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Lectures 11 and 12

VII. Free Superstrings

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Purely bosonic string theory, as defined and studied in the preceding lectures, has a number of fundamental flaws, which we shall now briefly summarize.

(i) Bosonic string theory in flat — or nearly flat — space-time possesses a tachyonic state. The presence of a tachyon leads to a violation of the physical principle of causality. It signals an instability of (nearly) flat space-time, which may be detected from the fact that the 1-loop order dilaton tadople amplitude is non-zero (in fact it is infinite). Ultimately, the presence of the tachyon leads to the divergence of transition amplitudes beyond tree level. Thus, bosonic string, in (nearly) flat space-time is in itself inconsistent.

(ii) Bosonic string theory does not possess any fermionic states (i.e. transforming under a spinor representation of the space-time Lorentz group). This fact by itself does not à priori lead to an inconsistency of the bosonic string. Rather, since Nature clearly displays fermionic states (electrons, quarks, etc.), any viable physical theory aimed at encompassing all physical particles in Nature, as well as their interactions, must contain fermions. From this point of view, the bosonic string cannot be a viable theory of Nature.

(iii) In addition, it is desirable to have a theory of closed strings only that contains Yang-Mills gauge states. (We have shown already how Yang-Mills particles arise in open string theories.)

(iv) Finally, we do not perceive 26 dimensions of space-time, but only 4, at distance scales accessible to present day experiments. Thus, it must be understood why and how the remaining dimensions have remained invisible.

Problems (i), (ii) and in some cases also (iii) may all be resolved by introducing certain additional dynamical degrees of freedom on the string worldsheet. In particular, extra degrees of freedom may be introduced that will result in fermionic string states in the physical Hilbert space. This extension will in general modify the critical dimension D of space-time.

There are several ways in which one can proceed to include fermionic string states in the Hilbert space of string states.

(1) *The Green-Schwarz (GS) formulation.*

The theory has space-time vector fields x^μ, transforming under the vector representation of $\mathrm{SO}(1, D-1)$ as well as space-time spinor fields θ^α transforming under a spinor representation of $\mathrm{Spin}(1, D-1)$. The ground state (i.e. lowest mass state of a single string) is in general degenerate, with both vector and spinor states. Bosonic and fermionic states are obtained by applying x^μ or θ^α to the ground states.

(2) *The Neveu-Schwarz-Ramond (RNS) formulation.*

The fundamental fields in the theory are space-time vectors x^μ, ψ^μ. There are two sectors in the Hilbert space of states, built on two distinct ground states: The Neveu-Schwarz ground state is a boson, while the Ramond ground state is a fermion. All bosonic states are obtained by applying x^μ and ψ^μ to the Neveu-Schwarz ground state; this generates the Neveu-Schwarz sector of the theory. All fermionic states are obtained by applying x^μ and ψ^μ to the fermionic ground state; this generates the Ramond sector of the theory.

By quantizing each formulation in the lightcone gauge, one demonstrates that (under certain coniditons) the GS and RNS formulations are equivalent. The Green-Schwarz formulation apears difficult (if not impossible) to quantize in a manifestly Lorentz covariant way.

One distinguishes the following five, consistent, tachyon free, space-time supersymmetric string theories in flat Minkowski space-time.

•	Type I	Open and closed unoriented strings, with Yang-Mills degrees of freedom via Chan-Paton rules: SO(32) gauge group only.
•	Type II A,B	Closed oriented strings only; A: opposite chirality Ramond ground states B: same chirality Ramond ground states.
•	Heterotic Spin(32)$/Z_2$, $E_8 \times E_8$	Type II left-movers, bosonic right movers, including gauge degrees of freedom: Spin(32)$/Z_2$ or $E_8 \times E_8$ gauge groups only.

The goal of this Chapter (§VII) and the next one (§VIII) is to define these theories, show their uniqueness and analyze the Hilbert space of states, in the RNS formulation. Space-time supersymmetry and the GS formulation will be discussed extensively in §X.

A) Degrees of Freedom of the RNS string

To obtain fermionic physical states, we add further degrees of freedom which are *spinors on the worldsheet* Σ. We begin by focussing on closed oriented worldsheets, without boundary.

Worldsheet spinors exist on Σ, provided a spin bundle $S = K^{1/2}$ can be constructed as the square root of the canonical bundle K. The worldsheet spinor fields are then sections of S. Spin bundles exist on Σ provided the second Stiefel-Whitney class $H^2(\Sigma, \mathbb{Z}_2)$, which is the first Chern class $\bmod 2$ of K, vanishes. This is indeed the case, since $c_1(K) = -\chi(\Sigma) = 2h - 2$ which is even. The different spin bundles are labeled by $H^1(\Sigma, \mathbb{Z}_2)$ and may be distinguished by $\pm$ sign assignments around the homology cycles of Σ. For compact Σ without boundary, there are 2^{2h} different spin structures.

Following the introductory remarks, the worldsheet spinors transform as Lorentz $SO(1, D-1)$ vectors. We denote them by $\psi^\mu_\pm(z, \bar z)$, where the subscript $\pm$ is the worldsheet spinor index. In addition, ψ^μ_+ and ψ^μ_- are to be constrained by a requirement inherited from the Minkowski signature worldsheet: each is a worldsheet Majorana-Weyl spinor, which means that each is a one component real spinor in a suitable basis.

On worldsheets with Euclidean signature, there are no Majorana-Weyl spinors. However, one may use a complex Weyl spinor ψ_+^μ, and take ψ_-^μ to be its complex conjugate. This construction preserves the number of degrees of freedom. Chiral splitting techniques, relying on the holomorphic dependence on moduli — as introduced and applied in §V — may then be invoked to separate correlation functions into parts arising from ψ_+ and ψ_- separately. Thus, henceforth, ψ_+^μ is a complex Weyl spinor, which is a section of a spin bundle $K^{1/2}$ for some given spin structure; ψ_-^μ is the complex conjugate of ψ_+^μ, and a section of $\bar{K}^{1/2}$ for the same spin structure.

It is natural (but, as we shall find later, it is incomplete) to let the dynamics of $\psi_\pm^\mu$ be governed by the action for massless spinors obtained from the bc system of §IV for $n = 1/2$. For $n = 1/2$, the bc system decomposes into two identical spinor systems in a conformally invariant way:

$$b = \frac{1}{\sqrt{2}}(\psi_+ + i\psi'_+) \qquad c = \frac{1}{\sqrt{2}}(\psi_+ - i\psi'_+)\ . \tag{7.A.1}$$

The action for $\psi_\pm^\mu$ is then found to be

$$S_\psi[\psi] = \frac{1}{4\pi}\int_\Sigma d\mu_g \left(\psi_+^\mu \nabla^z_{(1/2)}\psi_{+\mu} + \psi_-^\mu \nabla^{\bar z}_{(-1/2)}\psi_{-\mu}\right)\ . \tag{7.A.2}$$

It describes a conformal field theory with central charge $D \times \frac{1}{2} = \frac{1}{2}D$, namely $c = 1/2$ for each Majorana fermion. We record here the stress tensor for this system:

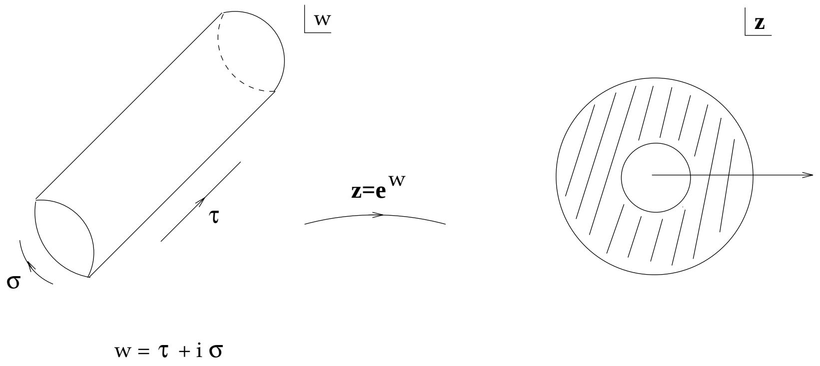
$$T_{zz}^{(\psi)} = \frac{1}{2}\partial_z \psi_+^\mu \psi_{+\mu} \tag{7.A.3}$$

as well as the field equations and the OPE (both can be deduced from the bc system)

$$\nabla^z \psi_+^\mu = 0\,; \qquad \psi_+^\mu(z)\psi_+^\nu(w) \sim \frac{1}{z-w}\,\eta^{\mu\nu} \tag{7.A.4}$$

and analogously for ψ_-^μ.

We now specialize to the case of a flat cylindrical worldsheet, or by conformal mapping the annular worldsheet Σ.



First of all, this topology supports a single homology cycle, and there are two different spin bundles, corresponding to $\pm$ assignments around this cycle. To preserve the action of Lorentz transformations under which ψ^μ transforms as a vector, all components of ψ^μ, $\mu = 0, \ldots, D-1$ must carry the same spin structure, i.e. be sections of the same spin bundle. It is conventional to denote these spin structures by Ramond (R) and Neveu-Schwarz (NS), defined as follows. (The correspondence between the assignments on the cylinder and annulus is unintuitive, so we shall be completely explicit here.)

On the Cylinder

$$\begin{aligned} \psi^\mu_\pm(\tau, \sigma + 2\pi) &= +\psi^\mu_\pm(\tau, \sigma) \qquad && \text{R: periodic on cylinder} \\ \psi^\mu_\pm(\tau, \sigma + 2\pi) &= -\psi^\mu_\pm(\tau, \sigma) \qquad && \text{NS: anti-periodic on cylinder} \end{aligned} \tag{7.A.5}$$

To obtain the corresponding boundary conditions on the annulus, we use the conformal mapping $z = e^w$. On spinors ψ_+, the effect is

$$\begin{aligned} \psi_+^{\text{annulus}}(z)(dz)^{1/2} &= \psi_+^{\text{cylinder}}(w)(dw)^{1/2} \\ \left(\frac{dz}{dw}\right)^{1/2} &= e^{w/2} \end{aligned} \tag{7.A.6}$$

Under a 2π rotation on the cylinder, which corresponds to a 2π rotation on the annulus, the transition factor $e^{\frac{1}{2}w}$ changes by a sign. Hence the (anti-) periodicity assignments are reversed between the cylinders and the annulus.

