f} = -1$
	λ	$\mathbf{1}$		1
infrared	ψ_M	$\wedge^2 F^* \ominus \mathbf{1}$		$1 - 2\frac{N_c+1}{N_f} = -1$

Here and below, $\mathbf{1}$ denotes the trivial representation. The non-zero combinations to 't Hooft's condition are given by

(6.20)

	ultraviolet	infrared
$\mathfrak{sp}(F)^2\mathfrak{u}(1)_R$	$2N_c c_2(F) \cdot (-1)$	$c_2(\wedge^2 F \ominus \mathbf{1}) \cdot (-1)$
$\mathfrak{u}(1)_R$	$4N_cN_f\ (-1) + N_c(2N_c+1)\ 1$	$(N_f(2N_f-1)-1)\ (-1)$
$\mathfrak{u}(1)_R^3$	$4N_cN_f\ (-1)^3 + N_c(2N_c+1)\ 1^3$	$(N_f(2N_f-1)-1)\ (-1)^3$

To calculate $c_2(\wedge^2 F \ominus \mathbf{1})$, we note that the root system of $\mathfrak{sp}(N_f)$ is $\{\epsilon_i \pm \epsilon_j, 2\epsilon_i \,|\, 1 \le i \ne j \le N_f\}$ and the positive roots are $\epsilon_i \pm \epsilon_j$ $(1 \le i \le j \le N_f)$. The highest weights of the representations of F and $\wedge^2 F \ominus \mathbf{1}$ are $\lambda = \epsilon_1$ and $\lambda_2 = \epsilon_1 + \epsilon_2$, respectively. Therefore $c_2(\wedge^2 F \ominus \mathbf{1}) = \frac{(\lambda_2+2\rho,\lambda_2)\ \dim(\wedge^2 F\ominus\mathbf{1})}{(\lambda_1+2\rho,\lambda_1)\ \dim F} c_2(F) = 2(N_f-1)c_2(F)$, where $\rho = \sum_{i=1}^{N_f}(N_f+1-i)\epsilon_i$ is half of the sum of positive roots. The two columns match when $N_f = N_c + 1$.

Finally, we propose a global anomaly matching condition and show that it is satisfied in the above situation. Consider two theories with the same (anomaly free) global symmetry group H but with different contents of light fermionic fields, which form representations R and R' of H, respectively. If the two theories describe the same low energy physics, then if H is hypothetically gauged, the determinant line bundles Det $\not{D}_R^+$ and Det $\not{D}_{R'}^+$ of the Dirac operators twisted by R and R' shall be isomorphic. 't Hooft's anomaly matching condition checks whether the curvatures of the two bundles are the same. This is a stringent condition for the two theories to describe the same low energy physics. However the two bundles can still differ by a flat bundle, which can be non-trivial if $\pi_4(H) \neq 1$. If the representation $R^* \oplus R'$ of H is real, then the determinant line bundle Det $\not{D}_{R^* \oplus R'}^+ \cong (\text{Det } \not{D}_R^+)^{-1} \otimes \text{Det } \not{D}_{R'}^+$ is trivial. Therefore the reality of the representation $R^* \oplus R'$ is a sufficient condition of global anomaly matching. In the above problem, suppose only the $Sp(F)$ factor of the global symmetry is gauged. In the ultraviolet theory, the representation is an even number $(2N_c)$ of copies of F^* and the trivial representations. In the infrared theory, the representation $\wedge^2 F^* \ominus \mathbf{1} \subset F^* \otimes F^*$ is real because F^* is quarternionic. So the global anomalies match.

$N_f = N_c + 2$

When $N_f = N_c + 2$, we obtain from (6.16) that $\wedge^{N_c+1} M \to 0$ as $m \to 0$. So there is no quantum correction to the classical moduli space $\mathcal{M}_c$. We claim that in the low energy theory, all components of M become dynamical and that there is a superpotential

$$W = -\frac{\text{Pf } M}{2^{N_c-1} \Lambda^{2N_c+1}}. \tag{6.21}$$

Indeed, varying W with respect to M, we get the constraint (5.2). If we add a mass term $W_{\text{tree}} = \langle M, m \rangle$ with $\frac{1}{2} \operatorname{rank} m = 1$, then $\frac{1}{2} \dim \hat{F} = N_c + 1$. At the extremum of $W + W_{\text{tree}}$, M is of the block-diagonal form (6.1). From varying $M^\perp$, we get

$$\text{Pf } \hat{M} = 2^{N_c-1} m^\perp \Lambda^{2N_c+1}. \tag{6.22}$$

By (6.7), this is exactly the constraint (6.17).

The most symmetric point on $\mathcal{M}_c$ is $M = 0$, where the full global symmetry $SU(F) \times U(1)_R$ is unbroken. This is also the most singular point. The singularity is the result of additional massless fields. In fact all the light fields in M are massless at the origin. We check this using 't Hooft's anomaly matching condition. Under the global symmetry, the light fermions in the ultraviolet and the infrared theories transform as

(6.23)

		$SU(F)$ $\times$	$U(1)_R$
ultraviolet	ψ_Q	F^*	$-\frac{N_c+1}{N_f}$
	λ	$\mathbf{1}$	1
infrared	ψ_M	$\wedge^2 F^*$	$1 - 2\frac{N_c+1}{N_f}$

The non-zero combinations to be matched are
(6.24)

	ultraviolet	infrared
$\mathfrak{su}(F)^3$	$2N_c\,(-d_F)$	$-d_{\wedge^2 F}$
$\mathfrak{su}(F)^2\mathfrak{u}(1)_R$	$2N_c c_2(F) \cdot (-\frac{N_c+1}{N_f})$	$c_2(\wedge^2 F) \cdot (1 - 2\frac{N_c+1}{N_f})$
$\mathfrak{u}(1)_R$	$4N_c N_f\ (-\frac{N_c+1}{N_f}) + N_c(2N_c+1)\ 1$	$N_f(2N_f-1)\ (1 - 2\frac{N_c+1}{N_f})$
$\mathfrak{u}(1)_R^3$	$4N_c N_f\ (-\frac{N_c+1}{N_f})^3 + N_c(2N_c+1)\ 1^3$	$N_f(2N_f-1)\ (1 - 2\frac{N_c+1}{N_f})^3$

We claim that $c_2(\wedge^2 F) = 2(N_f - 1)c_2(F)$ and $d_{\wedge^2 F} = 2(N_f - 2)d_F$. Indeed, let $\epsilon_1, \ldots, \epsilon_{2N_f}$ be the weights of the fundamental representation of $SU(F)$. Since $\sum_{i=1}^{2N_f} \epsilon_i = 0$, we have $\sum_{i<j}(\epsilon_i + \epsilon_j)^2 = 2(N_f - 1)\sum_{i=1}^{2N_f} \epsilon_i^2$ and $\sum_{i<j}(\epsilon_i + \epsilon_j)^3 = 2(N_f - 2)\sum_{i=1}^{2N_f} \epsilon_i^3$. Therefore the two columns match when $N_f = N_c + 2$.

Solution to Problem 4

$N_f \le N_c - 5$ and pure gauge theory

When $N_f \le N_c - 5$, it is possible to have a dynamically generated superpotential of the form (5.22). Since most of the analysis is parallel to the $Sp(N_c)$ gauge theories, we only give an outline here.

Suppose M acquires a large expectation value $\langle M\rangle$ of rank r. Let $\hat{F} = \ker\langle M\rangle$. Then $F = \hat{F} \oplus \hat{F}^\perp$ and the restriction $\langle M^\perp\rangle \in S^2\hat{F}^{\perp *}$ of $\langle M\rangle$ to $\hat{F}^\perp$ is non-degenerate. The low energy theory has $N_c - r$ colors and $\dim\hat{F} = N_f - r$ flavors. So $b_0 = 3(N_c - r - 2) - (N_f - r)$. Matching the coupling constant at $\mu = (\det\langle M^\perp\rangle)^{1/2r}$, we obtain

$$\Lambda^{3(N_c-2)-N_f} = \det\langle M^\perp\rangle \hat{\Lambda}^{3(N_c-r-2)-(N_f-r)}. \tag{6.25}$$

