Lecture 14

IX. Superstring Perturbation Theory

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We shall now develop a systematic treatment of superstring perturbation theory in the RNS formulation. Local $N = 1$ supersymmetry (SUSY) on the worldsheet plays a fundamental rôle, but this symmetry is not manifest in the component formalism, set up in §VII. The $N = 1$ superfield formalism, formulated in the language of $N = 1$ supergeometry, treats diffeomorphisms and local supersymmetry transformations on an equal footing, and renders both manifest. Their combined action makes up the group of super-diffeomorphisms of the worldsheet super-manifold, and local supersymmetry invariants can be constructed from super-diffeomorphism tensors. We first present the different ingredients needed and then apply these methods to the RNS superstrings. A general reference for this subject is E. D'Hhoker and D. H. Phong, "The Geometry of String Perturbation Theory", Rev. Mod. Phys. 60 (1988), p. 917.

A) $N = 1$ **supergeometry** (also called $N = 1$, $D = 2$ supergravity)

We consider a supermanifold Σ of dimension $(2|2)$, and use local coordinates $\xi^M = (\xi^m, \theta^\mu)$, $m = 1, 2$, $\mu = 1, 2$ with the grading

$$\xi^M \xi^N = (-)^{MN} \xi^N \xi^M \tag{9.A.1}$$

where $(-)^{MN}$ equals 1 unless $M = \mu$, $N = \nu$, in which case it equals -1. We introduce a local orthonormal *frame*

$$E^A \equiv d\xi^M E_M{}^A \qquad A = (a, \alpha) \quad a = z, \bar{z};\ \alpha = +, - \ , \tag{9.A.2}$$

where it is understood that a has the same grading as m, while α has the same grading as μ. We denote the inverse frame by $E_A{}^M$, and we have $E_A{}^M\, E_M{}^B = \delta_A{}^B$ and $E_M{}^A\, E_A{}^N = \delta_M{}^N$.

Next, we consider frame rotations by a group of $U(1)$ gauge transformations, under which the fields E^z, $E^{\bar{z}}$, E^+, E^- have weights -1, $+1$, $-\frac{1}{2}$, $+\frac{1}{2}$, respectively. A general superfield V has $U(1)$ weight $-n$ if it transforms as $(E^z)^{\otimes n}$. We introduce a $U(1)$ gauge field or *connection*

$$\Omega = d\xi^M \Omega_M \tag{9.A.3}$$

with associated covariant derivatives

$$\mathcal{D}_A^{(n)} V \equiv E_A{}^M (\partial_M + in\Omega_M) V \ . \tag{9.A.4}$$

Here $\partial_M \equiv \partial/\partial\xi^M$ and $i = \sqrt{-1}$. Torsion $T_{AB}{}^C$ and curvature R_{AB} superfield tensors are defined by the graded commutators

$$[\mathcal{D}_A, \mathcal{D}_B] V = T_{AB}{}^C \mathcal{D}_C V + in\, R_{AB} V \ . \tag{9.A.5}$$

The data of $N = 1$ supergeometry is the frame E^A and the $U(1)$ connection Ω, together with the following *constraints* on the torsion and curvature, arising in $[\mathcal{D}_\alpha, \mathcal{D}_\beta]$:

$$T^\gamma_{\alpha\beta} = 0; \quad T^c_{\alpha\beta} = 2\gamma^c_{\alpha\beta}; \quad R_{++} = R_{--} = 0 \tag{9.A.6}$$

Here, γ^c are the 2-dimensional Dirac matrices, satisfying $\{\gamma^a, \gamma^b\} = -\eta^{ab}$, with a convenient representation given by $\gamma^z_{++} = \gamma^{\bar{z}}_{--} = 1$, all other components $= 0$. The constraints of (9.A.6) are equivalent to the following set of constraints

$$\begin{aligned} &\mathcal{D}_+^2 = \mathcal{D}_z, \quad \mathcal{D}_-^2 = \mathcal{D}_{\bar{z}} \\ &\{\mathcal{D}_+, \mathcal{D}_-\} V = in\, R_{+-} V \ , \end{aligned} \tag{9.A.7}$$

where V is any superfield of weight n. The structure relations of the remaining covariant derivatives are defined by using the (graded) Jacobi identities. The only free quantity in this supergeometry is the *curvature superfield* $R_{+-} = R_{-+} \equiv R$.

