Lecture 13

VIII. Heterotic Strings

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So far, the only method we have discussed for obtaining massless Yang-Mills states was via the Chan-Paton rules in open string theory, where gauge charges were appended to the end points of open strings.

Heterotic string theories utilize a second method for obtaining massless Yang-Mills states. Heterotic string theories are constructed from *closed oriented strings* only. Internal degrees of freedom — required to obtain Yang-Mills states — are provided by degrees of freedom that live in the bulk of the worldsheet. The heterotic string construction relies on the independence of the oscillator modes of left and right moving sectors on closed oriented string worldsheets (the left and right momenta obey certain relations that couple left and right degress of freedom). The left moving sector of the heterotic string is built out of the (left-moving sector of the) Type II superstring, while the right moving sector of the heterotic string is built out of the (right-moving sector of the) bosonic string.

At first sight, this kind of construction would appear inconsistent. The Type II superstring part (left movers) exhibits $\mathcal{N}=1$ worldsheet supersymmetry, and $\mathcal{N}=1$ super-Virasoro invariance and has critical space-time dimension $D=10$, while the bosonic string part (right-movers) exhibits only Virasoro invariance, and has critical space-time dimension $D=26$. This mismatch of dimensions is put to excellent use in the following way. We let the heterotic string live in $D=10$ space-time dimensions, and we let the remaining 16 dimensions of right movers emerge as internal degrees of freedom, which are to be metamorphosed into Yang-Mills degrees of freedom.

This system of 16 free internal right movers forms a conformal field theory of central charge 16. It is a unitary CFT, since the 16 scalar fields emerge out of this construction with positive definite internal metric. (We shall later establish that the 16 scalars actually live on a torus, with constant internal metric.) An alternative way of seeing that this CFT must be unitary is by the fact that in the right moving sector, the Virasoro algebra has already been used up to eliminate the negative norm states from the 10-dimensional space-time part of the x-fields. Consistency of the construction will only be possible if the remaining internal part of the x-fields is unitary all by itself.

Poincaré invariance is required to hold only in the 10-dimensional subspace of the bosonic right movers which is naturally complemented by the 10-dimensional space of the bosonic left movers from the superstring. The internal degrees of freedom in the right moving sector are invariant under this 10-dimensional Poincaré symmetry, and are thus not constrained to be free fields on $\mathbb{R}^{16}$. In fact, it is consistent with conformal invariance in the right moving sector to consider a system of internal degrees of freedom described by a general, unitary conformal field theory of central charge 16.

On worldsheets of general topology, the separation of left and right movers, which is used in the very definition of the heterotic string, leads to anomalies in each sector separately under diffeomorphisms of Σ. Actually, the subgroup of diffeomorphisms connected to the identity, $\mathrm{Diff}_0(\Sigma)$, is anomaly free as long as the combined central charge of matter and ghosts vanishes in each sector. However, global diffeomorphisms, belonging to the modular group $\mathrm{Sp}(2h;\mathbb{Z})$ in general lead to global anomalies. Unitarity of the heterotic string (on worldsheets of genus $h \geq 1$) will require that such anomalies cancel, and this will impose severe restrictions on the allowed $c = 16$ CFT's that can govern the right moving sector of internal degrees of freedom. In fact, only gauge groups $E_8 \times E_8$ and $\mathrm{Spin}(32)/\mathbb{Z}_2$ will be allowed, as we shall show.

The benefits of the heterotic string construction should be quite evident. Due to space-time supersymmetry of the left movers, we automatically have a string theory with space-time fermions, but without a tachyon! In addition, the theory will possess large — and phenomenologically interesting — gauge symmetries. We now begin the construction of the heterotic string.

First, we shall investigate unitary $c = 16$ conformal field theories involving free fields only. Using Bose-Fermi correspondence, free fields may be formulated either in terms of 16 real scalar fields x^I, $I=1,\ldots,16$, which map Σ into a flat torus $\mathbb{R}^{16}/\Lambda$, for some lattice Λ, with Euclidean signature metric, or in terms of 32 Majorana spinors λ^α, $\alpha=1,\ldots,32$. Of course, we shall have to retain only the right-moving sector of these degrees of freedom. At the level of conformal field theory operators, one may simply retain the right moving operators, while at the level of transition amplitudes, we shall make use of a generalization of the chiral splitting theorem, established in §VI.

The actions in local complex coordinates z, $\bar z$ are given by

$$S_I[x^I,g] \equiv \frac{1}{4\pi}\int_\sigma d^2z\partial_z x^I\partial_{\bar z}x^I \tag{8.1}$$

$$S_I[\lambda^\alpha,g] \equiv \frac{1}{4\pi}\int_\Sigma d^2z(\lambda_+^\alpha\partial_{\bar z}\lambda_+^\alpha+\lambda_-^\alpha\partial_z\lambda_-^\alpha) \tag{8.2}$$

Here, we represent Majorana spinors by a complex Weyl spinor λ_+^α and its complex conjugate $\lambda_-^\alpha=(\lambda_+^\alpha)^*$. Summation over repeated indices I and α is assumed throughout.

A) Free Fermion Realization of Internal Degrees of Freedom

We start by analyzing the local properties (on Σ) of the conformal field theory described by $S_I[\lambda,g]$. We shall discuss the global issues, such as the spin structure of λ^α, shortly. We restrict attention to λ_-^α, since λ_+^α may be gotten by complex conjugation.