In fact, this relation remains valid for all values of N_c, N_f as long as $N_c - r \ge 5$. For future needs, we also consider here the cases $N_c - r \le 4$. For $N_c - r = 4$, in the low energy $SO(4)$ theory, both scales $\hat{\Lambda}_\pm$ of the $SU(2)_\pm$ factors are given by (6.25). If $N_c - r = 3$, we derive the relation in two stages: first from $N_c \ge 5$ to $N_c = 4$, then from $N_c = 4$ to $N_c = 3$. In the second stage, $N_c = 4$, $r = 1$. The low energy theory has 3 colors and $N_f - 1$ flavors. Above the scale $\hat{\Lambda}$, the gauge group is $SU(2)_+ \times SU(2)_-$; the coupling constants $g_\pm(\mu)$, or $\tau_\pm(\mu)$ run according (4.2) with $b_0 = 6 - N_f$. Since $SU(2)$ is diagonally embedded, its coupling constant is given by $\tau(\mu) = \tau_+(\mu) + \tau_-(\mu)$. Therefore the scale is $(\hat{\Lambda}_+^{6-N_f}\hat{\Lambda}_-^{6-N_f})^{1/2(6-N_f)} = (\hat{\Lambda}_+\hat{\Lambda}_-)^{1/2}$. At low energies, the $SU(2)$ gauge theory has $b_0 = 6 - 2(N_f - 1)$. Matching the coupling constant $g(\mu)$ at $\mu = \frac{1}{2}\langle M^\perp\rangle$, we get

$$4\hat{\Lambda}_+^{6-N_f}\hat{\Lambda}_-^{6-N_f} = \langle M^\perp\rangle^2 \hat{\Lambda}^{6-2(N_f-1)}. \tag{6.26}$$

Combining (6.25) and (6.26), we get, for any $N_c \ge 4$, N_f and $r = N_c - 3$,

$$4(\Lambda^{3(N_c-2)-N_f})^2 = (\det\langle M^\perp\rangle)^2 \hat{\Lambda}^{6-2(N_f-N_c+3)}. \tag{6.27}$$

When $N_f \le N_c - 5$ and $N_c - r \ge 5$, a comparison of the superpotential (5.22) above and below $\mu = (\det\langle M^\perp\rangle)^{1/2r}$ yields $A = \hat{A}$. If $N_f = N_c - 5$ and $r = N_f$, the low energy theory is a pure $SO(5) \cong Sp(2)/\mathbb{Z}_2$ gauge theory. Using (6.11), we get $\hat{A} = 3 \cdot 2^{\frac{1}{3}}$. So $A = 3 \cdot 2^{\frac{1}{3}}$ when $N_f = N_c - 5$.

Now we add a mass term $W_{\text{tree}} = \frac{1}{2}\langle M, m\rangle$ with $\operatorname{rank} m = r$. Again $F = \hat{F} \oplus \hat{F}^\perp$. Here $\dim\hat{F} = N_c - r$ and the projection $m^\perp \in S^2\hat{F}^\perp$ of m is non-degenerate. The low energy theory has the same number of colors and $N_f - r$ flavors. So $b_0 = 3(N_c - 2) - (N_f - r)$. Matching the coupling constant at $\mu = (\det m^\perp)^{1/r}$, we obtain

$$\hat{\Lambda}^{3(N_c-2)-(N_f-r)} = \det m^\perp \Lambda^{3(N_c-2)-N_f}. \tag{6.28}$$

This is valid for $N_c \ge 5$ and all N_f. When $N_c = 4$, the scales $\hat{\Lambda}_\pm$ of the $SU(2)_\pm$ factors are related to the scales $\Lambda_\pm$ of the ultraviolet theory by the same formula.

When $N_c = 3$, (6.28) should be replaced by

$$\hat{\Lambda}^{6-2(N_f-r)} = (\det m^{\perp})^2 \Lambda^{6-2N_f}. \tag{6.29}$$

When $N_f \leq N_c - 5$, the superpotential W before mass deformation is given by (5.22). Varying the heavy modes in M, the extremum of the total superpotential $W + W_{\mathrm{tree}}$ is reached when M is of the diagonal form (6.1) with

(6.30)

$$\langle M^{\perp}\rangle = \left(\frac{2A}{N_c - 2 - N_f}\right)^{\frac{N_c-2-N_f}{N_c-2-(N_f-r)}} \left(\frac{\det m^{\perp} \Lambda^{3(N_c-2)-N_f}}{\det \hat{M}}\right)^{\frac{1}{N_c-2-(N_f-r)}} (m^{\perp})^{-1}.$$

In terms of the of the light fields $\hat{M}$, this extremum has the same form as (5.22) with

$$\hat{A} = \frac{N_c - 2 - (N_f - r)}{2} \left(\frac{2A}{N_c - 2 - N_f}\right)^{\frac{N_c-2-N_f}{N_c-2-(N_f-r)}}. \tag{6.31}$$

If we start from $N_f = N_c - 5$, then $A = 3 \cdot 2^{\frac{1}{3}}$ and $\hat{A} = \frac{1}{2}(N_c - 2 - (N_f - r))2^{\frac{4}{N_c-2-(N_f-r)}}$. Equivalently, for a theory with $N_f \leq N_c - 5$, the coefficient A in (5.22) is given by

$$A = \tfrac{1}{2}(N_c - 2 - N_f)2^{\frac{4}{N_c-2-N_f}}. \tag{6.32}$$

For pure gauge theories, $N_f = 0$ and $A = \frac{1}{2}(N_c - 2)2^{\frac{4}{N_c-2}}$. As in (6.33), for the pure $SO(N_c)$ gauge theory with $N_c \geq 5$,

$$\langle\lambda\lambda\rangle = 2^{2+\frac{4}{N_c-2}}\pi^2 i\Lambda^3 = 2^{2+\frac{4}{N_c-2}}\pi^2 i\mu^3 e^{\frac{2\pi i}{N_c-2}\tau}; \tag{6.33}$$

this is $(N_c - 2)$-valued, corresponding to the $N_c - 2$ vacua. Finally, the result is consistent with another accidental isomorphism $SO(6) \cong SU(4)/\mathbb{Z}_2$. Indeed, when $N_c = 6$, $A = 4$; this agrees with the coefficient of the pure $SU(4)$ gauge theory.

$N_f = N_c - 4$

A generic flat deformation on the moduli space breaks the gauge group to $SO(4) \cong (SU(2)_+ \times SU(2)_-)/\mathbb{Z}_2$. From (6.25), the scales $\hat{\Lambda}_{\pm}$ of the two $SU(2)$ gauge theories (without matter) are related to the scale Λ of the ultraviolet theory by

$$\hat{\Lambda}_{\pm}^6 = \frac{\Lambda^{2(N_c-1)}}{\det M}. \tag{6.34}$$

In the infrared, the superpotential of the two $SU(2)_{\pm}$ gauge theories are $W_{\pm} = 2\hat{\Lambda}_{\pm}^3$, respectively. But both of them are two-valued, corresponding to the two vacua of each $SU(2)$ theory. We fix a square root. Then the superpotential of the original theory is

$$W = 2(\epsilon_+ + \epsilon_-)\left(\frac{\Lambda^{2(N_c-1)}}{\det M}\right)^{1/2}, \tag{6.35}$$

where $\epsilon_+, \epsilon_- = \pm 1$. There are two physically inequivalent branches $\epsilon_+ = \epsilon_-$ and $\epsilon_+ = -\epsilon_-$.

The superpotential (6.35) is compatible with mass deformation. In the branch $\epsilon_+ = \epsilon_-$, it is the continuation to $N_f = N_c - 4$ of (5.22). As before, adding a mass term $\frac{1}{2}\langle M, m\rangle$ of rank $m = r$ reduces the theory to $N_f = N_c - 4 - r$ with the same superpotential (5.22) in the low energies. In the other branch $\epsilon_+ = -\epsilon_-$, the total superpotential is $W_{\rm tree} = \frac{1}{2}\langle M, m\rangle$. The theory has no vacuum and hence does not contribute to the low energy theory with a smaller N_f.

For $\epsilon_+ = \epsilon_-$, the superpotential (6.35) lifts the classical vacuum degeneracy and the quantum theory has no vacuum. In the branch $\epsilon_+ = -\epsilon_-$, the vacuum degeneracy is not lifted by quantum effects. Classically there is a singularity at $M = 0$, where the full gauge symmetry is restored and all the gluons become massless. We claim that in quantum theory, the singularity is smoothed out and the color charges are confined. That is, there are no massless gluons and all the massless fermions are contained in M, even at the origin of $\mathcal{M}$. We check this claim by the 't Hooft and the global anomaly matching conditions. At $M = 0$, the identity component of the unbroken global symmetry is $SU(F) \times U(1)_R$. The light fermions in high and low energy theories transform according to

(6.36)

		$SU(F)$	$\times$	$U(1)_R$
ultraviolet	ψ_Q	F^*		$-\frac{N_c-2}{N_f}$
	λ	$\mathbf{1}$		1
infrared	ψ_M	S^2F^*		$1-2\frac{N_c-2}{N_f}$

