— DRAFT — IASSNS-HEP-97/72

Lecture I

Point Particles vs Strings

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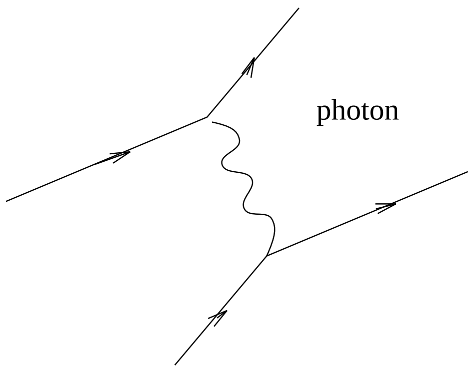
In physics, the structure of matter is understood in terms of the dynamics of smaller constitutents. Atoms are composed of nuclei and electrons, nuclei are made up of protons and neutrons, which in turn are bound states of quarks. To the present day limit of distance scales up to approximately 10^{-15} cm, electrons and quarks do not show any substructure, and are viewed as "elementary particles". (Which particles are elementary in this way may change over time, since at smaller distance scales substructure may be found in terms of more elementary particles.) In quantum field theory, such elementary particles are described as *points* in space.

Forces between particles result from exchanges between them of other elementary particles, and the fundamental interactions occur in a *local* way, at coincident time at the same point in space. For example electro-magnetic interaction forces between charged particles result from the exchange of photons, with local couplings to the charged particles.



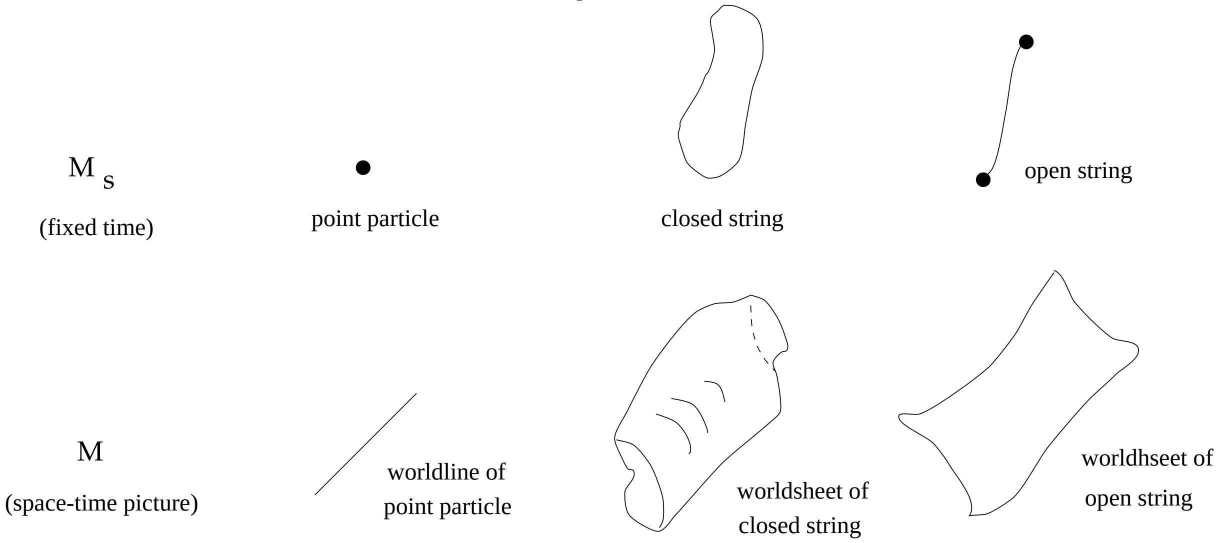
The photon thereby transfers momentum and produces a force, the Coulomb force.