The field equations $\partial_z\lambda_-^\alpha=0$ imply that $\lambda_-^\alpha(\bar z)$ is anti-holomorphic, and the OPE is given as usual. (We assume that all λ_-^α have the same spin structure.)

$$\lambda_-^\alpha(\bar z)\lambda_-^\beta(\bar w)\sim\frac{\delta^{\alpha\beta}}{\bar z-\bar w} \tag{8.A.1}$$

The stress tensor is

$$T_{\bar z\bar z}=\frac{1}{2}\partial_{\bar z}\lambda_-^\alpha\lambda_-^\alpha \tag{8.A.2}$$

The action S_I is also invariant under SO(32) rotations of the spinor λ_-^α. We denote the structure constants of SO(32) by f^{abc} and the representation matrices in the fundamental (32-dimensional) representation of SO(32) by t^a, $a,b,c=1,\ldots,\dim\,\mathrm{SO}(32)=496$:

$$[t^a,t^b]=if^{abc}t^c \qquad\qquad \mathrm{tr}\,t^at^b=+\frac{1}{2}\delta_{ab}\ . \tag{8.A.3}$$

The SO(32) Noether current $J^a_{\bar z}$ is anti-analytic and given by

$$J^a_{\bar z}(\bar z) = (t^a)_{\alpha\beta}\lambda^\alpha_-\lambda^\beta_-(\bar z) \tag{8.A.4}$$

and satisfies the following OPE:

$$J^a_{\bar z}(\bar z)J^b_{\bar z}(\bar w) \sim \frac{k\delta^{ab}}{(\bar z-\bar w)^2} + i\frac{f^{abc}}{\bar z-\bar w}J^c_{\bar z}(\bar w)\ . \tag{8.A.5}$$

Here, we have $k\delta^{ab} \equiv 2\,\mathrm{tr}\,t^a t^b$ and thus $k=1$. Notice that this OPE is precisely the one encountered in Problem Set #8. Here again, the stress tensor of (8.A.2) may be expressed in terms of the SO(32) current by the Sugawara construction:

$$T_{\bar z\bar z} = \frac{1}{2\kappa}\ :J^a_{\bar z}J^a_{\bar z}: \tag{8.A.6}$$

with $\kappa = +\frac{1}{2}C_2(G)+k$, where $C_2(G)$ is the quadratic Casimir of the group G. The central charge is given by

$$c = \frac{2k.\dim G}{2k+C_2(G)}\ . \tag{8.A.7}$$

For $G=\mathrm{SO}(32)$, $\dim G = 496$, $C_2(G)=60$ and we have $k=1$. (Witten established that Wess-Zumino-Witten models with $k=1$ are equivalent to free fermions for general G.)

We now make use of this free fermion model to introduce internal degrees of freedom in the right-moving sector of the heterotic string. Thus, we retain from λ^α only λ^α_-. We begin by investigating free heterotic strings, and work on a worldsheet with the topology of an annulus (equivalently a cylinder), where two spin structures are possible. We denote these by P and A, and define them in analogy with R and NS spin structures:

$$\begin{array}{llll} \mathrm{P}: & \lambda^\alpha_- & \text{periodic on cylinder} & \text{(cf. R-sector)} \\ \mathrm{A}: & \lambda^\alpha_- & \text{anti-periodic on cylinder} & \text{(cf. NS-sector)} \end{array} \tag{8.A.8}$$

A priori, not all λ^α_- have to carry the same spin structures. If all λ^α_- carry the same spin structure, then the free fermion theory will be invariant under SO(32). More generally, one may consider spinors λ^α_- belonging to different groups, where the spin structure in each group is independently chosen. For example if λ^α_- fall into 2 groups of size k and $32-k$, with $1\le k\le 31$, respectively, then the remaining symmetry will be $\mathrm{SO}(k)\times\mathrm{SO}(32-k)$, with four possible combinations of spin structures; PP, AP, PA, AA. We shall investigate these possibilities shortly.

The mode expansions of $\lambda_-^\alpha(\bar z)$ are given in terms of integer (P) or integer $+\frac{1}{2}$ (A) modes:

$$\begin{aligned} \mathrm{P}: \qquad & \lambda_-^\alpha(\bar z) = \sum_{n\in\mathbb{Z}} \lambda_n^\alpha \; \bar z^{-n-1/2} \\ \mathrm{A}: \qquad & \lambda_-^\alpha(\bar z) = \sum_{r\in\frac{1}{2}+\mathbb{Z}} \lambda_r^\alpha \; \bar z^{-r-1/2} \end{aligned} \tag{8.A.9}$$

with

$$\begin{aligned} \mathrm{P}: \qquad & \{\lambda_m^\alpha, \lambda_n^\beta\} = \delta^{\alpha\beta}\delta_{m+n,0} \qquad m,n\in\mathbb{Z} \\ \mathrm{A}: \qquad & \{\lambda_r^\alpha, \lambda_s^\beta\} = \delta^{\alpha\beta}\delta_{r+s,0} \qquad r,s\in\tfrac{1}{2}+\mathbb{Z} \end{aligned} \tag{8.A.10}$$