The non-zero combinations to be matched are given by

(6.37)

	ultraviolet	infrared
$\mathfrak{su}(F)^3$	$-N_c d_F$	$-d_{S^2F}$
$\mathfrak{su}(F)^2\mathfrak{u}(1)_R$	$N_c c_2(F)\cdot(-\frac{N_c-2}{N_f})$	$c_2(S^2F)\cdot(1-2\frac{N_c-2}{N_f})$
$\mathfrak{u}(1)_R$	$N_cN_f\ (-\frac{N_c-2}{N_f}) + \frac{1}{2}N_c(N_c-1)\ 1$	$\frac{1}{2}N_f(N_f+1)\ (1-2\frac{N_c-2}{N_f})$
$\mathfrak{u}(1)_R^3$	$N_cN_f\ (-\frac{N_c-2}{N_f})^3 + \frac{1}{2}N_c(N_c-1)\ 1^3$	$\frac{1}{2}N_f(N_f+1)\ (1-2\frac{N_c-2}{N_f})^3$

Let ϵ_i $(i = 1,\ldots,N_f)$ be the weights in the defining representation F. Using $\sum_{i=1}^{N_f}\epsilon_i = 0$, it is easy to check $\sum_{i\le j}(\epsilon_i+\epsilon_j)^2 = (N_f+2)\sum_{i=1}^{N_f}\epsilon_i^2$ and $\sum_{i\le j}(\epsilon_i+\epsilon_j)^3 = (N_f+4)\sum_{i=1}^{N_f}\epsilon_i^3$. So $c_2(S^2F) = (N_f+2)c_2(F)$ and $d_{S^2F} = (N_f+4)d_F$. It is now straightforward to check that the two columns of (6.37) match when $N_f = N_c - 4$, fulfilling 't Hooft's criterion. For $N_f = 2$, $SU(F) \cong Sp(1)$. In the ultraviolet theory, the number of copies of F^* is even $(N_c = 6)$. In the infrared, the representation $S^2F^* \subset F^* \otimes F^*$ is real. So the global anomalies also match.

$N_f = N_c - 3$

We first find the superpotential by flat deformations on the moduli space. A generic expectation value of Q breaks the gauge group to $SO(3)$. Let $\hat{F} \subset F$ a subspace of dimension 1. We consider the limit where a large expectation value $\langle M^\perp\rangle \in S^2\hat{F}^{\perp *}$ with rank $N_f - 1$ breaks the theory to $SO(4) \cong (SU(2)_+ \times SU(2)_-)/\mathbb{Z}_2$ with one quark $\hat{Q} \in \mathrm{Hom}(\hat{F}, C)$, which further break the gauge group to a diagonal subgroup $SU(2)/\mathbb{Z}_2$.[1] In the first stage of deformation, from (6.25), the scales $\hat{\Lambda}_\pm$

[1] The decomposition of $SO(4)$ into $SU(2)_+ \times SU(2)_-$ is canonical. The isotropy group of $\hat{Q} \in \mathrm{Hom}(\hat{F}, C)$ (in $SO(4)$) and its isomorphisms onto $SU(2)_\pm$ depend on $\hat{Q}$.

of the $SU(2)_+ \times SU(2)_-$ theory are given by

$$\hat{\Lambda}^5_\pm = \frac{\Lambda^{2N_c-3}}{\det\langle M^\perp\rangle}. \tag{6.38}$$

The effective $SU(2)_\pm$ theories have instanton generated superpotentials

$$\hat{W}_\pm = 2\frac{\hat{\Lambda}^5_\pm}{\det \hat{M}} = 2\frac{\Lambda^{2N_c-3}}{\det M}. \tag{6.39}$$

In the second stage, we have the low energy pure $SU(2)$ gauge theory whose scale $\hat{\Lambda}$ is given by (6.27). It has a superpotential

$$\hat{W} = 2\hat{\Lambda}^3 = 4\epsilon\frac{\Lambda^{2N_c-3}}{\det M}, \tag{6.40}$$

where $\epsilon = \pm 1$ comes from taking the square root of (6.27) and labels the two vacua of the pure $SU(2)$ theory. Combining (6.39) and (6.40), we find that the superpotential of the $N_f = N_c - 3$ theory is

$$W = \hat{W}_+ + \hat{W}_- + \hat{W} = 4(1+\epsilon)\frac{\Lambda^{2N_c-3}}{\det M}. \tag{6.41}$$

The first term in (6.41) is due to the instanton effect whereas the second term comes from gaugino condensation in the unbroken $SU(2)$ theory. As in $SU(N_c)$ and $Sp(N_c)$ gauge theories, the superpotential (6.41) is proportional to the exponential of the instanton action, and it is possible to form a diagram with two external fermion lines using the instanton vertex. The novelty here is that the unbroken gauge group $SU(2)$ is non-Abelian. However, when the gauge group is broken to the diagonal SU(2), an instanton in one $SU(2)_\pm$ cannot be rotated into the unbroken group. Therefore the instanton computation is still reliable because there is no infrared divergence.

We check that the result (6.41) is consistent with mass deformation. The theory has two branches: $\epsilon = 1$ and $\epsilon = -1$. After adding a mass term $W_{\text{tree}} = \frac{1}{2}\langle M, m\rangle$ with $\operatorname{rank} m = 1$, the two branches should correspond to those of $N_f = N_c - 4$. In fact, when $\epsilon = 1$, the total superpotential (6.41) is the continuation of (5.22), just as (6.35) when $\epsilon_+ = \epsilon_-$. So the extremum of $W + W_{\text{tree}}$ matches (6.35) in the branch $\epsilon_+ = \epsilon_-$. In particular, from (6.30), the extremum is reached when

$$\langle M^\perp\rangle = 4\left(\frac{\Lambda^{2N_c-3}}{m^\perp \det \hat{M}}\right)^{1/2}. \tag{6.42}$$

The ambiguity in the sign of the square root corresponds to the two possibilities $\epsilon_+ = \epsilon_- = \pm 1$. In the other branch $\epsilon = -1$, we claim that the theory has additional light fields $q \in F \otimes \wedge^{N_f} F$ coupling to M and that the superpotential near $M = 0$ behaves as

$$W \sim \frac{1}{2\mu} M(q,q), \tag{6.43}$$

where $\mu \in (\wedge^{N_f} F)^{\otimes 2}$ is a parameter of mass dimension 1. The R-charges of M, q are $-\frac{2}{N_f}$, $1 + \frac{1}{N_f}$, respectively. This is consistent with requirement that W has R-charge 2. Upon mass deformation, we write $F = \hat{F} \oplus \hat{F}^\perp$ as before. The extremum of $W + W_{\text{tree}}$ is reached at $q^\perp = \pm(-m\mu)^{1/2}$, $\hat{q} = 0$, $M(q, \cdot) = 0$. The light fields are in $M = \hat{M} \in S^2\hat{F}$. This matches the low energy behavior of $N_f = N_c - 4$. The signs of $q^\perp$ correspond to the choices of ϵ_+, ϵ_- in $\epsilon_+ = -\epsilon_- = \pm 1$.

We check the postulate on the existence of q using 't Hooft's and global anomaly matching conditions. In the low energy theory, the global symmetry $SU(F) \times U(1)_R$ is unbroken at $M = 0$. Also at $M = 0$, the fields q are massless. The R-charge of ψ_q is $\frac{1}{N_f}$. So their contribution to the left column of (6.37) is

(6.44)

	infrared: additional contribution from q
$\mathfrak{su}(F)^3$	d_F
$\mathfrak{su}(F)^2\mathfrak{u}(1)_R$	$c_2(F) \cdot \frac{1}{N_f}$
$\mathfrak{u}(1)_R$	$N_f\,(\frac{1}{N_f})$
$\mathfrak{u}(1)_R^3$	$N_f\,(\frac{1}{N_f})^3$

It is straightforward to check that for $N_f = N_c - 3$, each term in the left column of (6.37) is equal to the sum of the corresponding one in the right column and that in (6.44). When $N_f = 2$, the number of copies of F^* in the ultraviolet theory is odd ($N_c = 5$). In the infrared theory, the representation S^2F^* is real, but there is an additional quark q in the representation F. Therefore the global anomalies match.

$N_f = N_c - 2$

Classically, a generic expectation value $\langle M \rangle$ breaks the gauge group to the Abelian group $SO(2) \cong U(1)$. At the singularities, where $\langle M \rangle$ is of a lower rank, the unbroken gauge group is larger and hence there are more massless gauge bosons. At the quantum level, since no superpotential is generated, the vacuum degeneracy is not lifted. In fact when $N_f = N_c - 2$, $U(1)_X = U(1)_R$ is anomaly free and M is neutral, whereas a superpotential should have R-charge 2. We expect the classical picture to be valid when $\langle M \rangle$ is large. There the theory is in the Higgs phase. Modification is needed near the singularities due to strong coupling effects. In particular, the quantum moduli space may have different singularities.