The symmetry transformations on the data E^A and Ω that preserve the constraints (9.A.6) or equivalently (9.A.7) are

1) sDiff(Σ): super-diffeomorphisms $\xi^M \to \xi'^M(\xi^N)$

contains Diff(Σ), local $N=1$ susy, and further *algebraic* transformatons.

2) a) sWeyl(Σ): generated by a superfield Σ, contains Weyl(Σ) and further algebraic transformations.

b) $sU(1)$: generated by a superfield L, contains $U(1)$ gauge transformations and further algebraic transformations.

It is convenient to combine transformations a) and b) into complexified Weyl transformations, given in terms of a complex superfield Λ, defined by

$$\Lambda \equiv \Sigma - iL, \quad \Lambda^* \equiv \Sigma + iL\ . \tag{9.A.8}$$

The superderivatives transform under $s\mathrm{Weyl} \times sU(1)$ as

$$\begin{aligned} \mathcal{D}_+^{(n)} &= e^{n\Lambda - \frac{1}{2}\Lambda^*}\,\widehat{\mathcal{D}}_+^{(n)} e^{-n\Lambda} \\ \mathcal{D}_-^{(n)} &= e^{-n\Lambda^* - \frac{1}{2}\Lambda}\,\widehat{\mathcal{D}}_-^{(n)} e^{n\Lambda^*} \end{aligned} \tag{9.A.9}$$

and the curvature transforms as

$$R_{+-} = e^{-\Sigma}\left(\widehat{R}_{+-} - 2i\widehat{\mathcal{D}}_+\widehat{\mathcal{D}}_-\Sigma\right)\ . \tag{9.A.10}$$

From (9.A.9), the transformation rules for $E_\pm$ and $\Omega_\pm$ may be deduced, and from these combined with the torsion constraints (9.A.6), the transformation laws of E_z, $E_{\bar z}$, Ω_z and $\Omega_{\bar z}$ follow.

$N=1$ supergeometry admits an almost complex structure

$$J_M{}^N = E_M{}^a \epsilon_a{}^b E_b{}^N + E_M{}^\alpha \bar\gamma_\alpha{}^\beta E_\beta{}^N \tag{9.A.11}$$

where $\epsilon^2 = \bar\gamma^2 = -1$. In fact, J is a *complex structure* provided the constraints given in (9.A.6) are satisfied. (This may be viewed as one of the motivations for choosing those constraints.) We shall denote local complex coordinates by $(\xi, \bar\xi, \theta, \bar\theta)$ with $\xi = 1/\sqrt{2}\,(\xi^1 + i\xi^2)$, $\theta = 1/\sqrt{2}\,(\theta^1 + i\theta^2)$, and $\bar\xi$ and $\bar\theta$ their complex conjugates.

By a combined sDiff, sWeyl and $sU(1)$ transformation, any $N = 1$ supergeometry may be mapped on a locally flat (Euclidean) supergeometry, characterized by the superderivatives

$$\mathcal{D}_+ = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial\xi} \qquad \mathcal{D}_z = \mathcal{D}_+^2 = \frac{\partial}{\partial\xi}$$
$$\mathcal{D}_- = \frac{\partial}{\partial\bar\theta} + \bar\theta\frac{\partial}{\partial\bar\xi} \qquad \mathcal{D}_{\bar z} = \mathcal{D}_-^2 = \frac{\partial}{\partial\bar\xi} \; . \tag{9.A.12}$$

The residual symmetry group of $s\text{Diff} \times sU(1) \times s\text{Weyl}$ that leaves this supergeometry invariant is the group of *superconformal* transformations, which consists of superanalytic diffeomorphisms, $\xi \to \xi'(\xi,\theta)$, $\bar\xi \to \bar\xi'(\bar\xi,\bar\theta)$, $\theta \to \theta'(\xi,\theta)$; $\bar\theta \to \bar\theta'(\bar\xi,\bar\theta)$, combined with sWeyl and $sU(1)$ transformations.