As is expected in analogy with the R sector, there are zero modes in the P sector, generated by λ_0^α and obeying the Clifford algebra of SO(32). As a result, the P-ground state is a Spin(32) spinor, which we shall denote by $|\sigma;\mathrm{P}\rangle_R$, with σ running over the weights of the Spin(32) spinor representation (we shall see later that this spinor must be Weyl), and $\lambda_n^\alpha\,|\sigma;\mathrm{P}\rangle_R = 0$, for all $n\in\mathbb{N}$. (If several independent groups of λ_-^α's arise, SO(32) is reduced accordingly.) The P sector produces a Hilbert space

$$\mathcal{F}^{\mathrm{P}} \equiv \left\{ \{\lambda_{-n}^\alpha\}_{n\in\mathbb{N}}\,|\sigma;\mathrm{P}\rangle_R \right\} \;. \tag{8.A.11}$$

The A sector has a unique ground state denoted by $|0,\mathrm{A}\rangle_R$ and obeying $\lambda_r^\alpha\,|0,A\rangle_R = 0$ for all $r\in -\frac{1}{2}+\mathbb{N}$, and produces a Hilbert space

$$\mathcal{F}^{\mathrm{A}} \equiv \left\{ \{\lambda_{-r}^\alpha\}_{r\in-\frac{1}{2}+\mathbb{N}}\,|0,A\rangle_R \right\} \;. \tag{8.A.12}$$

The Virasoro generators are given as usual, and we shall only need L_0 and $\tilde L_0$ of these. We define those generators by normal ordering as usual:

$$L_0 = \frac{1}{2}p^2 + N; \qquad\qquad \tilde L_0 = \frac{1}{2}p^2 + \widetilde N \tag{8.A.13}$$

where the number operators are given by

$$\begin{aligned} \mathrm{R}: \qquad & N \equiv \sum_{n=1}^\infty (x_{-n}^\mu x_{n\mu} + n d_{-n}^\mu d_{n\mu}) \\ \mathrm{NS}: \qquad & N \equiv \sum_{n=1}^\infty x_{-n}^\mu x_{n\mu} + \sum_{r=\frac{1}{2}}^\infty r b_{-r}^\mu b_{r\mu} \end{aligned} \tag{8.A.14}$$

and

$$
\begin{aligned}
\mathrm{P}: \qquad & \widetilde{N} = \sum_{n=1}^{\infty} (\tilde{x}^{\mu}_{-n}\tilde{x}_{n\mu} + n\lambda^{\alpha}_{-n}\lambda^{\alpha}_{n}) \\
\mathrm{A}: \qquad & \widetilde{N} = \sum_{n=1}^{\infty} \tilde{x}^{\mu}_{-n}\tilde{x}_{n\mu} + \sum_{r=1/2}^{\infty} r\lambda^{\alpha}_{-r}\lambda^{\alpha}_{r}
\end{aligned}
\tag{8.A.15}
$$

It is understood in the above that the summations over the α-index run over the number of λ^{α}'s that belong either to the P or A sectors. We shall denote the eigenvalues in each sector by N_{R}, N_{NS}, N_{P}, N_{A}.

B) Free Fermion Realization of the $\mathrm{Spin}(32)/Z_2$ Heterotic String

We now assume that S_I has SO(32) symmetry, so that there are only 2 sectors: either all λ^{α}_{-} are P or all λ^{α}_{-} are A. The Virasoro conditions are easily worked out

$$
(L_0 - a)\ |\psi\rangle_L = 0 \qquad\qquad (\tilde{L}_0 - \tilde{a})\ |\psi\rangle_R = 0 \tag{8.B.1}
$$

The constants a and $\tilde{a}$ are given by:

$$
a = \begin{cases} 0 & \mathrm{R} \\ 1/2 & \mathrm{NS} \end{cases} \qquad\qquad \tilde{a} = \begin{cases} 1 & \mathrm{A} \\ -1 & \mathrm{P} \end{cases} \tag{8.B.2}
$$

The values for a were obtained in §VII. The values for $\tilde{a}$ follow in complete analogy with the corresponding values of a in the R, NS sectors, namely $1/24$ for each physical bosonic x, $-1/24$ for each integer moded physical fermionic ψ or λ, and $+1/48$ for each half integer moded physical fermionic ψ or λ.

We are now in a position to determine the massless spectrum. We have the following states

$$
\begin{pmatrix} N_{\mathrm{R}} = 0 \\ N_{\mathrm{NS}} = 1/2 \end{pmatrix}_L \otimes \begin{pmatrix} N_{\mathrm{A}} = 1 \\ \text{no P states allowed} \end{pmatrix}_R \tag{8.B.3}
$$

This may be seen from the expression for M^2 in each sector by combining (8.A.13–15) and (8.B.2)

$$
\begin{aligned}
M^2 &= 2N_{\mathrm{R}} = 2N_{\mathrm{NS}} - 1 && \text{left-movers} \\
&= 2N_{\mathrm{A}} - 2 = 2N_{\mathrm{P}} + 2 && \text{right-movers}
\end{aligned}
\tag{8.B.4}
$$

For reasons that will become fully apparent at string loop amplitudes only, a GSO projection must be carried out on the λ^α's, just as it was required in the R and NS sectors. In the functional integral formulations, the GSO summation will again correspond to a summation over the spin structures of the worldsheet spinor λ^α.