Quantum field theory of point particles (QFT) has been extremely successful in describing quantitatively three out of the four fundamental forces of Nature by *Yang-Mills theory*: electro-magnetic, weak and strong interactions. However, no consistent quantum field theory description appears to be available (or to exist?) for the force of gravity. One thus seeks an enlargement of the framework of QFT in which all forces of nature fit consistently with the known physical principles of quantum mechanics and (special and general) relativity.

String theory to date seems to be the only such candidate. In fact my task will be to show that consistent string theories automatically contain gravity, supersymmetry and under certain circumstances Yang-Mills.

A. **Strings**

In string theory, particles are not described as points, but instead as strings: one-dimensional extended objects. If M_s and $M(\sim \mathbb{R} \times M_s)$ denote the space and space-time manifolds respectively, then we picture strings as follows:



While the point particle sweeps out a one-dimensional *worldline*, the string sweeps out a *worldsheet*, i.e. a 2-dim real surface. For a *free string*, the topology of the worldsheet is a *cylinder* (in the case of a closed string) or a *sheet* (for an open string).

Roughly, different elementary particles correspond to different vibration modes of the string jus as different minimal notes correspond to different vibrational modes of musical string instruments.

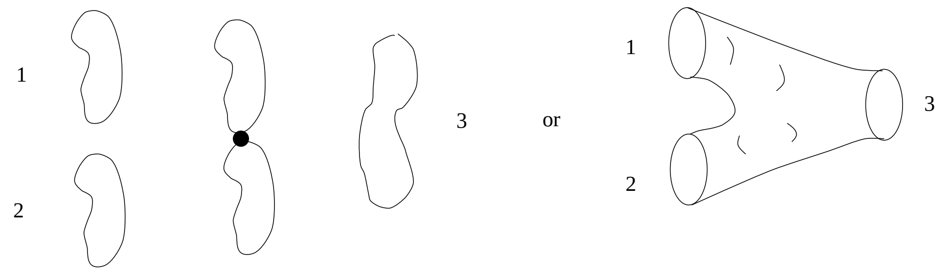
It will turn out that the physical size of strings is set by gravity, more precisely the Planch length $\ell_P = \sqrt{G_N \hbar c^{-3}} \sim 10^{-33}$ cm. This scale is so small that we effectively only see point particles at our distance scales. Thus, for length scales much larger than ℓP, we expect to recover a quantum field theory description of point particles, plus typical string corrections that represent physics at the Planck scale.

Extended objects arise in many areas of physics. For examples *flux tubes* arise in superconductivity, and are used to describe the physics of cosmic strings and vortices. However, those objects do have a transverse size, and are only *approximately one dimensional.* String theory considered here will always involve strictly one-dimensional objects, suitable for the problems of quantum gravity and particle physics. In certain instances, it may be possible to use the strictly one-dimensional string modules as approximations to the dynamics of flux tubes.

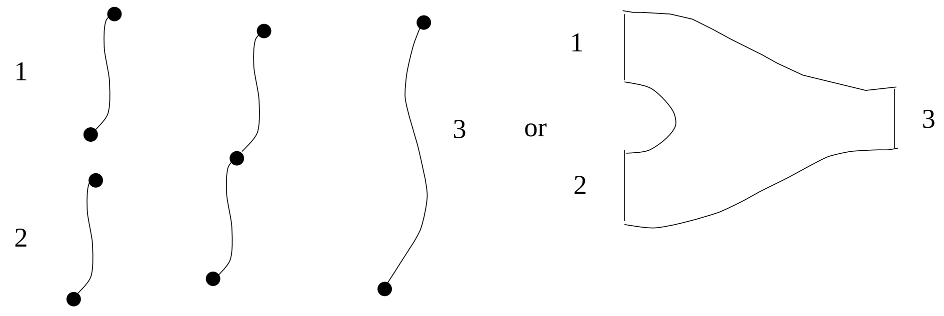
B. **Interactions**

While the string itself is extended, the fundamental interactions are *local*, just as for point particles. The interaction takes place when strings overlap in space at the same time.