On the Annulus

$$\begin{aligned} \psi^\mu_\pm(e^{2\pi i}) &= -\psi^\mu_\pm(z) \qquad && \text{R: anti-periodic on annulus} \\ \psi^\mu_\pm(e^{2\pi i}) &= +\psi^\mu_\pm(z) \qquad && \text{NS: periodic on annulus} \end{aligned} \tag{7.A.7}$$

We are now ready to decompose $\psi^\mu_\pm$ in modes and deduce the algebra of the modes, and the Virasoro algebra. We work on the annulus; translation to the cylinder is straightforward. In the Ramond sectors, we have

$$\begin{cases} \psi^\mu_+(z) = \displaystyle\sum_{n\in\mathbb{Z}} d^\mu_n z^{-n-1/2} \\ \psi^\mu_-(\bar z) = \displaystyle\sum_{n\in\mathbb{Z}} \tilde d^\mu_n \bar z^{-n-1/2} \end{cases} \qquad \text{R (annulus)} \tag{7.A.8}$$

$$\{d^\mu_m, d^\nu_n\} = \eta^{\mu\nu}\delta_{m+n,0} \qquad \{\tilde d^\mu_m, \tilde d^\nu_n\} = \eta^{\mu\nu}\delta_{m+n,0}\ . \tag{7.A.9}$$

And in the Neveu-Schwarz sector, we have*

$$\begin{cases} \psi^\mu_+(z) = \displaystyle\sum_{r\in\frac{1}{2}+\mathbb{Z}} b^\mu_r\, z^{-r-1/2} \\ \psi^\mu_-(\bar z) = \displaystyle\sum_{r\in\frac{1}{2}+\mathbb{Z}} \tilde b^\mu_r\, \bar z^{-r-1/2} \end{cases} \qquad \text{NS (annulus)} \tag{7.A.10}$$

$$\{b^\mu_r, b^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0} \qquad \{\tilde b^\mu_r, \tilde b^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0} \tag{7.A.11}$$

B) Ramond and Neveu-Schwarz Fock spaces

Let $\mathcal{F}_k$ and $\widetilde{\mathcal{F}}_k$ denote the Fock spaces of the bosonic degrees of freedom at momentum k for left and right movers, respectively. Let $\mathcal{F}^{\mathrm{R}}$ and $\mathcal{F}^{\mathrm{NS}}$ be the fock spaces for Ramond and Neveu-Schwarz degrees of freedom for left-movers and $\widetilde{\mathcal{F}}^{\mathrm{R}}$ and $\widetilde{\mathcal{F}}^{\mathrm{NS}}$ their right-moving counterparts. Then, the ful RNS Fock spaces for open and closed RNS strings are

$$\begin{aligned} \mathcal{F}_{\text{open}} &= \bigoplus_k \mathcal{F}^{\mathrm{RNS}}_k \\ \mathcal{F}_{\text{closed}} &= \bigoplus_k \mathcal{F}^{\mathrm{RNS}}_k \otimes \widetilde{\mathcal{F}}^{\mathrm{RNS}}_k \qquad k\in\mathbb{R}^D\ . \end{aligned} \tag{7.B.1}$$

Here we have defined the left and right moving RNS Fock spaces by

$$\begin{aligned} \mathcal{F}^{\mathrm{RNS}}_k &= \mathcal{F}_k \otimes (\mathcal{F}^{\mathrm{R}} \oplus \mathcal{F}^{\mathrm{NS}}) \\ \widetilde{\mathcal{F}}^{\mathrm{RNS}}_k &= \widetilde{\mathcal{F}}_k \otimes (\widetilde{\mathcal{F}}^{\mathrm{R}} \oplus \widetilde{\mathcal{F}}^{\mathrm{NS}})\ . \end{aligned} \tag{7.B.2}$$

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* It is conventional to denote the NS mode operators by b^μ_r, but this notation does not suggest a direct relation with the b operators of the bc system.

It thus suffices to construct $\mathcal{F}^{\rm R}$ and $\mathcal{F}^{\rm NS}$.

The *Neveu-Schwarz ground state* $|0;{\rm NS}\rangle \in \mathcal{F}^{\rm NS}$ is defined by**

$$b^\mu_r\ |0;\ {\rm NS}\rangle = 0 \quad r \in \frac{1}{2} + \mathbb{Z}^+ \tag{7.B.3}$$

The full ground state in $\mathcal{F}_k \otimes \mathcal{F}^{\rm NS}$ will be denoted by $|0,k;\ {\rm NS}\rangle \equiv |0,k\rangle \otimes |0;\ {\rm NS}\rangle$. We normalize the ground state by $\langle 0;\ {\rm NS}|0;\ {\rm NS}\rangle = 1$, and the same for $\tilde{}$. By examining the Lorentz spin operator in the NS sector

$$S^{\mu\nu}_{\rm NS} = \sum_{r\in\frac{1}{2}+\mathbb{Z}^+} \left(b^\mu_{-r}b^\nu_r - b^\nu_{-r}b^\mu_r\right)\ , \tag{7.B.4}$$

we see that $|0;\ {\rm NS}\rangle$ is a scalar under ${\rm SO}(1,D-1)$, and thus a *boson.* The NS Fock space $\mathcal{F}^{\rm NS}$ is now obtained by applying the algebra of raising operators $\{b^\mu_r, r<0\}$ to $|0;\ {\rm NS}\rangle$. All the states obtained this way transform under tensor representations of ${\rm SO}(1,D-1)$ and hence are *bosons.*

The *Ramond ground state* is degenerate, just as the ground state of the bc system was degenerate. We label the degenerate states in the ground state multiplet by an index α, and we have as usual

$$d^\mu_n\ |0,\alpha;\ R\rangle = 0 \qquad\qquad n \in \mathbb{N} \tag{7.B.5}$$

The full ground state in $\mathcal{F}_k \otimes \mathcal{F}^R$ will be denoted by $|0,k;\alpha;R\rangle \equiv |0,k\rangle \otimes |0,\alpha;R\rangle$. The mode d^μ_0 satisfies a Clifford algebra

$$\{d^\mu_0, d^\nu_0\} = \eta^{\mu\nu} \tag{7.B.6}$$

so that $|0,\alpha;R\rangle$ transforms under a spinor representation of the Lorentz group ${\rm Spin}(1,D-1)$. Here, α ranges over the weights of the Dirac spinor $\alpha = 1,\dots,2^{D/2}$ (we assume D even). The fact that the ground state $|0,\alpha;R\rangle$ is a spinor may also be confirmed from examining the expression for the Lorentz spin operator in the R sector:

$$S^{\mu\nu}_R = \frac{1}{2}[d^\mu_0, d^\nu_0] + \sum_{n\in\mathbb{N}}(d^\mu_{-n}d^\nu_n - d^\nu_{-n}d^\mu_n) \tag{7.B.7}$$

** We use the notation $\mathbb{N}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$ for the sets of positive, positive or zero and negative or zero integers, respectively.

On $|0,\alpha;R\rangle$, the operator $S_R^{\mu\nu}$ reduces to the familiar Lorentz generator in the spinor representation associated with the Clifford algebra of d_0^μ. Thus, the Ramond ground state is a Spin$(1,D-1)$ spinor, and a space-time *fermion.* $\mathcal{F}^R$ is obtained by applying the algebra of raising oprators $\{d_n^\mu,\ n<0\}$ to $|0,\alpha;R\rangle$ and all states obtained this way are spinors of SO$(1,D-1)$ and space-time fermions! The Ramond construction achieves the goal of introducing fermions into string theory.

Both Fock spaces $\mathcal{F}^R$ and $\mathcal{F}^{\rm NS}$ contain *negative norm states*, just as $\mathcal{F}_k$ contained negative norm states in the bosonic string. The reason is simply that both b_r^μ and d_n^μ transform under vector representations, just as the bosonic modes x_n^μ did. Thus, we have for some vector $\epsilon\in\mathbb{R}^D$:

$$\begin{aligned} &\|\epsilon\cdot b_r\ |0;{\rm NS}\rangle\ \|^2=\epsilon^2 \\ &\|\epsilon\cdot d_{-n}\ |0,\alpha;R\rangle\ \|^2=\epsilon^2\ \|\ |0;\alpha;R\rangle\ \|^2 \end{aligned} \tag{7.B.8}$$

which may become negative when ϵ is time-like and $\epsilon^2<0$.

C) Local supersymmetry on the worldsheet

In bosonic string theory, the negative norm states of x^μ are eliminated by the constraints of diffeomorphism invariance (expressed through the Virasoro algebra) which result from the quantization of the worldsheet metric.

In the RNS string, the bosonic fields x^μ of course still produce negative norm states, which are still eliminated by the Virasoro algebra constraints. But now, the RNS fields $\psi_\pm^\mu$ produce further negative norm states (independent of those of x^μ) which are not eliminated by the Virasoro algebra. In fact, since the fields $\psi_\pm^\mu$ are Grassmannian, the constraints needed to eliminate the states created by the time component $\psi_\pm^0$ will have to be Grassmannian. In addition, since we are to eliminate roughly an entire field component $\psi_\pm^0$, we need this invariance to be *local on* Σ. Grassmann odd local symmetries are so-called *local supersymmetries on* Σ, and actions invariant under those symmetries are so-called *supergravity actions.* They involve, in addition to x^μ, $\psi_\pm^\mu$ and the worldsheet metric g_{mn}, also a *gravitino field* $\chi_m^\pm$, which is a spinor-vector, i.e. $\chi\in K^{3/2}\oplus K^{1/2}\oplus\bar K^{1/2}\oplus\bar K^{3/2}$. This type of field was discovered by Rarita and Schwinger in their study of spin 3/2 fields in 4 space-time dimensions, and is therefore sometimes called the Rarita-Schwinger field.

We shall here give the appropriate supergravity action on any worldsheet, with arbitrary metric g_{mn} and χ_m^σ-field with the spinor index $\sigma = \pm$. To do so, we introduce a local frame $e_m{}^a$, where the m-indices refer to coordinate vectors, while the a-indices refer to the frame vectors $a = 1, 2$. We denote the inverse frame by $e_a{}^m$. We then have

$$g_{mn} = e_m{}^a e_n{}^b \delta_{ab}\,; \qquad\qquad e_a{}^m e_m{}^b = \delta_a{}^b \tag{7.C.1}$$

where δ_{ab} is the $O(2)$-invariant frame metric. Replacing the metric by the frame introduces an additional field. However, there is now also an entire group of local frame $O(2)$ rotations, which can always be used to reduce the number of components to that of the metric. We shall denote the spin connection by ω_m.