Over a smooth point M in the moduli space, we have a $U(1)$ gauge theory whose scale is set by M. By $N = 1$ supersymmetry, the effective gauge coupling $\tau_{\text{eff}}(M)$ is holomorphic in M. However, due to the periodicity in the (effective) θ-angle and the electric-magnetic duality, τ_{eff} needs not be single-valued in M; it is defined up to fractional linear transformations of $SL(2,\mathbb{Z})$. At the singularities, τ_{eff} is not defined because there are extra massless fields. Consequently, there are possible non-trivial $SL(2,\mathbb{Z})$ monodromies around the singularities. These information can be summarized by a family of elliptic curves over the smooth part $\mathcal{M}_{\text{smooth}}$ of the moduli space whose fibre over M has modular parameter $\tau_{\text{eff}}(M)$. The j-invariant of the family is a well-defined holomorphic function on $\mathcal{M}_{\text{smooth}}$. In our situation, the j-invariant is a function of $U = \det M$ because of the (anomaly free) $SU(F)$ symmetry. We construct a family of elliptic curves over the smooth part of $\mathcal{U} = (\wedge^{N_f} F^*)^{\otimes 2}$; the family over $\mathcal{M}_{\text{smooth}}$ is then the pull back by the determinant map.

We consider the range where M is of the form (6.1): the large expectation value $\langle M^\perp \rangle$ of rank $N_c - 3$ breaks the gauge group to $SO(3) \cong SU(2)/\mathbb{Z}_2$, which is further broken to $U(1)$ by $\hat{M}$. The $SO(3)$ theory in the intermediate stage has a single flavor. It is the $N = 2$ theory which gives rise to Donaldson and Seiberg-Witten invariants. Using (6.27), the scale $\hat{\Lambda}$ is given by

$$\hat{\Lambda}^4 = 4\left(\frac{\Lambda^{2N_c-4}}{\det\langle M^\perp\rangle}\right)^2. \tag{6.45}$$

The family of elliptic curve is

$$y^2 = (x-\hat{\Lambda}^2)(x+\hat{\Lambda}^2)(x-u), \tag{6.46}$$

where the parameter u is related to our theory by $u = \hat{M}/4$. After making a change of variables $x \mapsto x/4\det\langle M^\perp\rangle$, $y \mapsto y/(4\det\langle M^\perp\rangle)^{3/2}$, (6.46) becomes

$$y^2 = (x-8\Lambda^{2N_c-4})(x+8\Lambda^{2N_c-4})(x-U). \tag{6.47}$$

This is a good approximation for large U. When U goes around ∞, the monodromy is PT^{-2}. For large U, we have

$$e^{2\pi i\tau_{\text{eff}}(M)} = \left(\frac{\Lambda}{U^{1/2(N_c-2)}}\right)^{4N_c-8}. \tag{6.48}$$

The exact family of curves which agrees with (6.48) for large U must be of the form

$$y^2 = (x-8\Lambda^{2N_c-4})(x+8\Lambda^{2N_c-4})(x-U+\alpha\Lambda^{2N_c-4}). \tag{6.49}$$

The curve is singular at $U = (\alpha\pm 8)\Lambda^{2N_c-4}$. To determine α, we add a mass term $W_{\text{tree}} = \frac{1}{2}\langle M, m\rangle$ with $\operatorname{rank} m = 1$. Then the vacuum degeneracy is lifted except at two singularities, which shall correspond to the two branches $\epsilon = \pm 1$ of the $N_f = N_c - 3$ theory. The $\epsilon = 1$ branch has a superpotential

$$\tfrac{1}{2}M^\perp m^\perp = 8\frac{\hat{\Lambda}^{2N_c-3}}{\det\hat{M}}. \tag{6.50}$$

So $U = 16\hat{\Lambda}^{2N_c-3}/m^\perp = 16\Lambda^{2N_c-4}$. The $\epsilon = -1$ branch has $\det\hat{M} = 0$, hence $U = 0$. Therefore we have a unique solution $\alpha = 8$. After making another change of variable $x \mapsto x - 8\Lambda^{2N_c-4}$, we get the family of elliptic curves of our theory

$$y^2 = x(x-16\Lambda^{2N_c-4})(x-U). \tag{6.51}$$

The monodromies when U goes around $16\Lambda^{2N_c-4}$ and 0 are $(T^2S)T(T^2S)^{-1}$ and ST^2S^{-1}, and the fields that become massless at $U = 16\Lambda^{2N_c-4}$ and $U = 0$ are dyons and magnectic monopoles, respectively. Unlike the $N = 2$ gauge theory, the theory here does not have the $\mathbb{Z}_2$ symmetry that relates the two singular components in the modular space. We will see that the physics in these branches are inequivalent.

Taking M around the surface $\det M = 16\Lambda^{2N_c-4}$ in $\mathcal{M}$, U goes around $16\Lambda^{2N_c-4}$ once, so the monodromy is $(T^2S)T(T^2S)^{-1}$. From the one-loop β-function, we conclude that a pair of dyons $E^\pm$ of charges ± 1 become massless at $U = 16\Lambda^{2N_c-4}$. The effective action when U is near $16\Lambda^{2N_c-4}$ is of the form

$$W \sim (U - 16\Lambda^{2N_c-4})\, E^+ \cdot E^-, \tag{6.52}$$

where $\cdot$ is the pairing between fields of opposite $U(1)$ charges. After adding the mass term $W_{\text{tree}} = \frac{1}{2}\langle M, m\rangle$ with $\operatorname{rank} m = 1$, the $M^\perp$ equation of motion implies that $E^+\cdot E^- = -m^\perp/2\det\hat{M}$. The light fields are in $\hat{M}$ and the superpotential (6.50) agrees with the $\epsilon = 1$ branch of $N_f = N_c - 3$. The $U(1)$ gauge bosons becomes heavy due to the expectation values of $E^\pm$. But since $E^\pm$ are dyons, this is called oblique confinement.

We check 't Hooft's anomaly condition at a generic point M_0 on the moduli space. The light fermions in the infrared theory are the mesons M, the photino λ_0 and, if $\det M = 16\Lambda^{2N_c-4}$, the dyons $E^{\pm}$. When M_0 is non-degenerate and is compatible with the Hermitian form on F, the global symmetry is broken to $SO(F) \times U(1)_R$. The light fermions in the high and low energy theories transform as

(6.53)

		$SO(F)$ $\times$	$U(1)_R$
ultraviolet	ψ_Q	F^*	$-\frac{N_c-2}{N_f} = -1$
	λ	$\mathbf{1}$	1
infrared	ψ_M	S^2F^*	$1 - 2\frac{N_c-2}{N_f} = -1$
	λ_0	$\mathbf{1}$	1
	$(E^{\pm})$	$\mathbf{1}$	0

With or without $E^{\pm}$, the non-zero combinations to be matched are

(6.54)

	ultraviolet	infrared
$\mathfrak{so}(F)^2\mathfrak{u}(1)_R$	$N_c c_2(F) \cdot (-1)$	$c_2(S^2F) \cdot (-1)$
$\mathfrak{u}(1)_R$	$N_c N_f\ (-1) + \frac{1}{2}N_c(N_c-1)\ 1$	$\frac{1}{2}N_f(N_f+1)\ (-1) + 1$
$\mathfrak{u}(1)_R^3$	$N_c N_f\ (-1)^3 + \frac{1}{2}N_c(N_c-1)\ 1^3$	$\frac{1}{2}N_f(N_f+1)\ (-1)^3 + 1^3$

It is easy to check that the two columns match when $N_f = N_c - 2$. The global anomalies also match because all the representations of $SO(F)$ are of real type here. We compare this and the $Sp(N_c)$ theory with $N_f = N_c + 1$. There, the meason M is constrained on the surface (6.17) and there is no massless photon. In the low energies, the fermion ψ_M satisfies a traceless conditon. In the present theory, M can be deformed away from the singular branch and hence there is no traceless condition on ψ_M.