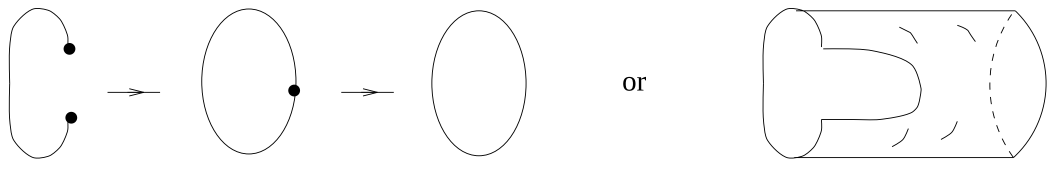
For *closed* strings



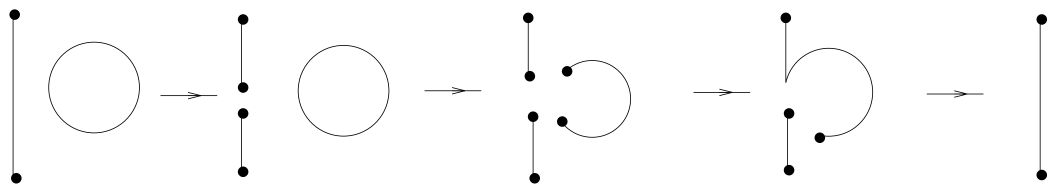
For *open* strings, end points can join



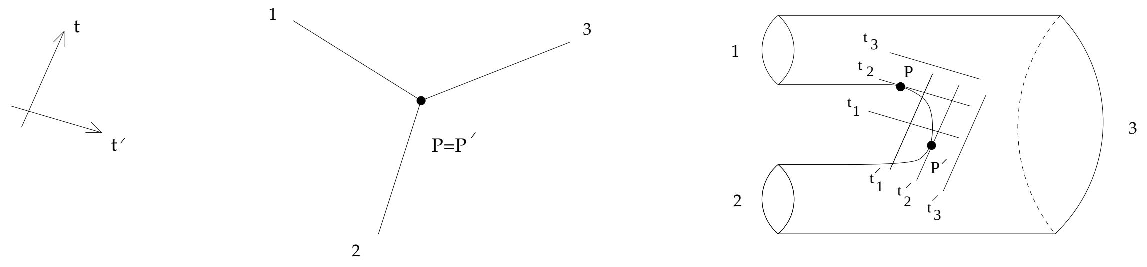
For consistency, as soon as an interacting theory contains open strings, it must also contain closed strings, since end points can join. Such theories are "*open string theories*".



On the other hand, it is consistent to have a theory of only closed strings, called *closed string theories.* Other interactions result from combining the interactions defined above



In point particle theories, the fundamental interactions are read off from the QFT Lagrangian. An interaction occurs at a *geometrical point*, where the worldlines join and cease to be a manifold. In Lorentz invariant theories (with M flat Minkowski space-time), the interaction point is Lorentz invariant. To specify how the point particles interact, additional data must be supplied at the interaction point, giving rise to *many possible* distinct quantum field theories.



In string theory, the interaction point *depends* upon the Lorentz frame chosen to observe the process. In the figure above, equal time slices are indicated from the point of view of two different Lorentz frames, schematially indicated by t and t'. The closed string interaction, as seen from frames t and t', occurs at times t_2 and t'_2 and at (distinct) points P and P' respectively.

Lorentz invariance of interaction forbids that any point on the worldsheet be singled out as interaction point. Instead, the interaction results purely from the joining and splitting of strings. While free closed strings are characterized by their topology being that of a cylinder, interacting strings are characterized by the fact that their associated worldsheet is connected to at least 3 strings, incoming and/or outgoing.

As a result, the free string determines the nature of the interactions completely, leaving only the string coupling constant undetermined.