We also have a 2-dimension Clifford algebra, defined by

$$\{\gamma^a, \gamma^b\} = -\delta^{ab}\ . \tag{7.C.2}$$

The $N = 1$ supergravity action, with a single supersymmetry, is given by (for flat space-time M):

$$\begin{aligned} S[x,\psi;g,\chi] = \frac{1}{4\pi}\int_\Sigma d\mu_g \Big[&\tfrac{1}{2} g^{mn}\partial_m x^\mu \partial_n x_\mu + \psi^\mu \gamma^a e_a{}^m \partial_m \psi_\mu \\ &- \psi^\mu\gamma^a\gamma^b\chi_a e_b{}^m \partial_m x_\mu - \tfrac{1}{4}\psi^\mu\gamma^a\gamma^b\chi_a(\chi_b\psi_\mu)\Big] \end{aligned} \tag{7.C.3}$$

(The generalization to arbitrary space-time manifolds M is straightforward.) For this action to make sense, and be the integral of a single-valued function, it is necessary that the spin structures of ψ^μ and χ_m^α be the same. The action is invariant under the following transformations.

1) Poincaré symmetry in M.

2) Diff(Σ).

3) Weyl(Σ) $\quad (\delta\psi^\mu = -\frac{1}{2}\,\delta\sigma\psi^\mu \qquad \delta\chi_m^\sigma = \frac{1}{2}\,\delta\sigma\chi_m^\sigma)$. $\qquad$ (7.C.4)

4) Lorentz(Σ) $\quad$ local $O(2)$ frame rotations.

5) super-Weyl $\quad (\delta e^a = 0 \qquad \delta\chi_m = \gamma_m \delta\lambda)$

(The spin structure of λ is the same as that of χ.)

6) Local supersymmetry, generated by a spinor ζ

$$\begin{cases} \delta x^\mu = \zeta\psi^\mu \\ \delta\psi^\mu = \gamma^m(\partial_m x^\mu - \frac{1}{2}\chi_m\psi^\mu)\zeta \\ \delta e_m{}^a = \zeta\gamma^a\chi_m \\ \delta\chi_m^\sigma = -2\nabla_m\zeta^\sigma \end{cases}$$

The spin structure of ζ is the same as that of ψ and χ. In view of Diff(Σ) and Weyl(Σ) invariance, the above action defines a conformal field theory; as we shall explain shortly it will in fact be *super-conformal.*

D) Functional Integral Representation of Transition Amplitudes

It will turn out to be useful in discussing the quantization of free strings to have available the general functional integral representation of transition amplitudes for interacting strings as well, on general worldsheet topologies. Thus, we briefly interrupt the discussion of free strings to include the functional integral representation.

We give first a definition of the transition amplitudes in terms of the fields x, ψ, g, χ. Later on, we shall make use of superfields to simplify the expressions. Let V_i be a collection of vertex operators for the RNS string (we shall postpone the construction of the vertex operators). The scattering amplitude of N strings represented by vertex oprators $V_1, \ldots, V_N$ is given by:

$$\begin{aligned} A = \sum_{\substack{\text{topologies} \\ h=0,\ldots,\infty}} \sum_{\substack{\text{spin} \\ \text{structures} \\ \nu,\bar\nu}} \omega(\nu,\bar\nu) \int_{\mathrm{Met}(\Sigma)} Dg \frac{1}{\mathcal{N}(g)} \int D\chi \\ \int_{\mathrm{Map}(\Sigma,M)} D\chi \int D\psi \cdot V_1 \ldots V_N e^{-S[x,\psi;g,\chi]} \end{aligned} \tag{7.D.1}$$

The measures are constructed with respect to the L^2 norms on the corresponding spaces of sections of $K^r \otimes \bar K^{\bar r}$ for a suitable choice of r, and $w(\nu,\bar\nu)$ is a weight factor that depends upon the spin structure, to be determined later.

1) *A quick derivation of the critical dimension*

Using our experience from the bosonic string, we may collect the various determinant factors that arise from integrating out x and ψ

$$\left(\frac{\text{Det}'\Delta_{(0)}}{\int_\Sigma d\mu_g}\right)^{-D/2}\left(\text{Det}\,\Delta^+_{(-1/2)}\right)^{D/2} \tag{7.D.2}$$

as well as from the Faddeev-Popov determinants arising from gauge fixing Diff(Σ) and local supersymmetry.

$$\left(\text{Det}'\Delta^-_{(-1)}\right)\left(\text{Det}'\,\Delta^-_{(-1/2)}\right)^{-1} \tag{7.D.3}$$

The inverse power occurs for the local supersymmetry determinant because local susy is Grassmann valued. (We have ignored finite-dimensional determinant contributions.) Now, we have the following contribution to the Weyl anomaly from each determinant

$$\begin{cases} C_{(0)} & = & 1 \\ C^+_{(-1/2)} & = & -1/2 \\ C^-_{(-1)} & = & 13 \\ C^-_{(-1/2)} & = & 11/2\,. \end{cases} \qquad \begin{cases} -D/2(-2C_{(0)}) & = & D \\ D/2(-2C^+_{(-1/2)}) & = & D/2 \\ 1(-2C^-_{(-1)}) & = & -26 \\ -1(-2\,C^-_{(-1/2)}) & = & 11 \end{cases} \tag{7.D.4}$$

Hence the total Weyl scaling factor of the determinant parts is $\frac{3}{2}(D-10)$, and the critical dimension is $D=10$. The super-Weyl symmetry (item 5) in (7.C.4) also has an anomaly, and one can show that it cancels in $D=10$.

2) *A quick look at super-moduli*

Later on, we shall carry out a careful study of the space of *super-geometries*, which is parametrized by (g_{mn}, χ_m^σ). For the time being, let us take a quick approach that will suffice for the study of free strings.

Given a metric g, and local complex coordinates z, $\bar z$, with $g = 2g_{z\bar z}|dz|^2$, let us analyze the action of super-Weyl and supersymmetry on $\chi_m^\pm$:

$$\begin{cases} \delta\chi_z^+ = -2\nabla_z\zeta^+ + \delta\lambda^- \\ \delta\chi_{\bar z}^+ = -2\nabla_{\bar z}\zeta^+ \end{cases} \tag{7.D.5}$$

and similarly for their complex conjugates. We see that by a super-Weyl transformation, we may always set $\chi_z^+ = 0$. Locally, we may also set $\chi_{\bar z}^+ = 0$, but globally, there will be obstructions, which are most easily identified when $\delta\chi$ is parametrized by

$$\begin{cases} \delta\chi_{\bar z}^+ = \text{Range}\,\nabla_{\bar z}^{(-1/2)} \oplus \text{Ker}\,\nabla_z^{(-3/2)} \\ \delta\chi_z^- = \text{Range}\,\nabla_z^{(1/2)} \oplus \text{Ker}\,\nabla_{\bar z}^{(3/2)}\ . \end{cases} \tag{7.D.6}$$

The space $\text{Ker}\,\nabla_{\bar z}^{(3/2)}$ is spanned by holomorphic 3/2 differentials.

Odd moduli are defined as the Grassmann odd coordinates that parametrize those parts of χ that are in the quotient of $\delta\chi$ by super-Weyl and supersymmetry:

$$\begin{cases} \delta m_\kappa = (\delta\chi_z^-, \phi_\kappa) \qquad \phi_\kappa \in \text{Ker}\,\nabla_{\bar z}^{(3/2)} \\ \delta \bar m_\kappa = (\delta\chi_{\bar z}^+, \phi_\kappa^*) \qquad \phi_\kappa^* \in \text{Ker}\,\nabla_z^{(-3/2)}\ . \end{cases} \tag{7.D.7}$$

The number of odd moduli is given by the dimension of $\text{Ker}\,\nabla_{\bar z}^{(3/2)}$:

$$\dim\,\text{Ker}\,\nabla_{\bar z}^{(3/2)} = \begin{cases} 0 & h=0 \\ \begin{cases} 0 \\ 1 \end{cases} & \begin{matrix} h=1,\ \text{even}\ \nu \\ h=1,\ \text{odd}\ \nu \end{matrix} \\ 2h-2 & h \geq 2\ . \end{cases} \tag{7.D.8}$$

Here, ν denotes the spin structure of χ.

Later on, we shall show that, together with the even moduli m_k, $\bar m_k$, the odd moduli m_κ, $\bar m_\kappa$ parametrize a so-called *super-moduli space* $s\mathcal{M}_k$, with supermoduli coordinates $(m_k, m_\kappa; \bar m_k, \bar m_\kappa)$. Here, $k = 1, \ldots, \dim\,\text{Ker}\,\nabla_{\bar z}{}^{(2)}$ and $\kappa = 1, \ldots, \dim\,\text{Ker}\,\nabla_{\bar z}{}^{(3/2)}$. This space may be defined as the quotient of the space of all $\{e_m^a, \chi_m^\sigma\}$, denoted by $s\text{Met}(\Sigma)$, by the group of superdiffeomorphisms $s\text{Diff}(\Sigma) = \{\text{Diff}(\Sigma), \text{local susy}\}$, of super-Weyl transformations $s\text{Weyl}(\Sigma) = \{\text{Weyl}(\Sigma), \text{super}\ \delta\lambda\}$ and local $O(2)$ frame rotations Lorentz(Σ):

$$s\mathcal{M}_h = s\text{Met}(\Sigma)/(s\text{Diff}(\Sigma) \rtimes s\text{Weyl}(\Sigma) \times \text{Lorentz}(\Sigma)) \tag{7.D.9}$$

E) Super-Virasoro Algebra and Physical Spectrum

Having clarified the rôle of χ^σ_m, we now apply the results obtained in §D to the free string with Σ of cylindrical topology. Using Diff(Σ), local supersymmetry, Weyl and super-Weyl invariance, in the critical dimension, we may choose a flat metric on Σ, $g = 2|dz|^2$ with vanishing $\chi^\sigma_m = 0$. However, just as was the case in quantizing the free bosonic string, we must retain the constraints arising from the integration over g and χ. They are given by

$$\begin{cases} T(z) = -\frac{1}{2}\partial_z x \cdot \partial_z x + \frac{1}{2}\partial_z \psi_+ \cdot \psi_+ \\ G(z) = \psi_+ \cdot \partial_z x \end{cases} \tag{7.E.1}$$

along with their complex conjugates $\bar{T}$, $\bar{G}$. Their OPE's are easily obtained.

$$\begin{aligned} T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) \\ T(z)G(w) &\sim \frac{3/2}{(z-w)^2}G(w) + \frac{1}{z-w}\partial_w G(w) \\ G(z)G(w) &\sim \frac{2c/3}{(z-w)^2} + \frac{2}{z-w}T(w) \; . \end{aligned} \tag{7.E.2}$$