Wess-Zumino gauge is obtained by fixing a gauge for the action of the *algebraic* parts of $s\text{Diff}(\Sigma)$, $sU(1)$ and sWeyl. The remaining independent fields are $e_m{}^a$, $\chi_m{}^\alpha$ and an auxiliary field A, and we have

$$\begin{aligned} E_m{}^a &= e_m{}^a + \theta\gamma^a\chi_m - \tfrac{1}{2}\theta\bar\theta e_m{}^a A \\ E_m{}^\alpha &= -\tfrac{1}{2}\chi_m{}^\alpha - \frac{i}{2}\theta^\beta(\gamma^a)_\beta{}^\alpha e_m{}^a A + \cdots \\ E_\mu{}^a &= (\gamma^a)_\mu{}^\beta\theta_\beta \\ E_\mu{}^\alpha &= \delta_\mu{}^\alpha\Big(1 + \frac{i}{4}\theta\bar\theta A\Big) \end{aligned} \tag{9.A.13}$$

This presentation of the supergeometry is so-called *in components.*

Integration over Σ is performed with the help of a super-measure $d\mu_E$, defined as follows:

$$d\mu_E \equiv s\det\, E_M{}^A d\xi\, d\bar\xi\, d\theta\, d\bar\theta \tag{9.A.14}$$

where the superdeterminant (also sometimes referred to as the Berezinian) is defined by

$$s\det\, E_M{}^A \equiv \det(E_m{}^a - E_M{}^\alpha(E_\mu{}^\alpha)^{-1}E_\mu{}^a)\cdot(\det\, E_\mu{}^\alpha)^{-1} \tag{9.A.15}$$

Of course, $d\mu_E$ is sDiff invariant. Inner products on superfields Ψ_1 and Φ_2 of weight n are defined by

$$\langle\Phi_1,\Phi_2\rangle = \int_\Sigma d\mu_E\, \Phi_1^*\Phi_2 \; . \tag{9.A.16}$$

B) Functional Integral Representation of Transition Amplitudes

The RNS worldsheet action, introduced in §VII in components, admits a simple expression in superfield form. We introduce a string coordinate superfield of $sU(1)$ weight 0; $X = \Sigma \to \mathbb{R}^{10}$

$$X^\mu = x^\mu + \theta^\mu \psi_+{}^\mu + \bar\theta \psi_-^\mu + \theta\bar\theta\, F^\mu \qquad \mu = 0, 1, \ldots, 9\ . \tag{9.B.1}$$

The action

$$S_X = \frac{1}{4\pi}\int_\Sigma d\mu_E \mathcal{D}_- X \cdot \mathcal{D}_+ X \tag{9.B.2}$$

reduces to the RNS action,* except for the fact that it has an entire term of the form F^2. This term is non-dynamical and the F field may be integrated out without any consequences.

The next action we need is that of the super (B, C) system: let B and C be superfields of $sU(1)$ weight n and $\frac{1}{2} - n$, respectively. The (B, C) action is then

$$S_{BC} = \frac{1}{2\pi}\int_\Sigma d\mu_E (B\mathcal{D}_-^{(n)} C + \bar B \mathcal{D}_+^{(-n)} \bar C)\ . \tag{9.B.4}$$

To make contact with the (b, c) action, we decompose the superfields B, C in components

$$\begin{aligned} B &= \beta + \theta b + \bar\theta b' + \theta\bar\theta\beta' \\ C &= c + \theta\gamma + \bar\theta\gamma' + \theta\bar\theta c' \end{aligned} \tag{9.B.5}$$

The fields b', β', γ' and c' are auxiliary, just as F was an auxiliary field in S_X, and we shall drop their contribution to S_{BC}. The $U(1)$ weight assignments in Wess-Zumino gauge are now obtained by using the fact that θ (resp. $\bar\theta$) have weight $-\frac{1}{2}$ (resp. $\frac{1}{2}$). Hence, the weights are, for $n = -1$:

field	β	b	γ	c
$U(1)$ weight	$3/2$	2	$-1/2$	-1

(9.B.6)

* In locally flat coordinates, this action equals in components

$$S_X = \frac{1}{4\pi}\int_\Sigma d^2 z (\partial_z x \cdot \partial_{\bar z} x - \psi_+ \cdot \partial_{\bar z}\psi_+ - \psi_- \cdot \partial_z \psi_- - F^2) \tag{9.B.3}$$

so that the sign of the ψ-kinetic term is opposite to the convention we used previously. Notice that this reverses the sign of the ψ stress tensor. Apologies! (Conventions are to be uniformized in the final write-up.)