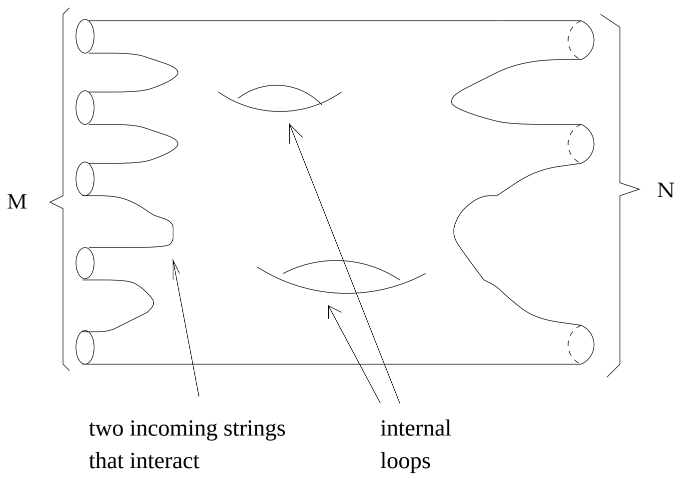
Additional structure

- Oriented strings: all worldsheets are assumed to be orientable.
- Non-oriented strings: worldsheets that are non-orientable such as the Möbius strip, Klein bottle, are included as possible configurations, as well.

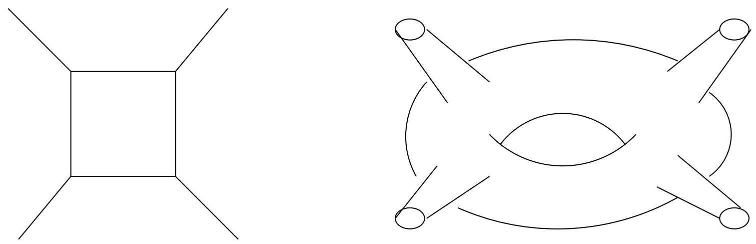
We shall almost exclusively deal with *oriented* strings only.

C. **Loop expansion – Topology of closed surfaces**

For simplicity consider closed oriented strings only, so that the associated worldsheet is also oriented. A general string configuration describing the process in which M incoming strings interact and produce N outgoing strings looks at the topological level like a closed surface with $M+N=E$ boundary components and any number of handles.

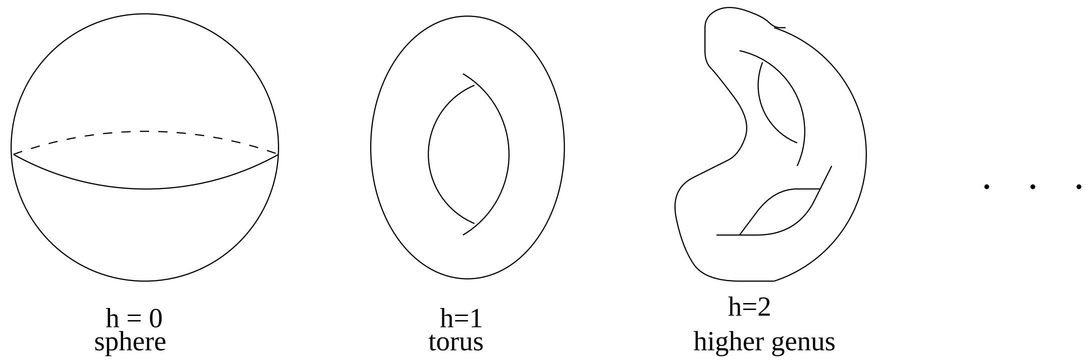


The internal loops may arise when virtual particle pairs are produced, just as in quantum field theory. For example, a Feynman diagram in quantum field theory that involves a loop is shown below together with its corresponding in string theory:



Surfaces associated with closed oriented strings have two topological invariants: the *number of boundary components* E (as we shall see later, boundary components may be shrunk to punctures, under certain conditions), and the number h of *handles* on the surface, which equals the *genus*.