Here $c = 3/2\, D$. The mode expansion must be carried out with care here, since we must distinguish between Ramond and Neveu-Schwarz sectors.

$$\begin{aligned} T(z) &= \sum_{m\in\mathbb{Z}} L_m z^{-m-2} \\ G(z) &= \sum_r G_r z^{-r-3/2} \qquad \begin{cases} r \in \frac{1}{2} + \mathbb{Z} & \underline{\mathrm{NS}} \\ r \in \mathbb{Z} & \underline{\mathrm{R}} \; . \end{cases} \end{aligned} \tag{7.E.3}$$

The (super) algebra formed by L_m and G_r is the $N = 1$ *super-conformal* or *super-Virasoro algebra*, with the following structure relations

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \\ [L_m, G_r] &= \left(\frac{1}{2}m - r\right) G_{r+m} \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{c}{3}(r^2 - \varepsilon)\delta_{r+s,0} \end{aligned} \tag{7.E.4}$$

where $\varepsilon = 0$ for <u>R</u> and $\varepsilon = 1/2$ for <u>NS</u>. (Notice that G_r and G_s must be both either <u>R</u> or <u>NS</u>!) It will be convenient in the sequel to have the expressions for L_n and G_r in terms of the modes x_n^μ, d_n^μ and b_r^μ. One readily finds by combining (7.E.1) and (7.E.3):

$$m \neq 0 \qquad \begin{cases} L_m = \frac{1}{2}\sum_n x_{m-n} \cdot x_n + \frac{1}{2}\sum_r r b_{m-r} \cdot b_r & \underline{\mathrm{NS}}\ . \\ L_m = \frac{1}{2}\sum_n x_{m-n} \cdot x_n + \frac{1}{2}\sum_n n d_{m-n} \cdot d_n & \underline{\mathrm{R}}\ . \end{cases} \tag{7.E.5}$$

$$\begin{cases} L_0 = \frac{1}{2}p^2 + \sum_{n=1}^{\infty} x_{-n} \cdot x_n + \sum_{r \in 1/2+\mathbb{Z}^+} r b_{-r} \cdot b_r & \underline{\mathrm{NS}}\ . \\ L_0 = \frac{1}{2}p^2 + \sum_{n=1}^{\infty} (x_{-n} \cdot x_n + n d_{-n} \cdot d_n) & \underline{\mathrm{R}}\ . \end{cases} \tag{7.E.6}$$

$$\begin{cases} G_r = \sum_m x_{-m} \cdot b_{r+m} & \underline{\mathrm{NS}} \quad r \in \frac{1}{2} + \mathbb{Z} \\ G_r = \sum_m x_{-m} \cdot d_{r+m} & \underline{\mathrm{R}} \quad r \in \mathbb{Z}\ . \end{cases} \tag{7.E.7}$$

The physical state conditions are enforced separately in the Ramond and Neveu-Schwarz sectors and may be formulated separately for left and right movers at fixed momentum. We have the following definitions for physical states in the left moving sector. (An analogous definition holds for the right-moving sector.)

Ramond sector

A left-moving state $|\psi; k; R\rangle_L$ is physical of momentum k in the Ramond sector ($|\psi; k; R\rangle_L \in \mathfrak{F}_{R,k}^{\mathrm{phys}}$) if

$$\begin{aligned} &\bullet \quad |\psi; k; R\rangle_L \in \mathfrak{F}_k \otimes \mathfrak{F}^R && \text{for some } k \in \mathbb{R}^{10} \\ &\bullet \quad L_n\, |\psi; k; R\rangle_L = G_n\, |\psi; k; R\rangle = 0 && \text{for all } n \in \mathbb{N}. \end{aligned} \tag{7.E.8}$$

and similarly $\widetilde{\mathfrak{F}}_{R,k}^{\mathrm{phys}}$ is defined from $\widetilde{\mathfrak{F}}_k \otimes \widetilde{\mathfrak{F}}^R$.

Neveu-Schwarz

A left-moving state $|\psi; k; \mathrm{NS}\rangle_L$ is physical of momentum k in the Neveu-Schwarz sector ($|\psi; k; \mathrm{NS}\rangle_L \in \mathfrak{F}_{\mathrm{NS},k}^{\mathrm{phys}}$) if

$$\begin{aligned} &\bullet \quad |\psi; k; \mathrm{NS}\rangle_L \in \mathfrak{F}_k \otimes \mathfrak{F}^{\mathrm{NS}} && \text{for some } k \in \mathbb{R}^{10} \\ &\bullet \quad \left(L_n - \frac{1}{2}a\delta_{n,0}\right) |\psi; k; \mathrm{NS}\rangle = 0 && \text{for all } n \in \mathbb{N} \\ &\bullet \quad G_r\, |\psi; k; \mathrm{NS}\rangle = 0 && \text{for all } r \in \mathbb{Z} + \frac{1}{2} \end{aligned} \tag{7.E.9}$$

and similarly $\widetilde{\mathcal{F}}^{\rm phys}_{{\rm NS},k}$ is defined from $\widetilde{\mathcal{F}}_k \otimes \widetilde{\mathcal{F}}^{\rm NS}$. From the Fock spaces of physical states, $\mathcal{F}^{\rm phys}_{R,k}$ and $\mathcal{F}^{\rm phys}_{{\rm NS},k}$, we construct the Fock spaces of open and closed physical strings, in analogy with (7.B.1) and (7.B.2):

$$
\begin{aligned}
\mathcal{F}^{\rm phys}_{\rm open} &= \bigoplus_{k\in\mathbb{R}^{10}} \left(\mathcal{F}^{\rm phys}_{{\rm NS},k} \oplus \mathcal{F}^{\rm phys}_{R,k}\right) \\
\mathcal{F}^{\rm phys}_{\rm closed} &= \bigoplus_{k\in\mathbb{R}^{10}} \left(\mathcal{F}^{\rm phys}_{{\rm NS},k} \oplus \mathcal{F}^{\rm phys}_{R,k}\right) \otimes \left(\widetilde{\mathcal{F}}^{\rm phys}_{{\rm NS},k} \oplus \widetilde{\mathcal{F}}^{\rm phys}_{r,k}\right) \ .
\end{aligned}
\tag{7.E.10}
$$

In complete analogy with the bosonic theory, we have a

No Ghost Theorem. *$\mathcal{F}^{\rm phys}_{\rm open}$ and $\mathcal{F}^{\rm phys}_{\rm closed}$ have non-negative norm for $D=10$, $a=1$. The positive definite Hilbert space of the RNS string is obtained by taking the quotient by the null space, which we denote by $\mathcal{F}^{\rm null}$:*

$$
\begin{aligned}
\mathcal{F}^{+}_{\rm open} &= \mathcal{F}^{\rm phys}_{\rm open}/\mathcal{F}^{\rm null}_{\rm open} \\
\mathcal{F}^{+}_{\rm closed} &= \mathcal{F}^{\rm phys}_{\rm closed}/\mathcal{F}^{\rm null}_{\rm closed}
\end{aligned}
\tag{7.E.11}
$$

We shall not prove this theorem here.

F) The spectrum of physical states at low mass

In the *Ramond sector*, the ground state obeys $x^\mu_n, d^\mu_n \ |0,k;\alpha;R\rangle = 0$ for all $n\in\mathbb{N}$ as well as the physical state condition from (7.E.8)

$$
G_0 \ |0,k;\alpha;R\rangle = 0 \tag{7.F.1}
$$

Since $L_0 = G_0^2$, the constraint by L_0 is automatically satisfied; similarly the constraints by G_n, L_n, $n \geq 1$ are automatically satisfied. Denoting the generators d^μ_0 of the 10-dimensional Clifford algebra by standard Dirac notation $d^\mu_0 = \frac{1}{\sqrt{2}}\Gamma^\mu$, we find that the G_0 condition amounts to the fact that the Ramond ground state satisfies the 10-dimension massless Dirac equation:

$$
\Gamma_\mu k^\mu \ |0,k;\alpha;R\rangle = 0 \ . \tag{7.F.2}
$$

This equation on the Ramond ground state confirms once more that it is a fermion. Higher mass states transform under higher spinor representations of Spin$(1,9)$, and are all *fermions* as well.

In the *Neveu-Schwarz sector*, the ground state obeys x^μ_n, b^μ_r $|0,k;\mathrm{NS}\rangle = 0$ for all $n \in \mathbb{N}$, $r \in \frac{1}{2} + \mathbb{Z}^+$ as well as the physical state condition from (7.E.9)

$$\Big(L_0 - \frac{1}{2}\Big)\ |0,k;\mathrm{NS}\rangle = 0 \tag{7.F.3}$$

The constraints by L_n, G_r, $n, r \geq \frac{1}{2}$ are again automatically satisfied. These conditions imply that $k^2 = 1$, i.e. $M^2 = -1$, so that this state is a *tachyon*! It is also a Lorentz scalar and thus a boson. The first excited state in the NS sector is obtained by applying $b^\mu_{-1/2}$, $|\zeta,k;\mathrm{NS}\rangle \equiv \zeta_\mu b^\mu_{-1/2}\ |0,k;\mathrm{NS}\rangle$ and obeys the constraints

$$\Big(L_0 - \frac{1}{2}\Big)\,,\, L_1\ |\zeta,k;\mathrm{NS}\rangle = 0 \tag{7.F.4}$$

We find $k^2 = 0$ and $\zeta \cdot k = 0$, so $|\zeta,k;\mathrm{NS}\rangle$ is a *massless vector state*, such as we need for Yang-Mills particles. The transversality condition $\zeta \cdot k = 0$ is also what we shall need for gauge invariance. It implies that the negative norm state is eliminated, and because $k^2 = 0$ that one further state is null. Thus, we are left with 8 components with positive norm, just as expected for a gauge particle. Higher mass states transform under higher tensor representations of $\mathrm{SO}(1,9)$, and are all bosons as well.

Let us note that by using lightcone gauge, we may construct all positive norm states by applying only the *transverse operators*

$$x^i_n, \quad d^i_n, \quad b^i_r \qquad\qquad i = 1, \ldots, 8 \tag{7.F.5}$$

to the ground states in each respective sector for $n, r < 0$. This method in particular allows for an easy counting of the number of physical states modulo null states, at each mass level (see Problem Set #9).

Actually, we have looked only at the left-moving sector of the Hilbert space. This suffices to construct the Hilbert space of the *open strings*. To summarize, we have found that this sector contains a NS tachyon, a massless Dirac spinor and a massless vector. Adding Chan-Paton rules to the open string, the single vector may be promoted to a Yang-Mills gauge field multiplet under the Chan-Paton gauge group. Similarly, the Dirac spinor (and the tachyon) will form multiplets under this group. We shall work out this construction for Type I strings shortly.