In the P sector, the projection is carried out with the help of the $(-1)^F$ operator, defined by

$$
\begin{aligned}
(-1)^F \equiv \bar{\lambda}_0 (-1)^{F_P} \qquad & F_P \equiv \sum_{n=1}^{\infty} \lambda^\alpha_{-n} \lambda^\alpha_n \\
\bar{\lambda}_0 \equiv \lambda^1_0 \lambda^2_0 \dots \lambda^{32}_0 &
\end{aligned}
\tag{8.B.5}
$$

which anti-commutes with all λ^α_n. We define the GSO projection on the Fock space of λ^α_- by requiring that only one eigenvalue of $(-1)^F$ be retained. Which eigenvalue is chosen is immaterial.

$$
\widetilde{\mathfrak{F}}^{\mathrm{P}}_{\mathrm{GSO}} \equiv \left\{ |\psi\rangle_{\mathrm{R}} \in \widetilde{\mathfrak{F}}^{p} \quad \text{such that} \quad (-1)^F \; |\psi\rangle_{\mathrm{R}} = |\psi\rangle_{\mathrm{R}} \right\} \; . \tag{8.B.6}
$$

In the A sector, the projection uses

$$
(-1)^F \equiv (-1)^{F_A} \qquad F_A \equiv \sum_{r=1/2}^{\infty} \lambda^\alpha_{-r} \lambda^\alpha_r \tag{8.B.7}
$$

which also anti-commutes with all λ^α_{-r}. We define the GSO projection on the Fock space of λ^α_- by requiring that only the $+$ eigenvalue of $(-1)^F$ is retained. This guarantees that the A-ground state is retained by the GSO projection.

$$
\widetilde{\mathfrak{F}}^{\mathrm{A}}_{\mathrm{GSO}} \equiv \left\{ |\psi\rangle_{\mathrm{R}} \in \widetilde{\mathfrak{F}}^{\mathrm{A}} \quad \text{such that} \quad (-1)^F \; |\psi\rangle_{\mathrm{R}} = |\psi\rangle_{\mathrm{R}} \right\} \; . \tag{8.B.8}
$$

Notice that in the NS and A sectors, the GSO projection is compatible with the physical state conditions. In the R and P sectors, the GSO projection simply determines the chirality.

It is now easy to list the massless states. In parentheses, we list the number of physical states in each sector.

Bosons NS sector only $N_{\rm NS} = 1/2,\ N_A = 1$

(1) $b^{\mu}_{-1/2}\tilde{x}^{\nu}_{-1}\ |0,k;{\rm NS}\rangle_{\rm L} \otimes |0,k\rangle_{\rm R} \otimes |0,{\rm A}\rangle_{\rm R}$

–	graviton	$G_{\mu\nu}$	(35)
–	anti-symmetric tensor	$B_{\mu\nu}$	(28)
–	dilaton	Φ	(1)

(2) $b^{\mu}_{-1/2}\lambda^{\alpha}_{-1/2}\lambda^{\alpha}_{-1/2}\ |0,k;{\rm NS}\rangle_{\rm L} \otimes |0,k\rangle_{\rm R} \otimes |0,{\rm A}\rangle_{\rm R}$

– SO(32) Yang-Mills field (adjoint representation) (8×496)

Fermions R sector only $N_{\rm R} = 0,\ N_{\rm A} = 1$

(1) $\tilde{x}^{\nu}_{-1}\ |0,\alpha;{\rm R}\rangle_{\rm L} \otimes |0,k\rangle_{\rm R} \otimes |0;{\rm A}\rangle_{\rm R}$

–	gravitino	$\chi^{\alpha}_{\mu}(56)$
–	dilatino	$\lambda_{\alpha}(\ 8)$

(2) $\lambda^{\alpha}_{-1/2}\lambda^{\beta}_{-1/2}\ |0,\alpha;k;{\rm R}\rangle_{\rm L} \otimes |0,k\rangle_{\rm R} \otimes |0;{\rm A}\rangle_{\rm R}$

– SO(32) gaugino (adjoint representation) $\psi^{\alpha}(8 \times 496)$

Next, we examine the first massive level, obtained for $M^2 = 2$:

$$\begin{pmatrix} N_{\rm R} = 1 \\ \\ N_{\rm NS} = 3/2 \end{pmatrix}_{\rm L} \otimes \begin{pmatrix} N_{\rm A} = 2 \\ \\ N_{\rm P} = 0 \end{pmatrix}_{\rm R} \tag{8.B.9}$$

The $N_{\rm R} = 1$ multiplet of left-movers has the structure

$$\begin{aligned} & x^{\mu}_{-1}\ |0,k,\alpha,{\rm R}\rangle_{\rm L} \\ & d^{\mu}_{-1}\ |0,k,\alpha;{\rm R}\rangle_{\rm L} \end{aligned} \tag{8.B.10}$$

and spans a massive Majorana fermion multiplet, with $2 \times 8 \times 8 = 128$ physical states. The $N_{\rm NS} = 3/2$ multiplet of left-movers has the following states

$$\begin{aligned} & x^{\mu}_{-1}b^{\nu}_{-1/2}\ |0,k,{\rm NS}\rangle_{\rm L} && (64) \\ & b^{\mu}_{-1/2}b^{\nu}_{-1/2}b^{\rho}_{-1/2}\ |0,k;{\rm NS}\rangle_{\rm L} && (56) \\ & b^{\mu}_{-3/2}\ |0,k;{\rm NS}\rangle_{\rm L} && (8) \end{aligned} \tag{8.B.11}$$

and has 128 physical bosonic states. Thus, the number of fermionic and bosonic physical states in the left-moving sector are equal, as is expected from space-time supersymmetry, as discussed also in §VII, The $N_{\mathrm{A}} = 2$ multiplet of right-movers has the following states