At the other singularity $U = 0$, a non-trivial loop around a point M_0 with $\operatorname{rank} M_0 = r$ is mapped one that goes $N_f - r$ times around 0. So the monodromy is $ST^{2(N_f-r)}S^{-1}$. Therefore the number of pairs of monopoles becoming massless at M_0 is $N_f - r$. This suggest a superpotential

$$W \sim \frac{1}{2\mu}\langle M, \tilde{Q}^+ \cdot \tilde{Q}^- \rangle, \tag{6.55}$$

where μ is of mass dimension 1 and $\tilde{Q}^{\pm}$ represent magnetic monopoles that become massless at the singularity. After adding a mass term $W = \frac{1}{2}\langle M, m \rangle$ with $\operatorname{rank} m = 1$, the fermionic content reduces to the $\epsilon = -1$ branch of $N_f = N_c - 3$. Again, the expectation values of $\tilde{Q}^{\perp}$ lift the $U(1)$ gauge boson. Consequently we see confinement in the original description. The light superfield q in the $N_f = N_c - 3$ theory can be constructed by

$$q = \frac{1}{2\sqrt{m^{\perp}\mu}}(\hat{\tilde{Q}}^+ \cdot (\tilde{Q}^-)^{\perp} - (\tilde{Q}^+)^{\perp} \cdot \hat{\tilde{Q}}^-). \tag{6.56}$$

We check 't Hooft's anomaly matching condition at $M = 0$. In addition to M, the fermionic component $\psi_{\tilde{Q}^\pm}$ of $\tilde{Q}^\pm$ and the photino λ_0 transform as

(6.57)

	$SU(F)$	$\times$	$U(1)_R$
$\psi_{\tilde{Q}^\pm}$	F		0
λ_0	$\mathbf{1}$		1

Their contribution to anomaly is

(6.58)

	infrared: additional contribution from λ_0, $\tilde{Q}^\pm$
$\mathfrak{su}(F)^3$	$2d_F$
$\mathfrak{su}(F)^2\mathfrak{u}(1)_R$	0
$\mathfrak{u}(1)_R$	1
$\mathfrak{u}(1)_R^3$	1^3

It is easy to check that the left column of (6.37) is the sum of the right column and (6.58). The global anomalies also match.

SOLUTIONS TO EXERCISES (BY SIYE WU)
Problems 5–6

Solution to Problem 5

The supersymmetric gauge theory with gauge group $G = Sp(C)$ and quarks $Q \in \mathrm{Hom}(F, C)$ is called the electric theory. When $N_f \geq N_c + 3$, we define a dual theory, called the magnetic theory, with gauge group $\tilde{G} = Sp(\tilde{C})$. The color space $\tilde{C}$ is an Hermitian vector space of dimension $2\tilde{N}_c = 2(N_f - N_c - 2)$ equipped with a compatible symplectic form $\tilde{\omega}_c$. The flavor space is

$$\tilde{F} = F^* \otimes (\wedge^{2N_f} F)^{\frac{1}{2(\tilde{N}_c+1)}}. \tag{7.1}$$

In addition to the dual quarks $\tilde{Q} \in \mathrm{Hom}(\tilde{F}, \tilde{C})$, the magnetic theory has elementary mesons $M \in \wedge^2 F^*$. The fields $\tilde{Q}$, M, their fermionic components $\psi_{\tilde{Q}}$, ψ_M and the gluino $\tilde{\lambda}$ of the magnetic theory transform under the group $SU(F) \times U(1)_R$ according to

(7.2)

	$SU(F)$	$\times$	$U(1)_R$
$\tilde{Q}$	F		$\frac{N_c+1}{N_f}$
$\psi_{\tilde{Q}}$	F		$\frac{N_c+1}{N_f} - 1$
M	$\wedge^2 F^*$		$2(1 - \frac{N_c+1}{N_f})$
ψ_M	$\wedge^2 F^*$		$1 - 2\frac{N_c+1}{N_f}$
$\tilde{\lambda}$	$\mathbf{1}$		1

The global symmetry $SU(F) \times U(1)_R$ is anomaly free in the dual theory; this can be checked by a simple calculation $(\frac{N_c+1}{N_f} - 1) \cdot 2N_f\, c_2(\tilde{C}) + c_2(\tilde{G}) = 0$. The fields $\tilde{Q}$ and M couple through a superpotential

$$W = \frac{1}{2\mu} \langle M, \tilde{Q}^* \tilde{\omega}_c \rangle, \tag{7.3}$$

where

$$\mu \in (\wedge^{2N_f} F^*)^{\frac{1}{N_c+1}} \tag{7.4}$$

has mass dimension 1. The superpotential has R-charge 2 and is invariant under $SU(F)$. Therefore the dual theory has the same global symmetry $SU(F) \times U(1)_R$.

The β-functions of the electric and magnetic theories are given by $b_0 = 3(N_c + 1) - N_f$ and $\tilde{b}_0 = 3(\tilde{N}_c + 1) - N_f = 2N_f - 3(N_c + 1)$, respectively. The scale $\tilde{\Lambda}$ of the dual theory should be related to Λ and μ. Because Λ, $\tilde{\Lambda}$ and μ all have mass dimension 1 and because of (7.4), (5.7) and its magnetic counterpart, the only possible relation among them is

$$\Lambda^{3(N_c+1)-N_f}\tilde{\Lambda}^{3(\tilde{N}_c+1)-N_f} = C(-1)^{\tilde{N}_c+1}\mu^{N_f}, \tag{7.5}$$

where C is a constant to be determined later.

We check that duality is an involution. In the double dual theory, the number of colors is $N_f - \tilde{N}_c - 2 = N_c$, and the flavor space is $\tilde{F}^* \otimes (\wedge^{2N_F}\tilde{F})^{\frac{1}{2(\tilde{N}_c+1)}} \cong F$. There are new mesons $\tilde{M} \in \wedge^2\tilde{F}^*$ and quarks $Q \in \mathrm{Hom}(F, C)$. The total superpotential is

$$W = \frac{1}{2\mu}\langle M, \tilde{M}\rangle + \frac{1}{2\tilde{\mu}}\langle \tilde{M}, Q^*\omega_c\rangle, \tag{7.6}$$

where $\tilde{\mu} \in (\wedge^{2N_f}\tilde{F}^*)^{\frac{1}{\tilde{N}_c+1}} \cong (\wedge^{2N_f}F^*)^{\frac{1}{N_c+1}}$. We propose that $\tilde{\mu} = -\mu$. Then the double dual theory has the same scale Λ as the electric theory, and the equation of motion of $\tilde{M}$ is $M = Q^*\omega_c$. Therefore we recover the electric theory.

The electric theory is infrared free when $b_0 \leq 0$, i.e., when $N_f \geq 3(N_c + 1)$. (When $b_0 = 0$, the two-loop β-function is positive.) It is ultraviolet free when the dual theory is infrared free, i.e., when $\tilde{b}_0 \leq 0$ or $N_f \leq \frac{3}{2}(N_c + 1)$. We claim that for $\frac{3}{2}(N_c+1) < N_f < 3(N_c+1)$, the theory flows to an interacting superconformal field theory in the infrared. The infrared behavior of the $Sp(N_c)$ theories with $N_f \geq N_c + 3$ and their duals is summarized in the following table.

	electric theory	magnetic theory
$N_c \geq 1,\ N_f \geq 3(N_c+1)$ $(\tilde{N}_c + 3 \leq N_f \leq \frac{3}{2}(\tilde{N}_c+1))$	infrared free	strongly coupled
$\frac{3}{2}(N_c+1) < N_f < 3(N_c+1)$ $(\frac{3}{2}(\tilde{N}_c+1) < N_f < 3(\tilde{N}_c+1))$	non-trivial infrared fixed point	non-trivial infrared fixed point
$N_c + 3 \leq N_f \leq \frac{3}{2}(N_c+1)$ $(\tilde{N}_c \geq 1,\ N_f \geq 3(\tilde{N}_c+1))$	strongly coupled	infrared free

That the electric and the magnetic theories describe the same low energy physics will be established through the following highly nontrivial consistency tests.

Chiral operators and the unitarity bound

Though their field contents seem rather different, the electric and the magnetic theories have the same gauge invariant operators. We construct the mapping of gauge invariant chiral operators. The composite meson $Q^*\omega_c$ of the electric theory corresponds to the elementary meson M of the magnetic theory, whereas the composite meson $\tilde{M} = \tilde{Q}^*\tilde{\omega}_c$ of the magnetic theory vanishes by the equation of motion of M.

At the non-trivial infrared fixed-points, the dimensions of the chiral fields can be calculated by their R-charges. In fact,

$$D(M) = \frac{3}{2}R(M) = 3\left(1 - \frac{N_c+1}{N_f}\right). \tag{7.7}$$

The unitarity bound $D(M) \geq 1$ is satisfied if $N_f \geq \frac{3}{2}(N_c+1)$.

't Hooft anomaly matching condition

Under $SU(F) \times U(1)_R$, the fermions $\psi_{\tilde{Q}}$, ψ_M, $\tilde{\lambda}$ transform as (7.2). At $M = 0$, all matter fields are massless and the whole global symmetry group is unbroken. The non-zero contributions to the 't Hooft condition are

(7.8)

	magnetic
$\mathfrak{su}(F)^3$	$2\tilde{N}_c\, d_F - d_{\wedge^2 F}$
$\mathfrak{su}(F)^2\mathfrak{u}(1)_R$	$2\tilde{N}_c\, c_2(F)\cdot(\frac{N_c+1}{N_f}-1) + c_2(\wedge^2 F)\cdot(1-2\frac{N_c+1}{N_f})$
$\mathfrak{u}(1)_R$	$4\tilde{N}_c N_f\,(\frac{N_c+1}{N_f}-1) + N_f(2N_f-1)\,(1-2\frac{N_c+1}{N_f}) + \tilde{N}_c(2\tilde{N}_c+1)\,1$
$\mathfrak{u}(1)_R^3$	$4\tilde{N}_c N_f\,(\frac{N_c+1}{N_f}-1)^3 + N_f(2N_f-1)\,(1-2\frac{N_c+1}{N_f})^3 + \tilde{N}_c(2\tilde{N}_c+1)\,1^3$

Using $d_{\wedge^2 F} = 2(N_f-2)d_F$ and $c_2(\wedge^2 F) = 2(N_f-1)$, the above matches the left column of (6.24).