For *closed strings*, we must put left and right moving sectors together, in order to obtain the full spectrum, with the condition that left and right momenta should match $p_L = p_R$, i.e. $x_0^\mu = \tilde{x}_0^\mu$. We have four sectors:

NS-NS*: bosons.*

- The ground state $|0,k;\mathrm{NS}\rangle_L \otimes |0,k;\mathrm{NS}\rangle_R$ is a tachyon with mass squared equal to $M^2 = -1$.
- The massless states are $\epsilon_{\mu\nu}(k) b^\mu_{-1/2} \tilde{b}^\nu_{-1/2} |0,k;\mathrm{NS}\rangle_L \otimes |0,k;\mathrm{NS}\rangle_R$ and contain

  the graviton (symmetric μ, ν),

  the anti-symmetric tensor (anti-symmetric μ, ν),

  the dilaton (tracepart).
- massive excited states.

R-NS *(and* NS-R*): fermions.*

The Ramond sector always has mass$^2 \geq 0$, and thus no tachyons occur in the R-NS and NS-R sectors.

- The ground state $|0,k,\alpha;\mathrm{R}\rangle_L \otimes \zeta_\mu(k)\tilde{b}^\mu_{-1/2} |0,k;\mathrm{NS}\rangle_R$ is massless. This state is reducible into a spinor λ called the *dilatino* and a spinor-vector, called the *gravitino.*
- massive excited states

R-R*: more bosons.*

- The ground state $|0,k,\beta;\mathrm{R}\rangle_L \otimes |0,k,\alpha;\mathrm{R}\rangle_R$ is massless.

  These states decompose into a sum of tensor representations of $\mathrm{SO}(1,9)$, that may be obtained as antisymmetrized products of the vector representation. In other words, these states transform as differential forms of degree $0,\dots,10$.
- massive excited states.

We shall establish in §X that the RNS spectrum (after the GSO projection has been carried out in §G) exhibits space-time supersymmetry, under which bosons and fermions are interchanged. We shall then find that the dilaton and the dilatino on the one hand, and the graviton and the gravitino on other other hand, form supersymmetry multiplets.

G) The GSO Projection, Space-time Supersymmetry

By combining the NS and R sectors of the RNS string, we have obtained a string theory with both bosons and fermions, with massless graviton, anti-symmetric tensor, dilaton and anti-symmetric tensor R-R states which are all bosons, plus gravitino and dilatino states which are fermions. This theory is still inconsistent, for two reasons. First, in the NS sector, there is still a *tachyon* present, which gives rise to the same type of problems as the tachyon did in the bosonic string. Second, there is a *massless gravitino*, which can interact consistently only by coupling to a conserved supercurrent. (The gravitino shares many properties with a gauge field, which must couple to a conserved current.) But conservation of a supercurrent means that the theory must exhibit invariance under supersymmetry transformations. Clearly, the RNS theory is not supersymmetric as it stands. The easiest way to see this is by noticing that the tachyon has no fermionic counterpart. On the other hand, if the theory could be made supersymmetric, the tachyon problem would be eliminated right away and then the gravitino could be coupled to a conserved supercurrent, thus eliminating both of the above problems.

Gliozzi-Scherk and Olive (1976) proposed a truncation of the RNS string theory that produces a spectrum with space-time ($\mathbb{R}^{10}$) supersymmetry. We shall introduce the GSO projection first at the level of the free string Fock space, and later on show how to enfore it in the functional integral formulation. It will turn out that the GSO projection corresponds there to carrying out a summation over all spin structures of Σ, with certain weight factors. The weight factors will be arranged in such a way that the amplitudes will be invariant under the action of the modular group of the worldsheet Σ. Such spin structure weight factors were already included in the functional integral formulation in (7.D.1) on general grounds.

The GSO projection is defined by using the *worldsheet fermion number operator F* in the form $(-1)^F$. This operator has the property

$$\big\{(-1)^F, \psi^\mu_\sigma(z,\bar z)\big\} = 0 \qquad (7.G.1)$$

both in the R and NS sectors. This property defines the operator $(-1)^F$ only up to a factor of ± 1, which one may think of as the assignment of the eigenvalue of $(-1)^F$ on the string ground state in each sector. We shall now provide a careful definition of $(-1)^F$ in each sector.

In the *Neveu-Schwarz sector*, the renormalized worldsheet fermion number operators in the left- and right-moving sectors are given by

$$F_{\mathrm{NS}} \equiv \sum_{r\in 1/2+\mathbb{Z}^+} b_{-r}\cdot b_r \qquad \widetilde{F}_{\mathrm{NS}} \equiv \sum_{r\in 1/2+\mathbb{Z}^+} \tilde{b}_{-r}\cdot \tilde{b}_r \ . \tag{7.G.2}$$

and the assignment of $(-1)^F$ on the NS ground state is -1. Thus, we may represent $(-1)^F$ by

$$(-1)^F \equiv (-1)^{F_{\mathrm{NS}}-1} \qquad (-1)^{\widetilde{F}} \equiv (-1)^{\widetilde{F}_{\mathrm{NS}}-1} \ . \tag{7.G.3}$$

The GSO *projection* in the Fock space of the NS sector is then given by

$$\mathfrak{F}_{\mathrm{NS,GSO}} \equiv \left\{ |\psi\rangle_L \in \mathfrak{F}^{\mathrm{NS}}, (-1)^F\ |\psi\rangle_L = |\psi\rangle_L \right\} \tag{7.G.4}$$

and similarly

$$\widetilde{\mathfrak{F}}_{\mathrm{NS,GSO}} \equiv \left\{ |\psi\rangle_R \in \widetilde{\mathfrak{F}}^{\mathrm{NS}}, (-1)^{\widetilde{F}}\ |\psi\rangle_R = |\psi\rangle_R \right\} \ . \tag{7.G.5}$$

Alternatively, $\mathfrak{F}_{\mathrm{NS,GSO}}$ consists of states obtained from the ground state in the NS sector by applying only an *even* number of b operators. (Similarly for $\widetilde{\mathfrak{F}}_{\mathrm{NS,GSO}}$). It is straightforward to see that the GSO projection is compatible with the physical state conditions, since

$$\left[(-1)^F, L_m\right] = 0 \qquad \left\{(-1)^F, G_r\right\} = 0 \ . \tag{7.G.6}$$

Thus, we define $\mathfrak{F}^{\mathrm{phys}}_{\mathrm{NS,GSO}}$ and $\widetilde{\mathfrak{F}}^{\mathrm{phys}}_{\mathrm{NS,GSO}}$ by imposing physical state conditions on $\mathfrak{F}_{\mathrm{NS,GSO}}$ and $\widetilde{\mathfrak{F}}_{\mathrm{NS,GSO}}$.

The tachyon ground state (as well as any state obtained by applying an even number of b-operators to it) is projected out by GSO, and the lowest mass state in $\mathfrak{F}^{\mathrm{phys}}_{\mathrm{GSO,NS}}$ is massless. It consists of a vector of $\mathrm{SO}(1,9)$:

$$b^\mu_{-1/2}\ |0,k;\mathrm{NS}\rangle$$

with 8 physical, positive norm components. The first excited states have the form

$$b^{\mu_1}_{-1/2} b^{\mu_2}_{-1/2} b^{\mu_3}_{-1/2}\ |0,k;\mathrm{NS}\rangle \qquad b^\mu_{-3/2}\ |0,k;\mathrm{NS}\rangle \tag{7.G.7}$$

and have mass $M^2 = 2$. (There are $8.7.6/6 = 56, +8 = 64$ such physical states.)

We now proceed to define the GSO projection in the *Ramond sector*. Supersymmetry in $\mathbb{R}^{10}$ will require that we have equal numbers of bosonic and fermionic physical states at each mass level. (We assume here that supersymmetry is manifest, and not spontaneously broken.) The original Ramond sector of the RNS string has a massless Dirac spinor ground state. In $D = 10$, such a spinor has $2^{D/2} = 2^5 = 32$ components, which does not match the 8 bosonic components of the NS sector. To have equality, we ought to reduce the number of components of the spinor from 32 to 8, of course, in a way consistent with $\mathrm{Spin}(1, 9)$ invariance!

For groups of the form $\mathrm{Spin}(1, D-1)$, with D even, we have the following reducibility criteria for various dimensions D. The Clifford algebra is defined by $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$. The chirality matrix $\bar{\Gamma}$, defined by

$$\bar{\Gamma} \equiv \Gamma^0 \Gamma^1 \ldots \Gamma^{D-1} \qquad\qquad \bar{\Gamma}^2 = (-1)^{D/2-1} \tag{7.G.8}$$

commutes with all the generators of $\mathrm{Spin}(1, D-1)$ in the Dirac spinor representation — which we denote by S. Thus, S may be reduced to a sum of *Weyl spinors* $S = S_+ \otimes S_-$, where both $S_\pm$ are of dimension $2^{D/2-1}$.

For $D = 2, 4 \pmod 8$, there is an independent reducibility criterion on Dirac spinors, which decomposes S into a sum of two *Majorana spinors* $S = S_M + iS'_M$. A Majorana spinor is of dimension $2^{D/2}$, but is defined to be *real in some basis*:

$$S_M^* = BS_M \qquad \text{with} \qquad \Gamma^{\mu *} = B\Gamma^\mu B^{-1} \tag{7.G.9}$$

A spinor may be reduced to *Majorana-Weyl* spinors provided the Weyl and Majorana conditions are compatible (i.e. $\bar{\Gamma}$ and B commute). This is possible only in $D = 2 \pmod 8$. A Majorana-Weyl spinor is a real spinor of dimension $2^{D/2-1}$. In $D = 10$, we see that the Majorana-Weyl condition reduces the dimensionality of the R-ground state precisely to an 8 component spinor, just as is needed for supersymmetry!