$$
\begin{array}{lll}
\tilde{x}^{\mu}_{-1}\tilde{x}^{\nu}_{-1}\ |0,k;\mathrm{A}\rangle_{\mathrm{R}} & (36) & \\
\tilde{x}^{\mu}_{-2}\ |0,k;\mathrm{A}\rangle_{\mathrm{R}} & (8) & \\
\tilde{x}^{\mu}_{-1}\lambda^{\alpha}_{-1/2}\lambda^{\beta}_{-1/2}\ |0,k;A\rangle_{\mathrm{R}} & (8\times 496) & \qquad (8.B.12)\\
\lambda^{\alpha}_{-1/2}\lambda^{\beta}_{-1/2}\lambda^{\gamma}_{-1/2}\lambda^{\delta}_{-1/2}\ |0,k;\mathrm{A}\rangle_{\mathrm{R}} & (32\times 31\times 30\times 29/24) & \\
\lambda^{\alpha}_{-1/2}\lambda^{\beta}_{-3/2}\ |0,k;\mathrm{A}\rangle_{\mathrm{R}} & (32\times 32) &
\end{array}
$$

with a total of 41016 states. Finally, the novel sector here is the P sector, whose analogue did not occur at the massless level. It produces a single Weyl spinor of Spin(32) with dimension $2^{15} = 32768$.

Now, consider the representations allowed in the spectrum:

- only 1 Weyl spinor of Spin(32);
- even numbers of λ's applied to ground state.

Thus, we obtain representations not of Spin(32), but only of Spin(32)$/Z_2$.

Physically, the spinor representation of Spin(32)$/Z_2$ is a very interesting object: it is a massive string state, but it is *stable* (in the 10-dimensional theory) since there are no massless states that are spinors, and since a spinor cannot decay into only vector representations.

C) Free Fermion Realization of the $E_8 \times E_8$ Heterotic String

In the free fermion construction of the Spin(32)$/Z_2$ theory, we have restricted λ^{α}_{-}'s to have either all P or all A spin structure in keeping with SO(32) invariance. We now investigate in what way this restriction may be relaxed, and we shall find one other consistent string theory: the $E_8 \times E_8$ model.

We assume that λ^{α}'s fall into two groups of sizes k and $32-k$ respectively and $0 < k < 32$, with all λ^{α}'s in each group carrying the same spin structure, either A or

P. The manifest symmetry of the action S_I with these spin structure assignments is then only $\mathrm{SO}(k) \times \mathrm{SO}(32-k)$, and the possible spin structures are AA, AP, PA, and PP. Here the first and second assignments refer to the spin structures of the first group of λ_-^α's transforming under $\mathrm{SO}(k)$, and the second group of λ_-^α's transforming under $\mathrm{SO}(32-k)$, respectively. (The case of more than two groups of λ_-^α's with independent spin structures may be treated analogously, but will ultimately not yield any consistent string theories.

The first modification in the setup of §B arises from the fact that the $\tilde{a}$ constants of the Virasoro conditions are now given by

$$\begin{cases} \tilde{a}_{\mathrm{AA}} = 1 \\ \\ \tilde{a}_{\mathrm{PP}} = -1 \end{cases} \qquad \begin{cases} \tilde{a}_{\mathrm{AP}} = \dfrac{8}{24} + \dfrac{k}{24} - \dfrac{32-k}{24} = -1 + \dfrac{k}{16} \\ \\ \tilde{a}_{\mathrm{PA}} = \dfrac{8}{24} - \dfrac{k}{24} + \dfrac{32-k}{48} = 1 - \dfrac{k}{16} \; . \end{cases} \tag{8.C.1}$$

To investigate which states are allowed, we examine the mass^2 conditions, by matching the M^2 eigenvalue in the left- and right-moving sectors

$$\begin{aligned} M^2 &= 2N_{\mathrm{R}} = 2N_{\mathrm{NS}} - 1 && \text{(left movers)} \\ &= 2\widetilde{N} - 2\tilde{a} && \text{(right movers)} \end{aligned} \tag{8.C.2}$$

Here, $\widetilde{N}$ denotes the eigenvalue of the number operator introduced in (8.A.15). Now, in view of the fact that N_{R} is integer ≥ 0 and that the GSO projection in the NS sector restricts to $2N_{\mathrm{NS}} - 1$ integer, it follows that M^2 is an even integer ≥ 0. By (8.C.2), $\widetilde{N} - \tilde{a}$ must be integer as well. Now, $\widetilde{N}$ is half-integer by construction; so if there are to be any states in the AP and PA sectors, we must have $\tilde{a}_{\mathrm{AP}} = -\tilde{a}_{\mathrm{PA}}$ half-integer.

If $k \neq 0 \pmod 8$, then we find from the above reasoning that there are no physical states in the AP and PA states. Thus, the only remaining sectors are AA and PP. But, then all λ_-^α's always have the same spin structure, either A or P, and the theory really has $\mathrm{SO}(32)$ symmetry, and reduces to the case studied in §B.

There remain 2 new, inequivalent cases: $\mathrm{SO}(16) \times \mathrm{SO}(16)$ and $\mathrm{SO}(8) \times \mathrm{SO}(24)$. The latter turns out to be inconsistent at the 1-loop level (as will be demonstrated in §IX), and we shall not investigate it any further here.