Deformation along the flat directions

The magnetic theory has the same space of ground states as the electric theory. In the electric theory, the classical moduli space $\mathcal{M}_c$ consists of M satisfying (5.2) as a classical constraint. Since the superpotential is absent, the quantum moduli $\mathcal{M}_q$ space is the same as $\mathcal{M}_c$; we denote both by $\mathcal{M}$. In the magnetic theory, the equations of motion of M set $\tilde{Q} = 0$. The same constraint (5.2) are enforced by non-perturbative quantum effects. In fact, if $\frac{1}{2}\operatorname{rank} M > N_c$, then the number of flavors in the low energy magnetic theory is $N_f - \frac{1}{2}\operatorname{rank} M < \tilde{N}_c + 2$. When $N_f - \frac{1}{2}\operatorname{rank} M \leq \tilde{N}_c$, a superpotential of the form (5.11) is generated, and consequently there is no vacuum. When $N_f - \frac{1}{2}\operatorname{rank} M = \tilde{N}_c + 1$, the light components of $\tilde{M} = \tilde{Q}^*\tilde{\omega}_c$ are constrained by an equation like (6.17) which implies $\tilde{Q} \neq 0$. Therefore the superpotential (7.6) does not have an extremum on the constraint surface and there is no vacuum in this case either. Thus $\frac{1}{2}\operatorname{rank} M \leq N_c$ in the low energy magnetic theory.

We show that the procedure of deforming M in $\mathcal{M}$ commutes with taking the dual. Suppose in the electric theory, M has an expectation value $\langle M\rangle$ with $\frac{1}{2}\operatorname{rank}\langle M\rangle = r$. At low energies, the gauge group is broken to $Sp(N_c - r)$, and there are $N_f - r$ light flavors. Using the previous notations, the mass scale $\hat{\Lambda}$ of the effective theory is given by (6.4). The original magnetic theory has $\tilde{N}_c$ colors and N_f flavors, and the scale $\tilde{\Lambda}$ is related to Λ by (7.5). Now the expectation value $\langle M\rangle$ gives mass to r components of the dual quark $\tilde{Q}$. The low energy magnetic theory has $\tilde{N}_c$ colors and $N_f - r$ flavors. Its scale $\hat{\tilde{\Lambda}}$ is given by

$$\hat{\tilde{\Lambda}}^{3(\tilde{N}_c+1)-(N_f-r)} = \operatorname{Pf}\left(\langle M^{\perp}\rangle/2\mu\right)\tilde{\Lambda}^{3(\tilde{N}_c+1)-N_f}. \tag{7.9}$$

It follows from (6.4) and (7.9) that the relation (7.5) is preserved at low energies. Thus the low energy magnetic theory is dual to the low energy electric theory.

Mass deformation

We now check that mass deformation commutes with taking the dual. Suppose a mass term $\frac{1}{2}\langle M, m\rangle$ with $\frac{1}{2}\operatorname{rank} m = r$ is added to the electric theory. We follow the previous notations. The low energy theory has quarks $\hat{Q} \in \mathrm{Hom}(\hat{F}, C)$. Its scale $\hat{\Lambda}$ is given by (6.7). When $r \leq N_f - N_c - 2$, it has a dual description as well. In the magnetic theory dual to the original electric theory, let $\hat{\tilde{F}} = \hat{F} \otimes (\wedge^{2N_f} F)^{\frac{1}{2(\tilde{N}_c+1)}}$. The equations of motion of the heavy components of M imply that

$$\tilde{Q}^* \tilde{\omega}_c = \begin{pmatrix} \hat{\tilde{M}} & \\ & -2\mu m^\perp \end{pmatrix}, \tag{7.10}$$

where $\hat{\tilde{M}} \in S^2 \hat{\tilde{F}}$ contains the light components. This breaks the $Sp(\tilde{N}_c)$ gauge symmetry to $Sp(\tilde{N}_c - r)$. Integrating out the heavy components in M, $\tilde{Q}$, we obtain a low energy effective superpotential

$$\hat{W} = \frac{1}{2\mu}\langle \hat{M}, \hat{\tilde{M}}\rangle. \tag{7.11}$$

This is exactly a magnetic theory of $\tilde{N}_c - r$ flavors and $N_f - r$ quarks. Its mass scale $\hat{\tilde{\Lambda}}$ is given by the magnetic counterpart of (6.4), i.e.,

$$\tilde{\Lambda}^{3(\tilde{N}_c+1)-N_f} = (-\mu)^r \operatorname{Pf} m^\perp \hat{\tilde{\Lambda}}^{3(\tilde{N}_c-r+1)-(N_f-r)}. \tag{7.12}$$

From (6.7) and (7.12), we see that the relation (7.5) is preserved at low energies. So the low energy theory of the magnetic theory after mass deformation is dual to that of the electric theory.

If $r = N_f - N_c - 2$, then in the low energy electric theory, the number of flavors is $N_f - r = N_c + 2$. In the low energy magnetic theory, the gauge symmetry is completely broken since $\tilde{N}_c - r = 0$. Therefore, the instanton contribution should be included in the magnetic theory just as in the case $N_f = N_c$ for the electric theory. Now the number of zero modes of $\psi_{\tilde{Q}}$ is $2N_f > 2\tilde{N}_c$; these $2(N_c+2)$ extra ones are absorbed by the interaction (7.3), generating a term proportional to the (N_c+2)-th power of $(M/2\mu)_{\text{light}}$ in the effective superpotential. So the instanton contribution is

$$\begin{aligned} 2^{\tilde{N}_c-1}\frac{\tilde{\Lambda}^{3(\tilde{N}_c+1)-N_f}\operatorname{Pf}(M/2\mu)_{\text{light}}}{\operatorname{Pf}(\tilde{M})_{\text{heavy}}} &= \frac{(-1)^r \tilde{\Lambda}^{3(\tilde{N}_c+1)-N_f}\operatorname{Pf}\hat{M}}{2^{N_c+3}\mu^{N_f}\operatorname{Pf} m^\perp} \\ &= -\frac{C \operatorname{Pf}\hat{M}}{2^{N_c+3}\operatorname{Pf} m^\perp \Lambda^{3(N_c+1)-N_f}} = -\frac{C\operatorname{Pf}\hat{M}}{2^{N_c+3}\hat{\Lambda}^{2N_c+1}}, \end{aligned} \tag{7.13}$$

where (7.5), (6.7) have been used. This is exactly (6.21) if we choose $C = 16$.

Solution to Problem 6

Recall that the electric theory has a gauge group $G = SO(C)$ and quarks $Q \in \mathrm{Hom}(F, C)$. The cases with $N_c \geq 4$ and $N_f \leq N_c - 2$ have been treated earlier. When $N_f \geq N_c - 1$, we define a magnetic theory whose gauge group is $\tilde{G} = SO(\tilde{C})$. Here the space of dual colors $\tilde{C}$ is a complex vector space of dimension $\tilde{N}_c = N_f - N_c + 4$. It has a linear metric $\tilde{g}_c$ and a compatible volume form $\tilde{v}_c$. The space of flavors is

$$\tilde{F} = F^* \otimes (\wedge^{N_f} F)^{\frac{1}{N_c-2}}. \tag{7.14}$$

The matter fields of the magnetic theory are quarks $\tilde{Q} \in \mathrm{Hom}(\tilde{F}, \tilde{C})$ and (elementary) mesons $M \in \wedge^2 F^*$.

We first treat the cases $N_c \geq 4$, $N_f \geq N_c$. We concentrate on the identity component $SU(F) \times U(1)_R$ of the global symmetry. The fields $\tilde{Q}$, M, their fermionic components $\psi_{\tilde{Q}}$, ψ_M and the gluino $\tilde{\lambda}$ transform as

(7.15)

	$SU(F)$	$\times$	$U(1)_R$
$\tilde{Q}$	F		$\frac{N_c-2}{N_f}$
$\psi_{\tilde{Q}}$	F		$\frac{N_c-2}{N_f} - 1$
M	S^2F^*		$2(1 - \frac{N_c-2}{N_f})$
ψ_M	S^2F^*		$1 - 2\frac{N_c-2}{N_f}$
$\tilde{\lambda}$	$\mathbf{1}$		1

It is easy to check that $SU(F) \times U(1)_R$ is anomaly free in the dual theory. The fields $\tilde{Q}$ and M couple through a superpotential

$$W = \frac{1}{2\mu}\langle M, \tilde{Q}^* \tilde{g}_c \rangle, \tag{7.16}$$

where

$$\mu \in (\wedge^{N_f} F^*)^{\frac{2}{N_c-2}} \tag{7.17}$$

has mass dimension 1. The superpotential has R-charge 2 and is invariant under $SU(F)$.