In the *Ramond sector*, the GSO projection is defined in terms of the worldsheet fermion number operator as well, with the following normalization.

$$\begin{aligned}(-1)^F &\equiv \bar{\Gamma}(-1)^{F_R} & \qquad (-1)^{\widetilde{F}} &= \bar{\Gamma}(-1)^{\widetilde{F}_R}\\ F_R &\equiv \sum_{n=1}^{\infty} d_{-n}\cdot d_n & \widetilde{F}_R &= \sum_{n=1}^{\infty} \tilde{d}_{-n}\cdot\tilde{d}_n\end{aligned}\tag{7.G.10}$$

The GSO projection in the Fock space of the R sector involves a choice of chirality of the R ground state in each sector (left movers and right movers). Let us denote these chiralities by χ_L and χ_R, respectively, with values $\pm$ each. We now define

$$\mathfrak{F}^{\mathrm{R}}_{\mathrm{GSO}}(\chi_L) \equiv \left\{\, |\psi, R\rangle_L \in \mathfrak{F}^R; \qquad (-1)^F\ |\psi, R\rangle_L = \chi_L\ |\psi; R\rangle_L \right\}$$

and (7.G.11)

$$\widetilde{\mathfrak{F}}^{\mathrm{R}}_{\mathrm{GSO}}(\chi_R) \equiv \left\{\, \psi; R\rangle_R \in \widetilde{\mathfrak{F}}^R; \qquad (-1)^{\widetilde{F}}\ |\psi; R\rangle_R = \chi_R\ |\psi; R\rangle_R \right\}\ .$$

This means that, given the chirality of the ground state, the chirality of the higher states are determined by the GSO projection.

1) *Chirality and Massive States in the Ramond Sector*

The above GSO projection in the Ramond sector selects the chirality of the massless ground state. But it also appears to select the chirality of massive states, which would appear to be in contradiction with the fact that they are massive. The resolution of this apparent contradiction lies in the fact that opposite chirality components of a massive state arise in different ways. Consider the first excited level for left movers; it has states (in the lightcone gauge)

$$x^i_{-1}\ |0, k; \alpha; R\rangle_L \qquad\qquad d^i_{-1}\ |0, k; \beta; R\rangle_L \tag{7.G.12}$$

The GSO condition (for example $\chi = +$) requires that

$$\begin{aligned}\bar{\Gamma}x^i_{-1}\ |0, k; \alpha; R\rangle_L &= +x^i_{-1}\ |0, k; \alpha; R\rangle_L\\ \bar{\Gamma}d^i_{-1}\ |0, k; \beta; R\rangle_L &= -d^i_{-1}\ |0, k; \beta; R\rangle_L\end{aligned}\tag{7.G.13}$$

Hence these two states have opposite chiralities and together form the 16 components of a massive Majorana spinor! The same phenomenon persists at higher mass levels, and is responsible for correctly filling out the chirality components of all massive states.

2) *Space-time supersymmetry*

Now that we have defined the GSO projection in all sectors of the RNS string, it remains to investigate whether the spectrum thus obtained is supersymmetric. To carry out this analysis, we would need to construct the supersymmetry operators, and this requires technology that we shall develop only in §X. Instead, we can check here that the GSO projection passes an elementary test necessary to have a supersymmetric spectrum. It is an elementary consequence of the structure relations of supersymmetry that the supercharges commute with the mass^2 matrix. Thus, the number of bosonic and fermionic states at each mass level must be the same. We now show that this is indeed the case for the RNS spectrum, after GSO projection.

To do so, we count the number of fermionic states (in the R sector) and compare with the number of bosonic states (in the NS sector). To count states, it suffices to use the lightcone construction, in terms of transverse operators x^i, b^i, d^i, with $i = 1, \ldots 8$, and we calculate the generating function in each sector. In the Ramond sector, we have

$$\begin{aligned} f_{\mathrm{R}}(q) &= 8 \mathrm{Tr}\, q^{2M^2} = \sum_{n=1}^{\infty} d_R(n) q^{2n} \\ &= 8 \prod_{m=1}^{\infty} (1-q^{2m})^{-8} (1+q^{2m})^8 \end{aligned} \tag{7.G.14}$$

Here, the factor $(1-q^{2m})^{-8}$ arises from the x^i oscillators and the factor $(1+q^{2m})^8$ from the d^i modes. The prefactor of 8 arises from the eight-fold degeneracy of the R ground state. Similarly, in the Neveu-Schwarz sector, we have

$$f_{\mathrm{NS}}(q) = \prod_{m=1}^{\infty} (1-q^{2m})^{-8} \frac{1}{2q} \left\{ \prod_{n=1}^{\infty} (1+q^{2n-1})^8 - \prod_{n=1}^{\infty} (1-q^{2n-1})^8 \right\} \tag{7.G.15}$$

Using the product formulas for Jacobi ϑ-functions, and the equality

$$\vartheta_2(0|\tau)^4 + \vartheta_4(0|\tau)^4 = \vartheta_3(0|\tau)^4 \tag{7.G.16}$$

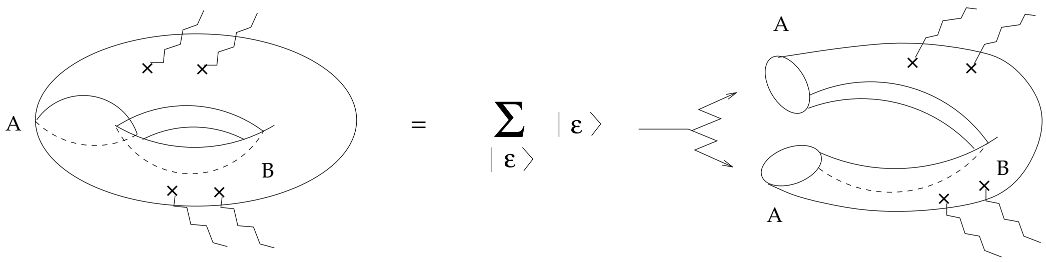
with $q = e^{2\pi i \tau}$, we see that $f_{\mathrm{R}}(q) = f_{\mathrm{NS}}(q)$. Thus, at each mass level, the number of bosonic and fermionic states agrees.

Notice that we have established evidence for the presence of a space-time supersymmetry in each sector, left-moving and right-moving independently. It is conventional to denote the number of independent supersymmetries by the letter N. In this sense, the open GSO projected RNS string has $N = 1$, while the closed GSO projected RNS string has $N = 2$.

3) *Summation over spin structures*

We are now ready to exhibit the relation between the GSO projection of the states in string transition amplitudes to any loop order, and summation over spin structures of string worldsheets. We concentrate on processes involving only NS-NS *external states*, which are created by vertex operators that are single-valued on the worldsheet. (Vertex operators for R states produce quadratic branch cuts and will be introduced in §IX.) Of course, the internal states will include both the R and the NS sectors.

We consider 1-loop amplitudes for closed, oriented strings first, and choose a homology basis of 1-cycles A and B. We analyze the contribution from a given spin structure.



The one loop amplitude may be *factorized* and expressed as a sum over intermediate states, as shown in the diagram above. In the functional integral, factorization is carried out by integrating over all field values (for x^μ and ψ^μ_σ) on the curve A. Integration over all RNS fields x^μ and ψ^μ_σ, with given spin structure, will give rise to the following structure.

The spin structure on the A-cycle determines the boundary condition on ψ^μ_+ (resp. ψ^μ_-) as either periodic (Ramond) or anti-periodic (Neveu-Schwarz). This information determines whether the state $|\varepsilon\rangle$ in the factorization is a fermion (R) or a boson (NS). In the full 1-loop amplitude, both bosons and fermions will contribute; thus, the spin structure on the A-cycle must be summed over with unit coefficients.

What is the physical significance of the spin structure along a B-cycle? Functional integration over worldsheet spinors ψ_+^μ (or ψ_-^μ) with *anti-periodic boundary conditions* along the B-cycle produces a sum over states $|\varepsilon\rangle$ in the factorization. On the other hand, functional integration over spinors ψ_+^μ (or ψ_-^μ) with *periodic boundary conditions* along the B-cycle produces a sum over states $|\varepsilon\rangle$ with an insertion of the operator $(-1)^F$ in the factorization. Here, F stands for worldsheet fermion number, just as we encountered in the definition of the GSO projection. This result may be established via the usual derivation of the Grassmann functional integral from canonical quantization of spinors. A summation over spin structures on the B-cycles with coefficients (± 1) produces the insertion of the projection operator

$$\frac{1}{2}(1+(-1)^F) \tag{7.G.17}$$

in the summation over states $|\varepsilon\rangle$ in the factorization, and this oprator precisely enforces the GSO projection. It is straightforward to extend the reasoning presented above to the case of worldsheets of arbitrary genus h. We shall leave it as an exercise to the reader.

Just as the GSO projection had to be enforced separately on left and right movers, we shall also have to separate left and right movers in loop amplitudes, using the chiral splitting theorem for the RNS string (which generalizes the results of §V), and sum independently over spin structures of left and right movers.

Finally, let us remark that we have shown that the GSO projection is enforced by separate summation over all spin structures of left and right movers with weights ± 1. The actual sign choice for each spin structure must be consistent with modular invariance of the full amplitude. There may be different consistent choices, yielding different string theories, as we shall see in §H.

H) Type II A, B Superstrings and Their Spectra

So far, we have analyzed the GSO projection in a single sector of RNS string theory: either left- or right-movers. We now consider closed oriented string theories built from GSO projected RNS strings. We shall temporarily ignore the physical state conditions, which may be easily imposed independently. There are 4 possible combinations, with the following Hilbert spaces:

$$\mathcal{F}^{\rm phys}_{\rm GSO}(\chi_L;\chi_R) = \bigoplus_{k\in\mathbb{R}^{10}} \mathcal{F}_k \otimes (\mathcal{F}^{\rm NS}_{\rm GSO} \oplus \mathcal{F}^{\rm R}_{\rm GSO}(\chi_L)) \otimes \tilde{\mathcal{F}}_k \otimes (\tilde{\mathcal{F}}^{\rm NS}_{\rm GSO} \oplus \mathcal{F}^{\rm R}_{\rm GSO}(\chi_R)) \ . \tag{7.H.1}$$

Now, a space-time parity operation reverses the chirality of all spinors and in particular, of the R ground states: $\chi_L \to -\chi_L$, $\chi_R \to -\chi_R$. Thus the string theory with assignment (χ_L, χ_R) defines the same string theory as the one with assignment $(-\chi_L, -\chi_R)$. So, instead of 4 combiantions, we are left with only 2 physically inequivalent string theories:

Type II A**:** $\chi_L = -\chi_R$

$$\mathcal{F}^{\rm phys}_{\rm II\ A} = \mathcal{F}^{\rm phys}_{\rm GSO}(+,-) \qquad (\sim \mathcal{F}^{\rm phys}_{\rm GSO}(-,+)) \ . \tag{7.H.2}$$

Type II B**:** $\chi_L = \chi_R$

$$\mathcal{F}^{\rm phys}_{\rm II\ B} = \mathcal{F}^{\rm phys}_{\rm GSO}(+,+) \qquad (\sim \mathcal{F}^{\rm phys}_{\rm SO}(-,-)) \ . \tag{7.H.3}$$

The Type II A theory is parity invariant upon exchange of left and right movers, while Type II B is parity violating. One also says that *Type* II B *is chiral*, since it singles out one of the chiralities, while *Type* II A *is non-chiral.*

To gain insight into these theories, we derive their physical spectra, with the help of the lightcone gauge formulation, in terms of the modes x^i_n, b^i_r, d^i_n with $i = 1, \dots, D-2 = 8$. (See Problem Set #9.) We restrict attention first to massless states. We begin by noticing that the GSO projection eliminates the tachyon, but leaves the massless states unchanged in the NS sector. In the Ramond sector, the GSO projection selects the chirality of the ground state, which is the only state needed for massless string states. We shall choose the chiralities to be $+$ in the left moving sector and $-$ for Type II A and $+$ for Type II B in the right moving sectors, respectively. A list of massless states in each sector, together with their standard identifications in terms of fields familiar from quantum field theory, is given in the table below. In parentheses, we indicate the number of physical degrees of freedom present in each state.