When $E = 0$, we just have the topological classification of compact oriented surfaces without boundary:



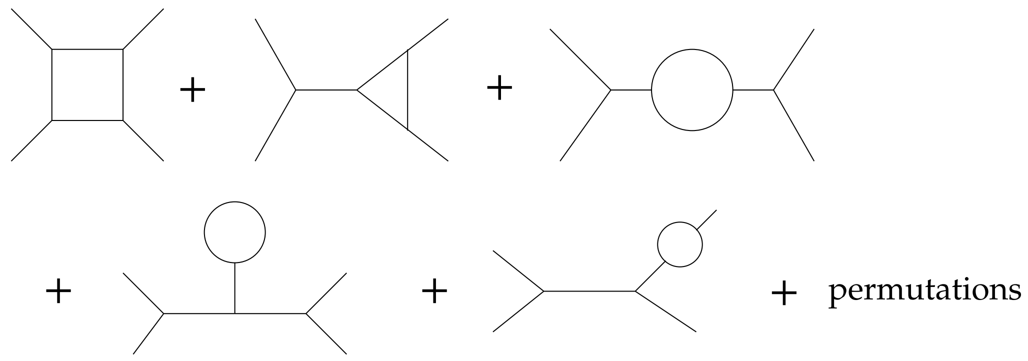
Rendering $E > 0$ is achieved by removing E discs from the surface.

Recall that in quantum field theory an expansion in powers of Planck's constant $\hbar$ yields an expansion in the number of *loops* of the associated Feynman diagram, for a given number of external states:

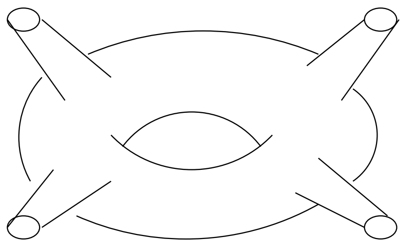
$$\hbar^{E+h-1} \begin{cases} \hbar & \text{for every propagator} \\ \hbar^{-1} & \text{for every vertex} \\ -1 & \text{for overall momentum conservation.} \end{cases}$$

Thus, in string theory we expect that, for a given number of external strings E, the topological expansion genus by genus should correspond to a *loop* expansion as well. (See Kazhdan's 4-th lecture.)

In quantum field theory, there are in general many Feynman graphs that correspond to an amplitude with a given number of external particles and a given number of loops, e.g. for $E = 4$ external particles and $h = 1$ loop in ϕ^3 theory we have



The same process in string theory (for closed oriented strings) is described by just a single diagram

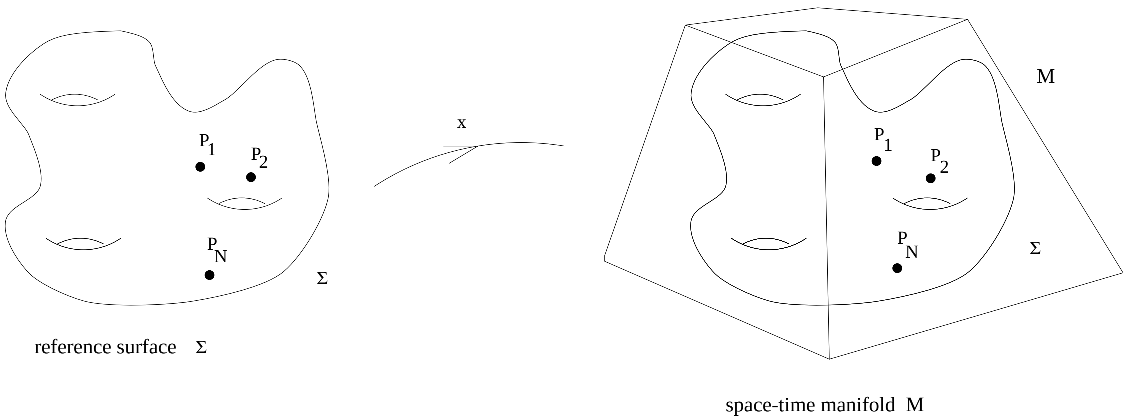


Later on in the lectures we will show that the single string diagram reproduces all the field theory diagrams. The different field theory diagrams arise from different cells in a *cell-decomposition of the moduli space of surfaces.*

D. **Transition Amplitudes for Strings**

The fundamental quantities we are interested in examining in a quantum field theory are the transition amplitudes for processes in which a number of M incoming particles scatter to produce N outgoing particles. (The square modulus of the amplitude yields the probability for this process to take place.)