When $N_f \geq N_c$, the β-functions of the electric and magnetic theories are given by $b_0 = 3(N_c - 2) - N_f$ and $\tilde{b}_0 = 3(\tilde{N}_c - 2) - N_f = 2N_f - 3(N_c - 2)$, respectively. The scale $\tilde{\Lambda}$ of the magnetic theory is given by

$$\Lambda^{3(N_c-2)-N_f} \tilde{\Lambda}^{3(\tilde{N}_c-2)-N_f} = C(-1)^{\tilde{N}_c} \mu^{N_f}, \tag{7.18}$$

where C is a constant to be determined. It is easy to check that duality is an involution just as the symplectic case. The electric theory is infrared free when $N_f \geq 3(N_c - 2)$. It is ultraviolet free when the dual theory is infrared free, i.e., when $N_f \leq \frac{3}{2}(N_c - 2)$. We claim that when $\frac{3}{2}(N_c - 2) < N_f < 3(N_c - 2)$, the theory flows to an interacting superconformal field theory in the infrared. The infrared behavior of the $SO(N_c)$ theories with $N_f \geq N_c + 3$ and their duals is summarized in the following table.

	electric theory	magnetic theory
$N_c \geq 4$, $N_f \geq 3(N_c - 2)$ $(\tilde{N}_c \leq N_f \leq \frac{3}{2}(\tilde{N}_c - 2))$	infrared free	strongly coupled
$\frac{3}{2}(N_c - 2) < N_f < 3(N_c - 2)$ $(\frac{3}{2}(\tilde{N}_c - 2) < N_f < 3(\tilde{N}_c - 2))$	non-trivial infrared fixed point	non-trivial infrared fixed point
$N_c + 3 \leq N_f \leq \frac{3}{2}(N_c - 2)$ $(\tilde{N}_c \geq 4$, $N_f \geq 3(\tilde{N}_c - 2))$	strongly coupled	infrared free

Chiral operators and the unitarity bound

We construct the mapping of gauge invariant chiral operators. Obviously, the composite mesons $Q^*\omega_c$ of the electric theory correspond to the elementary meson M of the magnetic theory. Recall the baryons $B = Q^*v_c \in \wedge^{N_c}F^*$ in the electric theory. Since the (fermionic) superfield strength W_α is valued in the Lie algebra $\mathfrak{so}(C) \cong \wedge^2 C$, we can define additional gauge invariant fields $b = Q^*v_c(W_\alpha, W^\alpha, \cdot) \in \wedge^{N_c-4}F^*$ and $w_\alpha = Q^*v_c(W_\alpha, \cdot) \in \wedge^{N_c-2}F^*$. The field q in the $N_f = N_c - 3$ theory is proportional to b.

In the dual theory, the baryons $\tilde{B} \in \wedge^{\tilde{N}_c}\tilde{F}^* \cong \wedge^{N_f-N_c+4}F \otimes (\wedge^{N_f}F^*) \otimes (\wedge^{N_f}F^*)^{\frac{2}{\tilde{N}_c-2}}$ and $\tilde{b} \in \wedge^{\tilde{N}_c-4}\tilde{F}^* \cong \wedge^{N_f-N_c}F \otimes (\wedge^{N_f}F^*) \otimes (\wedge^{N_f}F^*)^{-\frac{2}{\tilde{N}_c-2}}$, $\tilde{w}_\alpha \in \wedge^{\tilde{N}_c-2}\tilde{F}^* \cong \wedge^{N_f-N_c+2}F \otimes (\wedge^{N_f}F^*)$. It is easy to check that $R(B) = R(\tilde{b}) = \frac{N_c(N_f-N_c+2)}{N_f}$, $R(b) = R(\tilde{B}) = \frac{(N_c-2)(N_f-N_c+4)}{N_f}$ and $R(w_\alpha) = R(\tilde{w}_\alpha) = 1 + \frac{(N_c-2)(N_f-N_c+2)}{N_f}$. So B and $\tilde{b}$, b and $\tilde{B}$, w_α and $\tilde{w}_\alpha$ have the same quantum numbers, respectively. The mapping of chiral operators in the electric and magnetic theories are given by $B \mapsto \tilde{b}$, $b \mapsto \tilde{B}$ and $w_\alpha \mapsto \tilde{w}_\alpha$, upon choosing a trivialization of $\wedge^{N_f}F$.

At non-trivial infrared fixed-points, the dimension of any chiral field is given by Φ is $D(\Phi) = \frac{3}{2}R(\Phi)$. Since $R(Q) = 1 - \frac{N_c-2}{N_f}$ and $R(W_\alpha) = 1$, we have $D(B), D(b), D(w_\alpha) \geq 1$ and $D(M) = 3(1 - \frac{N_c-2}{N_f}) \geq 1$ if $N_f \geq \frac{3}{2}(N_c - 2)$. The unitarity bounds are satisfied.

't Hooft anomaly matching condition

Under the global symmetry, the light fermions $\psi_{\tilde{Q}}$, ψ_M, $\tilde{\lambda}$ transform according to (7.15). At $M = 0$ in $\mathcal{M}_{\mathrm{q}}$, the whole group is unbroken. The non-zero combinations in the 't Hooft condition are

(7.19)

	magnetic
$\mathfrak{su}(F)^3$	$\tilde{N}_c\, d_F - d_{S^2F}$
$\mathfrak{su}(F)^2\mathfrak{u}(1)_R$	$\tilde{N}_c\, c_2(F) \cdot (\frac{N_c-2}{N_f} - 1) + c_2(S^2F) \cdot (1 - 2\frac{N_c-2}{N_f})$
$\mathfrak{u}(1)_R$	$\tilde{N}_c N_f\, (\frac{N_c-2}{N_f} - 1) + \frac{1}{2}N_f(N_f+1)\, (1 - 2\frac{N_c-2}{N_f}) + \frac{1}{2}\tilde{N}_c(\tilde{N}_c - 1)\, 1$
$\mathfrak{u}(1)_R^3$	$\tilde{N}_c N_f\, (\frac{N_c-2}{N_f} - 1)^3 + \frac{1}{2}N_f(N_f+1)\, (1 - 2\frac{N_c-2}{N_f})^3 + \frac{1}{2}\tilde{N}_c(\tilde{N}_c - 1)\, 1^3$

Using $d_{S^2F} = (N_f + 4)d_F$ and $c_2(S^2F) = (N_f + 2)c_2(F)$, the above matches the left column of (6.37).

Deformation along the flat directions

First, the dual theory has the same moduli space of vacua. In the electric theory, the condition $\operatorname{rank} M \leq N_c$ is a classical constraint. If $\operatorname{rank} M > N_c$ in the magnetic theory, then the number of light flavors is $N_f - \operatorname{rank} M < \tilde{N}_c - 4$. So a superpotential of the form (5.22) is generated and consequently the quantum theory has no vacuum. If $\operatorname{rank} M = N_c$, then according to (5.12), there are two choices of sign of $B = \pm(\wedge^{N_c}M)^{1/2}$ in the electric theory. In the dual theory, the number of light flavors is $N_f - \operatorname{rank} M = \tilde{N}_c - 4$. So there are two branches. One branch has a superpotential (6.35) with $\epsilon_+ = \epsilon_-$ and hence does not have a vacuum. The other branch has two vacua, labeled by $\epsilon_+ = -\epsilon_- = \pm 1$. Therefore the magnetic theory also has two vacua for each M with $\operatorname{rank} M = N_c$.

We check that flat deformations on the moduli space commutes with taking the dual. If M acquires an expectation value $\langle M \rangle$ with $\operatorname{rank} \langle M \rangle = r$, then the low energy electric theory has N_c colors and $N_f - r$ flavors. In the magnetic theory, the expectation value $\langle M \rangle$ gives mass to some of the dual quarks. The low energy magnetic theory has $\tilde{N}_c - r$ colors and $N_f - r$ flavors. This is dual to the low energy electric theory. Further, using (6.25) and the magnetic analog of (6.28), one can show that the relation (7.18) is preserved in the low energy theories.