We recognize the (b,c) system of anti-commuting ghost fields, and we now also find a (β,γ) system of commuting ghost fields (called superghost) which has precisely the correct weight assignments to be the ghost system for local supersymmetry. The $(\beta\gamma)$ fields are worldsheet spinors, and their spin structure must be the same as that of $\psi_+{}^\mu$, in order for $(\beta\gamma)$ to be $N=1$ local supersymmetry ghosts. The BC action may be worked out in components and becomes in WZ gauge:

$$S_{BC} = \frac{1}{2\pi}\int_\Sigma d^2z[b\nabla^z_{(-1)}c + \beta\nabla^z_{(-1/2)}\gamma - \chi_{\bar z}{}^+ S^{\bar z}_+ + c.c.] \qquad (9.B.7)$$

where the BC supercurrent is

$$S_{z+} \equiv +\tfrac{1}{2}b\gamma - \frac{3}{2}\beta\nabla_z c - \nabla_z\beta c \qquad (9.B.8)$$

This current is the $N=1$ superpartner of the stress tensor and it is conserved; in fact it is analytic.

The combined X, (B,C) system for $D=10$ components of X is superconformal invariant, and so the super-conformal field theory associated with this system depends upon $N=1$ supergeometry only through its *supermoduli*. Recall that supermoduli space is defined by

$$s\mathcal{M}_h \equiv \frac{\{E_M{}^A, \Omega_M \text{ satisfying constraints of (9.A.6)}\}}{s\mathrm{Diff}\times s\mathrm{Weyl}\times sU(1)} \qquad (9.B.9)$$

and is a space of dimension

$$\dim\, s\mathcal{M}_h = \begin{cases} (0|0) & h=0 \\ \begin{cases}(1|0) & h=1 \quad \text{even spin structure} \\ (1|1) & h=1 \quad \text{odd spin structure}\end{cases} \\ (3h-3|2h-2) & n\geq 2 \end{cases} \qquad (9.B.10)$$

The holomorphic cotangent space to $s\mathcal{M}_h$ is spanned by super-holomorphic 3/2 differentials Φ_K, which satisfy

$$\mathcal{D}_-^{(3/2}\Phi_K = 0 \qquad \begin{matrix}\Phi_K \text{ odd graded} \\ K=1,\ldots,\dim\, s\mathcal{M}_h\end{matrix} \qquad (9.B.11)$$

Super-Beltrami differentials are defined as the duals to Φ_K, and may be parametrized by the super-moduli variations of $E_M{}^A$:

$$\mu_K \equiv E_-{}^M \frac{\partial E_M{}^z}{\partial m_K} \qquad K=1,\ldots,\dim\, s\mathcal{M}_h\ . \qquad (9.B.12)$$

It is instructive to work out the μ_K in Wess-Zumino gauge:

$$\mu_K = \bar{\theta} \left(e_{\bar{z}}{}^n \frac{\partial e_n{}^z}{\partial m_K} - \theta \frac{\partial \chi_{\bar{z}}{}^+}{\partial m_K} \right) \ . \tag{9.B.13}$$

The first term on the r.h.s. is recognized as the ordinary Beltrami differential, for m_K an even modulus, while the second term corresponds to a novel odd Beltrami differential for m_K an odd modulus.

We are now ready to state a preliminary *Definition* of transition amplitudes for RNS superstring amplitudes. This preliminary definition is the natural generalization of the "summation over surfaces" definition adopted for bosonic string amplitudes in §I. It is preliminary in the sense that it treats left and right chirality worldsheet spinors with the same spin structure. In the final definition of transition amplitudes, these spin structures will have to be summed over independently for left and right movers, and this will be presented in §IX.F, after the Chiral Splitting Theorem has been established. The preliminary definition will be useful, however, in order to introduce superghosts, and it is needed in order to set up the Chiral Splitting Theorem. Transition amplitudes are defined by

$$A_h = \int DE_M{}^A D\Omega_M \delta\, (\text{constraints}) \int DX^\mu V_1 \ldots V_N e^{-S_X} \tag{9.B.14}$$

for certain vertex operators V_i, to be specified shortly.