We now analyze the $SO(16) \times SO(16)$ model, and concentrate on the representations of $SO(16) \times SO(16)$, under which the massless gauge fields (space-time vectors) transform. These states belong to the multiplet with $N_{NS} = 1/2$ in the left-moving sector, and to those states with $\widetilde{N} = \tilde{a}$ and *with zero $\tilde{x}$ oscillator number*, in the right moving sector. There are now 4 contributions, from the AA, AP, PA and PP sectors. Below, we list the allowed states and indicate their respective representation of $SO(16) \times SO(16)$ (by listing the dimension of the representations) under each set of states.

$$\begin{array}{ll} \mathrm{AA}: & \lambda^{\alpha}_{-1/2}\lambda^{\beta}_{-1/2}\ |0,\mathrm{A}\rangle_{\mathrm{R}} \otimes |0,\mathrm{A}\rangle_{\mathrm{R}} \\ & (120,1) \oplus (1,120) \oplus (16,16) \\ \mathrm{AP}: & |0,\mathrm{A}\rangle_{\mathrm{R}} \otimes |0,\sigma,\mathrm{P}\rangle_{\mathrm{R}} \\ & (1,128) \oplus (1,128') \\ \mathrm{PA}: & |0,\sigma,\mathrm{P}\rangle_{\mathrm{R}} \otimes |0,\mathrm{A}\rangle_{\mathrm{R}} \\ & (128,1) \oplus (128',1) \\ \mathrm{PP}: & \text{no massless states} \end{array} \qquad (8.C.3)$$

Here, we have decomposed the Dirac spinor of $SO(16)$ (actually Spin(16)) into its two irreducible Weyl components denoted by 128 and 128′.

Now, all these states are to correspond to massless Yang-Mills fields, and they can form a consistent quantum field theory only if they correspond to the adjoint representation of some compact Lie algebra. (A theory of vector fields that does not arise as a Yang-Mills theory will not produce sufficient gauge invariance to eliminate the negative norm states created by the vector field.) But, there is no Lie algebra that contains all the states listed in (8.C.3). From the multiplet structure, it is clear that the Lie algebra has rank 16, with $SO(16) \times SO(16)$ as a maximal subalgebra. If all states were to occur, then the Lie algebra should have dimension 1008, but there is no such Lie algebra.

Again, to obtain a consistent theory, we have to project à la GSO. Of course, we can project out the AP and PA sectors, to recover the $\mathrm{Spin}(32)/Z_2$ theory. But, there is another solution: if we project out the $(16,16)$ representation, then we ought to obtain a factorized group, as is evident from the representation contents. Projecting out the (16, 16) representation is the only possible consistent projection that will not lead back to Spin(32)/Z_2. This can be seen as follows. The representation $(1, 120) \oplus (120, 1)$ in the AA sector is the adjoint representtion of SO(16) × SO(16), and cannot be projected out if the Yang-Mills multiplet has to have at least SO(16) × SO(16) symmetry. But then, if we were to retain the (16, 16) representation, the AA sector by itself would produce the gauge group SO(32). Including any of the representations in the AP and PA sectors would upset the Lie aglebra structure. Thus, it remains only to project out the (16, 16).

Projecting out the (16, 16) representation may be achieved by using the following construction. We have naturally 2 SO(16) × SO(16)-invariant fermion number operators.

$$\begin{aligned} (-1)^{F_1} &\equiv (-1)^{\sum\limits_{r>0} \lambda^{\alpha_1}_{-r}\lambda^{\alpha_1}_{r}} \qquad && \alpha_1 = 1, \ldots, 16 \\ (-1)^{F_2} &\equiv (-1)^{\sum\limits_{r>0} \lambda^{\alpha_2}_{-r}\lambda^{\alpha_2}_{r}} \qquad && \alpha_2 = 17, \ldots, 32 \end{aligned} \tag{8.C.4}$$

The product $(-1)^{F_1}(-1)^{F_2}$ is just the operator $(-1)^F$ of (8.B.7), in the A sector. The states (16, 16) are now eliminated by imposing the GSO projection

$$(-1)^{F_1} = 1 \qquad \underline{\text{and}} \qquad (-1)^{F_2} = 1\ . \tag{8.C.5}$$

This projection preserves all of the AP and PA sectors.

Having projected out (16, 16), we are now left with the following representation content in each factor:

$$120 \oplus 128 \oplus 128' \tag{8.C.6}$$

This content still does not close into a Lie algebra. But if we retain only one of the two SO(16) Weyl spinors, say 128, then we obtain the SO(16) contents of the adjoint representation of the exceptional Lie algebra E_8. Projecting out (128′) is easily achieved by the following GSO projection in the P sector. We define

$$\begin{aligned} (-1)^{F_1} &\equiv \lambda^1_0\lambda^2_0\ldots\lambda^{16}_0 \quad (-1)^{\sum\limits_{n>0} \lambda^{\alpha_1}_{-n}\lambda^{\alpha_1}_{n}} \qquad && \alpha_1 = 1, \ldots, 16 \\ (-1)^{F_2} &\equiv \lambda^{17}_0\ldots\lambda^{32}_0 \quad (-1)^{\sum\limits_{n>0} \lambda^{\alpha_2}_{-n}\lambda^{\alpha_2}_{n}} \qquad && \alpha_2 = 17, \ldots, 32\ , \end{aligned} \tag{8.C.7}$$

and require in the P sector

$$(-1)^{F_1} = 1 \qquad\qquad (-1)^{F_2} = 1 \ . \tag{8.C.8}$$

The Yang-Mills states that we are left with after this projection now correspond to the Lie algebra $E_8 \times E_8$.