The only way we have today to define string theory is by giving a *rule* for the evaluation of transition amplitudes, order by order in the loop expansion, i.e. genus by genus. The rule is to assign a relative weight to a given configuration and then to sum over all configurations. To make this more precise, we first describe the space of configurations.



We assume that Σ and M are differentiable manifolds, of dimensions 2 and D respectively, and that x is a continuous map from Σ to M. If ξ^m, $m = 1, 2$, are local coordinates on Σ and x^μ, $\mu = 1, \dots, D$, are local coordinates on M then the map x may be described by functions $x^\mu(\xi^m)$ which are continuous.

To each configuration is associated a weight

$$e^{-S[x,\Sigma,M]}, \qquad S \in \mathbb{C}$$

and the transition amplitude for specified external strings (incoming and outgoing) is obtained by summing over all surfaces Σ and all possible maps x.

$$A = \sum_{\text{surfaces } \Sigma} \sum_x e^{-S[x,\Sigma,M]}$$

We now need to specify each of these ingredients:

1) We assume M to be a D-dimensional Riemannian manifold, with metric $(\ ,\ )_G$. A special case is flat Euclidean space-time $\mathbb{R}^D$. The space-time metric is assumed *fixed.*

$$ds^2 = (dx, dx)_G = \sum_{\mu,\nu=1}^{D} G_{\mu\nu}(x) dx^\mu \otimes dx^\nu \tag{1.1}$$

The physical world is of course not described by a Riemann manifold, but rather by a pseudo-Riemann space, such as Minkowski space-time. We shall always take as a starting definition the Riemannian situation, and analytically continue to pseudo-Riemannia spaces to obtain physical transition amplitudes. For flat Euclidean space-time, this continuation is analogous to the correspondence between Schwinger and Wightman functions in QFT (see Kazhdan's lectures).

2) The metric G on M induces a metric on Σ: $\gamma = x^*(G)$

$$\gamma = \gamma_{mn} d\xi^m \otimes d\xi^n$$
$$\gamma_{mn} = G_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^m} \frac{\partial x^\nu}{\partial \xi^n}$$

This metric is non-negative, but depends upon x. It is advantageous to introduce an intrinsic Riemannian metric g on Σ, independently of x; in local coordinates, we have

$$g = g_{mn}(\xi) d\xi^m \otimes d\xi^n. \tag{1.2}$$

3) The worldsheet action $S[x, \Sigma, M]$ will in general depend on the map x. We choose S to be intrinsic, with the following properties

$$\begin{cases} \text{Diff}^+(\Sigma) & \text{invariant} \\ \text{Diff(M)} & \text{invariant} \\ \text{local in its dependence on } x, g, \text{ and } G \\ \text{renormalizable as a QFT.} \end{cases}$$

A natural intrinsic candidate for S is the area of $x(\Sigma)$, which gives the *Nambu-Goto action*

$$\text{Area}\,(x\,(\Sigma)) = \int_\Sigma d\mu_\gamma = \int_\Sigma d^2\xi \sqrt{\det\ \gamma_{mn}} \tag{1.3}$$

It depends only upon G and x, but not on g. The difficulty with this action is that it is non-polynomial in x and its derivatives, even in flat space-time. Its quantization is difficult to define unambiguously.