$L-R$	Type II A	Type II B
NS-NS (bosons)	$b^i_{-1/2}\tilde b^j_{-1/2}\ \|0,k;\mathrm{NS}\rangle_L\otimes\|0,k;\mathrm{NS}\rangle_R$ • (35) $G_{\mu\nu}$ (graviton) • (28) $B_{\mu\nu}$ • (1) Φ (dilaton)	idem
R-NS (fermions)	$\|0,k;\alpha;R\rangle_L\otimes\tilde b^i_{-1/2}\ \|0,k;\mathrm{NS}\rangle_R$ • (56) χ^α_μ (gravitinos) chirality $+$ • (8) λ_α (dilatinos) chirality $-$	idem chirality $+$ chirality $-$
NS-R (fermions)	$b^i_{-1/2}\ \|0,k;\mathrm{NS}\rangle_L\otimes\|0,k;\alpha;R\rangle_R$ • (56) χ^α_μ (gravitinos) chirality $-$ • (8) λ_α (dilatinos) chirality $+$	idem chirality $+$ chirality $-$
R-R (bosons)	$\|0,k;\alpha;R\rangle_L\otimes\|0,k;\beta;R\rangle_R$ • (8) $A^{(1)}_\mu$ • (56) $A^{(3)}_{\mu\nu\rho}$	$\|0,k;\alpha;R\rangle_L\otimes\|0,k;\beta;R\rangle_R$ • (1) $A^{(0)}$ • (28) $A^{(2)}_{\mu\nu}$ • (35) $A^{(4)}_{\mu\nu\rho\sigma}$ *self-dual F*

The first entry that requires clarification is the R-NS (and NS-R) sector: the dilatino and gravitino have opposite chiralities. To see how this happens, consider the tensor roduct of a spinor of definite chirality (say + as in R-NS) and a vector, which yields a field $\bar\chi^\alpha_\mu$ of chirality + with 64 physical components. It decomposes into irreducible spinors by

$$\bar\chi^\alpha_\mu=\chi^\alpha_\mu+(\Gamma_\mu)^{\alpha\beta}\lambda_\beta$$

where χ^α_μ is the gravitino field, obeying $(\Gamma^\mu)_{\beta\alpha}\chi^\alpha_\mu=0$, of chirality +, and λ is the dilatino of chirality −, since $\{\bar\Gamma,\Gamma_\mu\}=0$.

The next entry that requires clarification is the R-R sector. To see why the tensors that are indicated are the ones that arise, we study the tensor product decomposition properties of two spinors. This may be done either on SO(1, 9) spinors, or on the reduced SO(8) spinors. Notice that a Majorana-Weyl spinor of the Poincaré group of SO(1, 9) is *irreducible* under its SO(8) stabilizer subgroup, where it is simply a Weyl spinor.

If S_1 and S_2 are two Majorana-Weyl spinors of *the same chirality*, then they reduce to SO(8) Weyl spinors of the same chirality. Their product decomposes into anti-symmetric tensors of *even* rank

$$S_1 \otimes S_2 = A^{(0)} \oplus A^{(2)} \oplus A_+^{(4)} \tag{7.H.5}$$

(For Weyl spinors, the components $A^{(6)}$, $A^{(8)}$ are dependent on $A^{(0)}$ and $A^{(2)}$, respectively.) $A_+^{(4)}$ stands for the part with self-dual field strength, $F = dA^{(4)}$ with ${}^*F = F$.

If S_1 and S_2 are two Majorana-Weyl spinors of *opposite chirality*, they reduce to SO(8) Weyl spinors of opposite chirality. Their product decomposes into anti-symmetric tensors of *odd* rank

$$S_1 \otimes S_2 = A^{(1)} \oplus A^{(3)} \ . \tag{7.H.6}$$

(For Weyl spinors, the componens $A^{(5)}$ and $A^{(7)}$ are dependent on $A^{(3)}$ and $A^{(1)}$.)

The counting of the number of physical states for each of the fields $A^{(1)}_\mu$, $A^{(2)}_{\mu\nu}$, $A^{(3)}_{\mu\nu\kappa}$ and $A^{(4)}_{\mu\nu\kappa\rho}$ indeed confirms that they are all to be viewed as *gauge fields*, in fact, as differential forms $A^{(p)}$, invariant under

$$A^{(p)} \to A^{(p)} + dC^{(p-1)} \tag{7.H.7}$$

where $C^{(p-1)}$ is a form of degree $p - 1$. Thus, they are all Abelian gauge fields.

D-branes

It can be shown that the fundamental strings are neutral under these gauge fields, and do not carry their quantum numbers. A gauge field of the type $A^{(p)}$ couples naturally, however, to an extended object of space-time dimension p called a $D(p-1)$ brane. By analogy with electromagnetism, one says that the $D(p-1)$ brane carries the electric charge of $A^{(p)}$. By Poincaré duality of their field strength $F^{(p)} = dA^{(p)}$, the field $*F^{(p)} \equiv F^{(10-p)}$ may also naturally couple to extended objects of space-time dimensions $8 - p$ again called a $D(8-p)$-brane.

$$\begin{array}{llll} A^{(1)} & \sim & D0 & \text{branes} \\ A^{(2)} & \sim & D1 & \text{branes} \quad (\text{strings}) \\ A^{(3)} & \sim & D2 & \text{branes} \quad (\text{membranes}) \\ A^{(4)} & \sim & D3 & \text{branes} \end{array} \tag{7.H.8}$$

Polchinski showed that D-branes indeed carry the electric charges of these gauge fields.

The massive spectra of Type II A*and Type* II B*coincide.*

We conclude this section on Type II A, B superstrings with a discussion of their massive spectra. The full spectra are obtained by tensor product of left- and right-moving sectors as usual. The spectrum of each sector (say left movers) is identical to the spectrum of an $\mathcal{N}=1$ chiral open superstring. While its massless spinor states are chiral, its massive states cannot be chiral. Instead, we have already exhibited the chiral pairing of states of opposite chirality previously. For example, at the first excited level in the R sector, we have

$$x^i_{-1}\ |0,k;\alpha;R\rangle \qquad\qquad d^i_{-1}\ |0,k;\beta;R\rangle \tag{7.H.9}$$

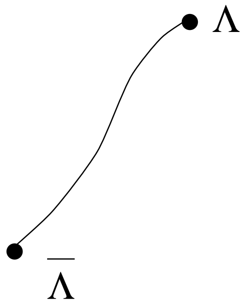
which are of opposite chirality and combine into a representation of massive Majorana fermions. Thus, the massive spectrum is insensitive to the choice of chirality for the massless ground state. As a result *the massive spectra of* Type II A *and* Type II B *superstrings coincide.* In other words, Type II A and Type II B superstrings differ only in their massless spectrum.

In fact, one can show that upon compactifying Type II A on a circle of radius R_A and Type II B on a circle of radius R_B, one obtains identical theories, including the massless sector, when $R_A R_B = \alpha'$. This identity may be seen as a result from a transformation between the two theories, called T-duality, which we investigate in Problem Set #10.

I) Type I Superstring

The type I superstring is constructed from open and closed strings, including orientable as well as unorientable worldsheets, and with Chan-Paton Yang-Mills charges on the open string end points.

We begin by reviewing the results from §II on Chan-Paton charges: the end points of open strings carry a multiplet of charges, transforming under representations Λ and $\bar{\Lambda}$ of a compact gauge group G. These charges may be obtained by quantization of Grassmann variables λ^i that move at the end points of the open strings.



For the open string excitation states to be precisely those of Yang-Mills gauge states, it is necessary that $\Lambda \otimes \bar{\Lambda}$ be equal to the adjoint representation of G, possibly after imposing certain reality and/or symmetrization rules. In fact, there are only 3 series of solutions, given by classical groups, and the followng rules:

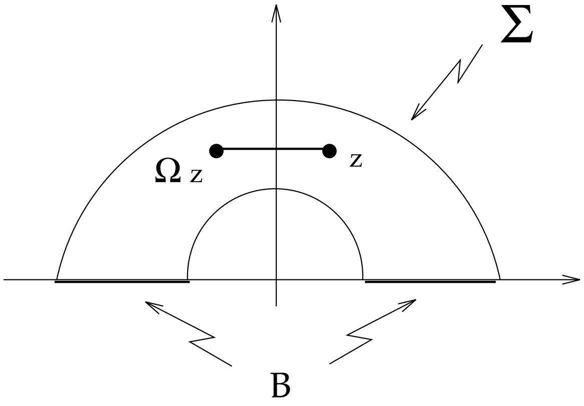
$U(N)$	$\bar{\Lambda} = \Lambda^*$	orientable strings
$\mathrm{SO}(N)$	$\bar{\Lambda} = \Lambda$	unorientable strings
$\mathrm{Sp}(N)$	$\bar{\Lambda} = \Lambda$	unorientable strings

(We shall examine the interplay between Chan-Paton charges and orientability shortly.)

Next, we need to clarify the construction of string theories with unorientable strings. One defines the *worldsheet parity operator* Ω (sometimes also called the twist operator) by its action on worldsheet coordinates:

$$\begin{cases} \Omega z = -\bar{z} \\ \Omega \bar{z} = -z \\ \Omega^2 = 1 \end{cases} \tag{7.I.1}$$

The action of Ω is exhibited in the figure below for an open string represented as a half annulus, with boundary B.