In the functional integral formulation, the above GSO projections correspond to an *independent* summation over all spin structures, separately for the two groups of 16 spinors λ^{α_1} and λ^{α_2}. We shall show in the next chapter that this summation leads to modular invariant amplitudes. Insistence on modular invariance will ultimately be what selects out the two gauge groups $\mathrm{Spin}(32)/Z_2$ and $E_8 \times E_8$, as we shall start to see in the next section, and prove in full in §IX.

We leave it as an exercise to the reader to show that the GSO projected states at the massless level are those of the SO(32) heterotic string, but with the gauge group $\mathrm{Spin}(32)/Z_2$ replaced by $E_8 \times E_8$. Notice that the adjoint representations of both groups have the same dimensions 496, and that both groups have the same rank 16.

D) Bosonic Realizations of the $\mathrm{Spin}(32)/Z_2$ and $E_8 \times E_8$ Strings

For the $\mathrm{Spin}(32)/Z_2$ heterotic string, we exhibited all the generators of the associated Kac Moody algebra, in the form of fermion bilinear operators

$$J^a(\bar z) = t^a_{\alpha\beta} \lambda^\alpha_- \lambda^\beta_- (\bar z) \ . \tag{8.D.1}$$

Indeed, all λ_- have the same spin structure, and so the current $J^a(\bar z)$ is a single valued 1-form. For the $E_8 \times E_8$ construction, however, we can only exhibit an $\mathrm{SO}(16) \times \mathrm{SO}(16)$ current algebra this way, since products of λ_-'s with opposite spin structure will not produce single-valued currents. Thus, while the adjoint representation of SO(16) is represented this way, the spinor of SO(16) cannot be represented by currents of the form (8.D.1).

A general construction of vertex operators for simply laced algebras (all of whose roots have equal length) allows us to represent all current generators. This construction was developed by Lepowski and Wilson and by Frenkel, Kac and Segal; we shall not reproduce it here in detail. The key ingredient is that scalar fields compactified on a torus may produce enhanced symmetries at special radii of the torus. For a single scalar, this is familiar from Witten's lectures.

The starting point is a bosonized realization of the internal degrees of freedom λ_-^α in terms of right-moving scalars $x_R^I(\bar{z})$, $I = 1, \ldots, 16$. These x^I are assumed to be propagating on a 16-dimensional torus $\mathbb{R}^{16}/2\pi\Gamma$, specified by a lattice Γ, with basis vectors e_i^I, $i = 1, \ldots, 16$. We define the metric of the lattice Γ by the matrix of inner products of basis vectors ${e_i}^I$:

$$g_{ij} = \sum_{I=1}^{16} e_i^I e_j^I \ . \tag{8.D.2}$$

By construction, this lattice has positive definite metric.

To describe the $x_R^I(\bar{z})$, we begin by describing a full $x^I = x_L^I + x_R^I$ field propagating on $\mathbb{R}^{16}/2\pi\Gamma$. The solution for x^I on a worldsheet of a cylinder is:

$$x^I(z, \bar{z}) = q^I - i x_0^I \ln z - i \tilde{x}_0^I \ln \bar{z} + \text{ oscillators} \ . \tag{8.D.3}$$

Now, x^I is valued in $\mathbb{R}^{16}/2\pi\Gamma$, and this allows x^I to be multiple valued by shifts in $2\pi\Gamma$ as $z \to e^{2\pi i} z$:

$$x^I \left(e^{2\pi i} z, e^{-2\pi i} \bar{z}\right) = x^I(z, \bar{z}) + 2\pi \left(x_0^I - \tilde{x}_0^I\right) \tag{8.D.4}$$

with

$$x_0^I - \tilde{x}_0^I = \sum_{i=1}^{16} n_i e_i^I \in \Gamma \qquad\qquad n_i \in \mathbb{Z} \ . \tag{8.D.5}$$

To see the effect of this multi-valuedness, it is helpful to consider the topology of the cylinder instead of the torus. Thus we set $z = \exp(\tau + i\sigma)$ with $\sigma \in [0, 2\pi]$, so that

$$x^I(\tau, \sigma) = q^I - i(x_0^I + \tilde{x}_0^I)\tau + (x_0^I - \tilde{x}_0^I)\sigma + \text{oscillators} \ . \tag{8.D.6}$$

Clearly now, as $\sigma \to \sigma + 2\pi$, and we wind around the string once, the integers n_i indicate how many times the string wraps around direction i of th torus $\mathbb{R}^{16}/2\pi\Gamma$. Such winding configurations can occur because the torus $\mathbb{R}^{16}/2\pi\Gamma$ is non-simply connected, and these configurations are called *winding modes*, with winding numbers n_i.

Now x_0^I and $\tilde{x}_0^I$ are momenta of the string, and $\frac{1}{2}(x_0^I + \tilde{x}_0^I)$ is the total momentum. It must be valued in the lattice dual to Γ, which we denote by Γ^*, and which is generated by the basis of vetors e_i^{*I}, dual to the e_i^I:

$$\sum_I e_i^{*I} e_j^I = \delta_{ij} \ . \tag{8.D.7}$$

Thus, we must have

$$x_0^I + \tilde{x}_0^I = \sum_{i=1}^{16} m_i e_i^{*I} \in \Gamma^* \qquad\qquad m_i \in \mathbb{Z} \ . \tag{8.D.8}$$