Instead, we take as starting point the *Polyakov action*

$$\begin{aligned} S[x, g, G] &= \kappa \int_\Sigma (dx, *dx)_G \\ &= \kappa \int_\Sigma d\mu_g g^{mn} \partial_m x^\mu \partial_n x^\nu G_{\mu\nu}(x) \end{aligned} \tag{1.4}$$

Here κ is the *string tension*, which is a positive constant with dimension of inverse length square. The stationary points of S with respect to g are at $g^0 = e^\phi \gamma$ for some function ϕ on Σ, and thus $S[x, g^0, G] \sim \text{Area}\,(x(\Sigma))$.

The Polyakov action leads to well-defined transition amplitudes, obtained by integration over the space $\text{Met}(\Sigma)$ of all positive metrics on Σ for a given topology, as well as over the space of all maps Map (Σ, M): We *define*

$$A = \sum_{\substack{\text{topologies}\\ \Sigma}} \int_{\text{Met}(\Sigma)} \frac{1}{\mathcal{N}(g)} \int_{\text{Map}(\Sigma,M)} Dx\, e^{-S[x,g,G]}, \tag{1.5}$$

and make the following remarks.

- $\mathcal{N}$ is a normalization factor, to be determined later.
- The measures Dg and Dx are constructed from $\text{Diff}^+(\Sigma)$ and Diff(M) invariant L^2 norms on Σ and M.
- For fixed metric g, the action S is well-known: its stationary points are the harmonic maps x: $\Sigma \to M$. Here, however, g varies and in fact is to be integrated over. (This is a reflection of the difference between quantum field theory and strings.)
- For a general metric G, the action S defines a *non-linear sigma model*, which is renormalizable because the dimension of Σ is 2. It would not in general be renormalizable in dimension higher than 2, which is usually regarded as an argument against the existence of fundamental membrane theories.

E. **Weyl Invariance and Vertex Operator Formulation**

The action S is also invariant under *Weyl rescalings* of the metric g by a positive function on Σ:

$$g \to e^{2\sigma} g \qquad\qquad \sigma\colon \Sigma \to \mathbb{R}.$$

In general, Weyl invariance of the full amplitude may be spoiled by anomalies. We shall establish later on that Weyl invariance is a crucial ingredient to obtain a consistent theory.

Assuming Weyl invariance of the full amplitude, the integral defining A may be *simplified* in two ways.

1) The integration over $\mathrm{Met}(\Sigma)$ effectively collapses to an integration over the *moduli space of surfaces*, which is finite dimensional, for each genus h.

2) The boundary components of Σ — characterizing external string states — may be mapped to regular points on an underlying compact surface without boundary by conformal transformations. The *data*, such as momenta and other quantum numbers of the external states, are mapped into *vertex operators*. The amplitudes are now

$$A = \sum_{h=0}^{\infty} \int_{\mathrm{Met}(\Sigma)} Dg \frac{1}{\mathcal{N}(g)} \int_{\mathrm{Maps}(\Sigma,\mathrm{M})} Dx\, V_1 \ldots V_N e^{-S} \tag{1.6}$$

for suitable vertex operators $V_1, \ldots V_N$. We shall construct such vertex operators explicitly in Lecture 3.

F. More general actions

The action S given above is not the most general one that satisfies the criteria we listed. Generalizations are possible when M carries extra structure.

1) M carries a 2-form $B \in \Omega^{(2)}(M)$, the so-called it anti-symmetric tensor field. The resulting contribution to the action is also that of a "non-linear sigma model""

$$S_B[x, B] = \int_\Sigma x^*(B) = \int_\Sigma dx^\mu \wedge dx^\nu B_{\mu\nu}(x) \tag{1.7}$$

2) M may carry a *dilaton field* $\Phi \in \Omega^{(0)}(M)$ so that

$$S_\Phi[x, \Phi] = \int_\Sigma d\mu_g R_g \Phi(x). \tag{1.8}$$

where R_g is the Gaussian curvature of Σ for the metric g.