We begin by considering the bosonic string. Worldsheet parity is a symmetry of the bosonic string action, when x^μ transforms as

$$\Omega x^\mu(z,\bar{z}) = x^\mu(-\bar{z},-z) \ . \tag{7.I.2}$$

For *open strings*, we have the mode expansion

$$x^\mu(z,\bar{z}) = q^\mu - ip^\mu \ \ln|z|^2 + i\sum_{n\neq 0}^{\infty}\frac{1}{n}\,x_n^\mu(z^{-n}+\bar{z}^{-n}) \tag{7.I.3}$$

so that

$$\Omega q^\mu = q^\mu; \quad \Omega p^\mu = p^\mu; \quad \Omega x_m^\mu = (-)^m x_m^\mu \ . \tag{7.I.4}$$

Thus, to define the worldsheet parity of each state in $\mathcal{F}_{\rm open}$, it suffices to have the Ω assignment of the (tachyonic) ground state. Now, Ω is a symmetry of the worldsheet action and thus preserved in string interactions. Since there is a non-zero 3 tachyon vertex, we must have $\Omega^3 = 1$, so that $\Omega = 1$ on the tachyon:

$$\Omega \ |0,k\rangle = + \ |0,k\rangle \ \ . \tag{7.I.5}$$

As a result, the Ω-assignment of Yang-Mills states is negative:

$$\Omega \ x_{-1}^\mu \ |0,k\rangle = -x_{-1}^\mu \ |0,k\rangle \tag{7.I.6}$$

For *closed strings*, we have the mode expansion

$$x^\mu(z,\bar z) = q^\mu - ip^\mu \ \ln|z|^2 + i\sum_{m\neq 0}\frac{1}{m}(x^\mu_m z^{-m} + \tilde x^\mu_m \bar z^{-m}) \tag{7.I.7}$$

and

$$\Omega q^\mu = q^\mu;\ \Omega p^\mu = p^\mu \qquad \begin{cases} \Omega x^\mu_m = (-)^m \tilde x^\mu_m \\ \Omega \tilde x^\mu_m = (-)^m x^\mu_m \end{cases} . \tag{7.I.8}$$

The tachyon, dilaton and graviton now have $\Omega = 1$, but the antisymmetric tensor state $x^{[\mu}_{-1}\ \tilde x^{\nu]}_{-1}\ |0;k\rangle$ has $\Omega = -1$, because Ω interchanges the indices μ and ν.

In addition to acting on the string coordinate x^μ, Ω will also act on the Chan-Paton charges, since the end points of the open string are exchanged. We consider here SO(N) or Sp(N) Chan-Paton charges, where the representations Λ are real or pseudo-real. We introduce a basis of open string states with Chan-Paton charges

$$|\varepsilon, k; ij\rangle \tag{7.I.9}$$

where k denotes the momentum, the indices $i,j = 1,\ldots,\dim\Lambda = \dim\bar\Lambda = N$ label the Chan-Paton charges, and ε label the remaining quantum numbers, such as polarizations. A general state may be described by a matrix of Chan-Paton charges λ_{ij}, and we shall specialize here to the ground state:

$$\lambda_{ij}\ |0,k;ij\rangle\ . \tag{7.I.10}$$

Now apply Ω to this state, and use the fact that charges ij are interchanged,

$$\Omega\,\lambda_{ij}\ |0,k;ij\rangle = \lambda'_{ij}\ |0,k;ij\rangle\ . \tag{7.I.11}$$

Clearly, λ' must be linear in λ^T, so we should have

$$\lambda' = M\lambda^T M'\ . \tag{7.I.12}$$

Now, parity is a symmetry of the worldsheet action, and thus of the transition amplitudes, so we must have an additional reqirement on the Chan-Paton charges:

$$\mathrm{tr}\lambda_1\ldots\lambda_p = \mathrm{tr}\,\lambda_1^T\ldots\lambda_p^T\ . \tag{7.I.13}$$

Hence $M' = M^{-1}$. In addition, from $\Omega^2 = 1$, we have

$$\lambda = M(M\lambda^T M^{-1})^T = M(M^{-1})^T \lambda (M(M^{-1})^T)^{-1} \ . \tag{7.I.14}$$

Assuming that the set of matrices λ is complete, we have by Schur's lemma that $M^T = \pm M$. Up to changes of bases, we then have 2 possibilities

$$\begin{aligned} 1) \quad M^T = M = I_N : \quad & \Omega\lambda_{ij} \ |0, k; ij\rangle = \lambda^T_{ij} \ |0, k; ij\rangle \\ & \Omega\lambda_{ij} x^\mu_{-1} \ |0, k; ij\rangle = -\lambda^T_{ij} x^\mu_{-1} \ |0, k; ij\rangle \end{aligned} \tag{7.I.15}$$

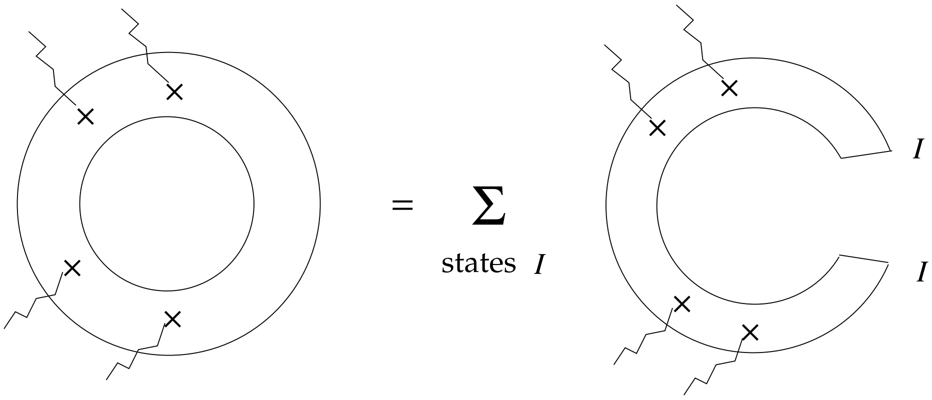
$$\begin{aligned} 2) \quad M^T = -M = J = \mathrm{diag}(\sigma_2 \ldots \sigma_2) : & \\ \Omega\lambda_{ij} \ |0, k; ij\rangle &= (J\lambda^T J)_{ij} \ |0, k; ij\rangle \\ \Omega\lambda_{ij} x^\mu_{-1} \ |0, k; ij\rangle &= -(J\lambda^T J)_{ij} x^\mu_{-1} \ |0, k; ij\rangle \ . \end{aligned} \tag{7.I.16}$$

Now, we are in a position to identify the rôle of *unorientable* strings. Ω is a (global) symmetry of the worldsheet action and of string interactions. This symmetry may be *gauged*, which means that the Hilbert space of states is projected to states that are invariant under Ω. The relevant projection operator is

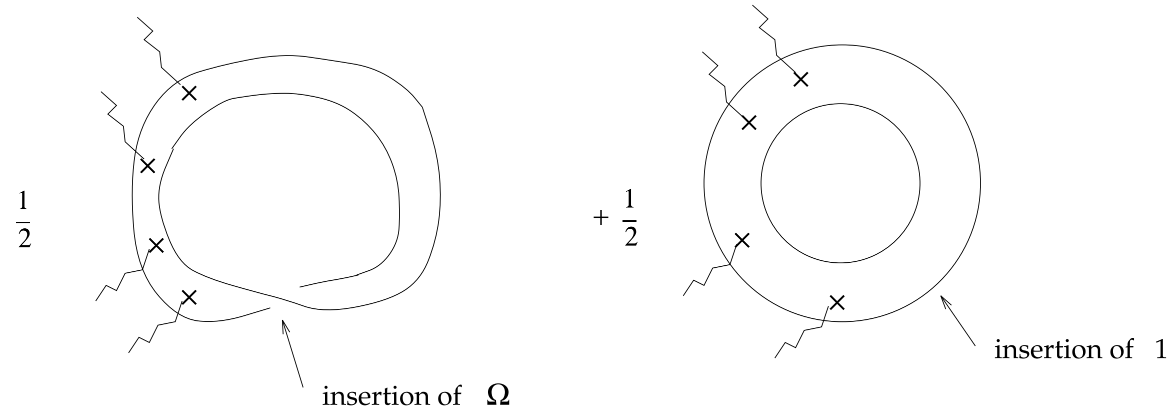
$$\frac{1}{2}(1 + \Omega) \ . \tag{7.I.17}$$

In the closed (unoriented) string, this projection eliminates the anti-symmetric tensor; in the open string, with Chan-Paton charges, as described above, it will project out the tachyon in 1) and 2) provided we require $\lambda^T = -\lambda$ in 1), i.e. SO(N) gauge group, and $\lambda^T = -J\lambda J$ in 2), i.e. Sp(N) gauge group. We see that we then retain the Yang-Mills vector states in the spectrum.

Non-orientable worldsheets now arise in the following way. An open string worldsheet, to tree level, with initial state I and final state F and possibly some vertex operators inserted, will give rise to a one loop diagram by summing over all possible intermediate states. For oriented open strings, we have



But when the open string spectrum is projected onto $\Omega = 1$ states only, then the projection operator $\frac{1}{2}(1+\Omega)$ must be inserted in the trace. Then, the contributions arise from the annulus and the Möbius string:



In the GSO projected RNS superstring, the worldsheet parity operator exchanges left and right moving ψ_σ's and thus interchanges left and right moving sectors of the GSO projected RNS spectrum. The unoriented $(\Omega = 1)$ projection of the closed string is achieved by imposing symmetry between left and right moving sectors. This can be achieved starting from the Type II B superstring, and retaining only states invariant under interchange of left and right movers. There remains in the *Type I closed string sector*

$$\begin{array}{llllll}
\mathrm{NS-NS} & \text{bosons} & (35) & G_{\mu\nu}; & (1) & \Phi \\
\mathrm{R-R} & \text{bosons} & (28) & A^{(2)}_{\mu\nu} & & \\
 & & & & & \\
\mathrm{NS-R} & \text{fermions} & & & & \\
\mathrm{R-NS} & \text{fermions} & (56) & \chi^\alpha_\mu, & (8) & \lambda_\alpha\ .
\end{array} \qquad (7.I.18)$$

In the *open superstring sector*, we have an $N = 1$ super-Yang-Mills multiplet in $D = 10$, with gauge group SO(32). (The restriction to SO(32) arises only at 1-loop level, and will not be justified further here.)

$$\begin{array}{lll} \mathrm{NS}: & 8 \times 496 & A_\mu \\ \mathrm{R}: & 8 \times 496 & \psi^\sigma \end{array} \qquad (7.I.19)$$