The discreteness of the momenta results from the compactness of the torus $\mathbb{R}^{16}/2\pi\Gamma$. These modes are called *Kaluza-Klein modes*, and the integers m_i are the Kaluza-Klein momentum mode numbers. Now, the Virasoro constraints are just as in the bosonic string, except for the fact that we have split momenta and oscillators into 10 dimensions of space-time and 16 internal bosons. We reflect this splitting in the Virasoro

$$\begin{aligned} L_0 &= \frac{1}{2}p^2 + \frac{1}{2}x_0^I x_0^I + N \qquad & N &= \sum_{n=1}^{\infty}(x_{-n}\cdot x_n + x_{-n}^I x_n^I) \\ \tilde{L}_0 &= \frac{1}{2}p^2 + \frac{1}{2}\tilde{x}_0^I \tilde{x}_0^I + \widetilde{N} \qquad & \widetilde{N} &= \sum_{n=1}^{\infty}(\tilde{x}_{-n}\cdot \tilde{x}_n + \tilde{x}_{-n}^I \tilde{x}_n^I) \end{aligned} \tag{8.D.9}$$

The relevant Virasoro constraints are unmodified: $L_0 = \tilde{L}_0 = 1$, so that we have

$$\begin{aligned} &\frac{1}{2}x_0^I x_0^I - \frac{1}{2}\tilde{x}_0^I \tilde{x}_0^I + N - \widetilde{N} = 0 \\ &M^2 = \frac{1}{2}x_0^I x_0^I + \frac{1}{2}\tilde{x}_0^I \tilde{x}_0^I + N + \widetilde{N} - 2 \ . \end{aligned} \tag{8.D.10}$$

Now, we replace left-movers of the bosonic string by the left-movers of the RNS, GSO projected string. Thus, the Virasoro and mass constraints become:

$$\begin{aligned} &L_0 = \frac{1}{2}p^2 + N \qquad \begin{cases} N_{\mathrm{R}} = \sum_{n=1}^{\infty}(x_{-n}\cdot x_n + n d_{-n}\cdot d_n) & \text{(R)} \\ N_{\mathrm{NS}} = \sum_{n=1}^{\infty}(x_{-n}\cdot x_n) + \sum_{r=1/2}^{\infty} r b_{-r}\cdot b_r & \text{(NS)} \end{cases} \\ &\tilde{L}_0 = \frac{1}{2}p^2 + \frac{1}{2}\tilde{x}_0^I \tilde{x}_0^I + \widetilde{N} \qquad \widetilde{N} = \sum_{i=1}^{\infty}(\tilde{x}_{-n}\cdot \tilde{x}_n + \tilde{x}_{-n}^I \cdot \tilde{x}_n^I) \end{aligned} \tag{8.D.11}$$

Thus, we have

$$
\begin{aligned}
M^2 &= 2N_{\rm R} = 2N_{\rm NS} - 1 \\
&= \tilde{x}_0^I \tilde{x}_0^I + 2\widetilde{N} - 2
\end{aligned}
\tag{8.D.12}
$$

Furthermore, $\tilde{x}_0^I$ must now live on Γ and on Γ^*, and the length of any allowed $\tilde{x}_0^I \tilde{x}_0^I$ must be *even*. For general lattices Γ, Γ and Γ^* will not have any common points. Thus, the above condition puts severe constraints on what lattices are allowed for Γ. We assume that 16 independent directions for $\tilde{x}_0^I$ are allowed, which essentially implies that Γ must be *self-dual*. (We cannot fully prove at this point that self-duality is necessary; the latter condition will be firmly established from the requirement of modular invariance of the transition amplitudes at loop order $h \geq 1$, as we shall see in §IX. For the time being, we assume self-duality.)

There are then only two possible lattices:

$$
\begin{array}{llll}
E_8 \times E_8 & \text{root lattice,} & g_{ij} & \text{Cartan matrix} \\[1ex]
\mathrm{Spin}(32)/Z_2 & \text{root lattice,} & g_{ij} & \text{Cartan matrix}
\end{array}
\tag{8.D.13}
$$

The states of the heterotic strings are reproduced as follows: for massless states, we have

$$
\begin{pmatrix} N_{\rm R} = 0 \\ N_{\rm NS} = 1/2 \end{pmatrix}_L \otimes \left\{ \begin{pmatrix} \widetilde{N} = 1 \\ \tilde{x}_0^I \tilde{x}_0^I = 0 \end{pmatrix}_R \oplus \begin{pmatrix} \widetilde{N} = 0 \\ \tilde{x}_0^I \tilde{x}_0^I = 2 \end{pmatrix}_R \right\}
$$

$$
\begin{pmatrix} \widetilde{N} = 1 \\ \tilde{x}_0^I \tilde{x}_0^I = 0 \end{pmatrix} \qquad
\begin{array}{lcl}
\tilde{x}_{-1}^\mu \text{ states} & \to & N = 1 \text{ supergravity multiplet} \\
\tilde{x}_{-1}^I \text{ states} & \to & 16 \text{ states of super-Yang-Mills multiplet}
\end{array}
$$

$$
\begin{pmatrix} \widetilde{N} = 0 \\ \tilde{x}_0^I \tilde{x}_0^I = 2 \end{pmatrix} \qquad
\begin{array}{l}
480 \text{ states of } N = 1 \text{ super-Yang-Mills multiplet} \\
\text{obtained as } \underline{\text{winding}} \text{ states}
\end{array}
$$

The total number of Yang-Mills states is $480+16 = 496$, which coincides with the dimension of $E_8 \times E_8$ and $\mathrm{Spin}(32)/Z_2$.

In Problem Set #10, toroidal compactifications of the heterotic strings, their relation by T-duality and their moduli spaces are discussed.