3) There may be a *tachyon field* $T \in \Omega^{(0)}(M)$ contributing

$$S_T[x, T] = \int_\Sigma d\mu_g T(x). \tag{1.9}$$

G. Transition Amplitude for a single point particle

The transition amplitude for a single point particle could in fact be obtained in a way analogous to how we prescribed string amplitudes. Let space-time be again a Riemannian manifold M, with metric $(\ ,\ )_G$. The prescription for the transition amplitude of a particle travelling from a point $y \in M$ to a point y' to M is expressible in terms of a sum over all (continuous) paths connecting y to y':

$$A(y,y') = \sum_{\substack{\text{paths} \\ \text{joining } y \text{ and } y'}} e^{-S[\text{path}]}$$

Paths may be parametrized by maps from $C = [0,1]$ into M with $x(0) = y$, $x(1) = y'$. A simple worldline action for a massless particle is obtained by introducing a metric g on $[0,1]$

$$S[x,g] = \frac{1}{2}\int_C d\tau\, g(\tau)^{-1}\dot{x}^\mu \dot{x}^\nu G_{\mu\nu}(x) \qquad\qquad \dot{x} = \frac{dx}{d\tau}$$

which is invariant under Diff$^+$(C) and Diff(M).

The analogous prescription for the point particle transition amplitude is

$$A(y,y') = \int_{\text{Met(C)}} Dg \frac{1}{\mathcal{N}} \int_{\text{Diff(C,M)}} Dx\, e^{-S[x,g]}$$

Notice that for string theory, we had a prescription for transition amplitudes valid for all topologies of the worldsheet. For point particles, there is only the topology of the interval C, and we can only describe a single point particle, but not interactions with other point particles. To put those in, we would have to supply additional information.

It is very instructive to work out the amplitude A by carrying out the integrations. The only Diff$^+$(C) invariant of g is the length $L = \int_0^1 d\tau\, g(\tau)$; all else is generated by Diff$^+$(C), and may be gauge-fixed following the standard Faddeev-Popov procedure. Defining the normalization factor to be the volume of Diff(C): $\mathcal{N} = \text{Vol(Diff(C))}$ we have $Dg = Dv\, dL$ and the transition amplitude becomes

$$\begin{aligned} A(y,y') &= \int_0^\infty dL \int Dx\, e^{-\frac{1}{2L}\int_0^1 d\tau (\dot{x},\dot{x})_G} \\ &= \int_0^\infty dL \Big\langle y'|e^{-L\Box}|y\Big\rangle = \Big\langle y'|\frac{1}{\Box}|y\Big\rangle. \end{aligned}$$

Thus, the amplitude is just the Green function at (y,y') for the Laplacian (for Euclidean space-time) or the d'Alembertian (in the Minkowski case), and corresponds to the propagation of a massless particle.

H. **Generalized Point Particle Propagation**

The manifold M may carry structure in addition to the metric $(\ ,\ )_G$, which is reflected in the point particle action.

1. An $U(1)$ line bundle on M with connection A gives rise to

$$S_A = ie \int_C x^*(A) = ie \int_C d\tau\, \dot{x}^\mu\, A_\mu(x), \tag{1.10}$$

   i.e. point particle with charge e in the presence of a $U(1)$ gauge field.

2. An SU(N) vector bundle on M with connection A. To describe a particle in the defining representation ρ of SU(N), introduce worldline fermions $\lambda^i(\tau)$ $i = 1, \ldots, N$ and the action

$$S_{A_\rho} = \int_C \bar{\lambda}(d + x^*(A_\rho))\lambda \tag{1.11}$$

3. The spin bundle over M with spin connection ∂, describes space-time spinors of M. Introduce worldline fermions ψ and the action

$$S_\omega = \int_C (\psi, \partial\psi)_G \tag{1.12}$$

   The total particle worldline action $S + S_\omega$ has worldline supersymmetry. (See Witten's problem sets.)