Mass deformation

If we add a mass term $W_{\text{tree}} = \frac{1}{2}\langle M, m\rangle$ with $\operatorname{rank} m = r$ in the electric theory, then the low energy theory has N_c colors and $N_f - r$ flavors. Its scale $\hat{\Lambda}$ is given by (6.28). When $r \le N_f - N_c + 1$, this low energy theory has a dual theory. The latter has $\tilde{N}_c - r$ colors and $N_f - r$ flavors. Its scale $\hat{\tilde{\Lambda}}$ is given by the magnetic analog of (6.25). It is easy to show that the relation (7.18) is preserved in low energy theories and so is the form of the superpotential (7.16) after integrating out the heavy fields.

When $r = N_f - N_c + 1$, the number of colors in the low energy magnetic theory is $\tilde{N}_c - r = 3$. Just as the electric theory when $N_f = N_c - 3$, we consider an intermediate stage in which the gauge group is broken to $SO(4) \cong (SU(2)_+ \times SU(2)_-)/\mathbb{Z}_2$. When the groups $SU(2)_\pm$ are further broken, there are instanton generated superpotentials. Now the number of zero modes of $\psi_{\tilde{Q}}$ is $2N_f > 2(\tilde{N}_c - 3)$. The extra $2(N_c - 1)$ ones are absorbed by the interaction (7.16), generating a term proportional to the $(N_c - 1)$-th power of $(\mu^{-1}M)_{\text{light}}$. Thus the superpotential contains an extra term

$$4\frac{\tilde{\Lambda}^{3(\tilde{N}_c-2)-N_f}\det(\mu^{-1}M)_{\text{light}}}{(\det \tilde{M})_{\text{heavy}}} = \frac{4(-1)^r\tilde{\Lambda}^{3(\tilde{N}_c-2)-N_f}\det\hat{M}}{\mu^{N_f}\det m^{\perp}}$$
$$= -\frac{4C\det\hat{M}}{\det m^{\perp}\Lambda^{3(N_c-2)-N_f}} = -\frac{4C\det\hat{M}}{\hat{\Lambda}^{2N_c-5}}. \tag{7.20}$$

Here we have used (7.18) and (6.28). So the total superpotential in the low energy magnetic theory is

$$\hat{\tilde{W}} = \frac{1}{2\mu}\langle \hat{M}, \hat{\tilde{Q}}^*\tilde{g}_c\rangle - \frac{4C\det\hat{M}}{\hat{\Lambda}^{2N_c-5}}. \tag{7.21}$$

Finally, we derive the relation between the scales of the low energy theories. The scale $\hat{\Lambda}$ of the low energy electric theory is given by (6.28). The scale $\hat{\tilde{\Lambda}}$ of the low energy magnetic theory is given by the magnetic analog of (6.27), i.e.,

$$4(\tilde{\Lambda}^{3(\tilde{N}_c-2)-N_f})^2 = \det(-\mu m^{\perp})^2\hat{\tilde{\Lambda}}^{6-2(N_f-\tilde{N}_c+3)}. \tag{7.22}$$

Using (6.28) and (7.22), the relation (7.18) reduces at low energies to

$$(\hat{\Lambda}^{2N_c-5})^2\hat{\tilde{\Lambda}}^{6-2(N_c-1)} = 4C^2\mu^{2N_c-2}. \tag{7.23}$$

$N_f = N_c - 1$

We consider the theory with $N_f = N_c - 1$ and scale Λ. The electric theory is strongly coupled at low energies. From (7.21), the superpotential of the magnetic theory is

$$W = \frac{1}{2\mu}\langle M, \tilde{Q}^* \tilde{g}_c\rangle - \frac{4C \det M}{\Lambda^{2N_c - 5}}. \tag{7.24}$$

From (7.23), the scale of the dual theory is given by

$$(\Lambda^{2N_c-5})^2 \tilde{\Lambda}^{6-2(N_c-1)} = 4C^2 \mu^{2N_c-2}. \tag{7.25}$$

In fact, up to a constant, (7.25) is the only possible relation consistent with (7.17), (5.17), the magnetic analog of (5.18), and the fact that Λ, $\tilde{\Lambda}$, μ all have mass dimension 1. Under a flat deformation, (7.25) is satisfied by the low energy electric and magnetic theories.

We show that adding the mass term $W_{\text{tree}} = \frac{1}{2}\langle M, m\rangle$ with $\operatorname{rank} m = 1$ reduces the theory to $N_f = N_c - 2$. The total superpotential is $W + W_{\text{tree}}$. Following previous notations, we have $\tilde{F} = \hat{\tilde{F}} \oplus \hat{\tilde{F}}^{\perp}$ with $\dim \hat{\tilde{F}}^{\perp} = 1$. The dual quarks have the decomposition $\tilde{Q} = (\hat{\tilde{Q}}, \tilde{Q}^{\perp}) \in \operatorname{Hom}(\hat{\tilde{F}}, \tilde{C}) \oplus \operatorname{Hom}(\hat{\tilde{F}}^{\perp}, \tilde{C})$. The $M^{\perp}$ equations of motion imply

$$(\tilde{Q}^{\perp})^* \tilde{g}_c = \frac{8C\mu \det \hat{M}}{\Lambda^{2N_c-5}} - \mu m^{\perp}. \tag{7.26}$$

In particular, $\operatorname{im}(\hat{\tilde{Q}}^{\perp})$ is not isotropic in $\tilde{C}$. We write $\tilde{C} = \hat{\tilde{C}} \oplus \hat{\tilde{C}}^{\perp}$, where $\hat{\tilde{C}}^{\perp} = \operatorname{im}(\hat{\tilde{Q}}^{\perp})$. The value of $\tilde{Q}^{\perp}$ breaks the dual gauge group $SO(\tilde{C})$ to $SO(\hat{\tilde{C}}) \cong SO(2)$. The equations of motion imply that M is block-diagonal and that $\tilde{g}_c(\hat{\tilde{Q}}, \hat{\tilde{Q}}^{\perp}) = 0$, i.e., $\hat{\tilde{Q}} \in \operatorname{Hom}(\hat{\tilde{F}}, \hat{\tilde{C}})$. Let $\hat{\tilde{C}} = \hat{\tilde{C}}^{+} \oplus \hat{\tilde{C}}^{-}$, where the weights of $SO(\hat{\tilde{C}})$ in $\hat{\tilde{C}}^{\pm}$ are ± 1, respectively. We write $\hat{\tilde{Q}} = (\hat{\tilde{Q}}^{+}, \hat{\tilde{Q}}^{-})$, where $\hat{\tilde{Q}}^{\pm} \in \operatorname{Hom}(\hat{\tilde{F}}, \hat{\tilde{C}}^{\pm})$. Then the superpotential $W + W_{\text{tree}}$ reduces in the low energies to

$$\hat{W} \sim \frac{1}{2\mu}\langle \hat{M}, \hat{\tilde{Q}}^{+} \cdot \hat{\tilde{Q}}^{-}\rangle, \tag{7.27}$$

where $\cdot$ is the pairing between $\hat{\tilde{C}}^{+}$ and $\hat{\tilde{C}}^{-}$. This matches (6.55). Due to the instanton effects in symmetry breaking, it is only an approximate result near $\hat{M} = 0$. When $\det \hat{M}$ is large, the quarks $\hat{\tilde{Q}}$ become heavy. After integrating out the heavy fields, the effective superpotential is

$$W = \frac{1}{2\mu} M^{\perp} \left(4u - \frac{8C\mu \det \hat{M}}{\Lambda^{2N_c-5}} + \mu m^{\perp} \right), \tag{7.28}$$

where $u = \tilde{g}_c(\tilde{Q}^{\perp}, \tilde{Q}^{\perp})/4$. The effective theory, which is the $SO(\tilde{C}) \cong SO(3)$ gauge theory with a single flavor $\tilde{Q}^{\perp}$, has $N = 2$ supersymmetry. Its scale $\tilde{\Lambda}'$ is given by

$$\tilde{\Lambda}'^4 = \det(\mu^{-1}\hat{M})^2 \tilde{\Lambda}^{6-2(N_c-1)} = \left(\frac{2C\mu \det \hat{M}}{\Lambda^{2N_c-5}} \right)^2. \tag{7.29}$$

The superpotential (7.28) lifts the vacuum degeneracy of the u-plane except at the two singularities $u = \pm \tilde{\Lambda}'^2 = \pm 8C\mu \det \hat{M}/\Lambda^{2N_c-5}$. There is still no vacuum with the plus sign. With the minus sign, there is a vacuum if $\hat{M}$ satisfies

$$\det \hat{M} = \frac{m^{\perp} \Lambda^{2N_c-5}}{16C} = \frac{\hat{\Lambda}^{2N_c-4}}{16C}. \tag{7.30}$$

This matches the non-zero singular branch of the $N_f = N_c - 2$ theory when $C = 2^{-8}$.

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