In the critical dimension, $D = 10$, and for super-conformally invariant vertex operators V_i, we have the following *Theorem*

$$\begin{aligned} A_h = \int_{s\mathcal{M}_h} \prod_K dm_K \int DX^u \int D(B\bar{B}) \int D(C\bar{C}) V_1 \ldots V_N e^{-S_X - S_{BC}} \\ \cdot \prod_K \delta(\langle \mu_{\scriptscriptstyle K}, B\rangle) \delta(\langle \bar{\mu}_{\scriptscriptstyle K}, \bar{B}\rangle) \end{aligned} \tag{9.B.15}$$

The proof of this result is much more involved than in the case or bosonic strings, because the inner products $\langle \Phi_1, \Phi_2\rangle$ are no longer positive on superfields, and extra care must be used in dealing with the $\mathcal{D}_+$ and $\mathcal{D}_-$ operators. We shall not prove it here; we refer the reader to Rev. Mod. Phys. **60** (1988) p. 917.

C) Superconformal Field Theory (some basics)

A conformal field theory with an additional $N = 1$ local supersymmetry invariance is automatically invariant under superconformal transformations, and is an $N = 1$ *superconformal field theory.* Each conformal field has a superpartner (which is also a conformal field), and together both fields fit into a superconformal field; X^μ, B, C are examples of such fields. Conserved currents are other examples of conformal fields that generalize to supercurrents. Conservation of a supercurrent J is given by

$$\mathcal{D}_- J = 0 \tag{9.C.1}$$

In local super-complex coordinates, this means that

$$J = J(z, \theta) = J_{1/2}(z) + \theta J_1(z) \tag{9.C.2}$$

with $J_{1/2}$ and J_1 holomorphic in z. The associated charge is defined by

$$Q_J = \oint dz\, d\theta\, J = \oint dz\, J_1(z) \tag{9.C.3}$$

Important examples of supercurrents include the stress tensor, which is built up out of the ordinary stress tensor T_{zz} and the (worldsheet) supercurrent S_{z+}:

$$T(z, \theta) \equiv S_{z+}(z) + \theta T_{zz}(z)\ . \tag{9.C.4}$$

For the X-system, T is given by

$$\begin{aligned} T^{(X)} &= -\tfrac{1}{2}\mathcal{D}_+ X \cdot \mathcal{D}_+^2 X \\ &= -\tfrac{1}{2}\psi_+ \cdot \partial_z x + \theta\left(-\tfrac{1}{2}\partial_z x \cdot \partial_z x - \tfrac{1}{2}\partial_z\psi_+ \cdot \psi_+\right) \end{aligned} \tag{9.C.5}$$

We recognize indeed the super-current and the $x\psi$ stress tensor. For the (B, C) system, T is given by

$$\begin{aligned} T^{(BC)} &= -C\mathcal{D}_+^2 B + \frac{1}{2}\mathcal{D}_+ C \mathcal{D}_+ B - \frac{3}{2}\mathcal{D}_+^2 C B \\ &= -c\partial_z\beta + \frac{1}{2}\gamma b - \frac{3}{2}\partial_z c\beta + \theta(T^{(bc)} + T^{(\beta\gamma)}) \end{aligned} \tag{9.C.6}$$

The OPE of T with superfields of weight $(h,0)$ is determined by Ward identities, and we have the asymptotic expansions

$$T(z,\theta)\Phi(w,\zeta) \sim n\frac{\theta-\zeta}{(z-w-\theta\zeta)^2}\Phi(w,\zeta)+\tfrac{1}{2}\frac{1}{z-w-\theta\zeta}\mathcal{D}_+\Phi(w,\zeta) + \frac{\theta-\zeta}{z-w-\theta\zeta}\mathcal{D}_+^2\Phi(w,\zeta) \tag{9.C.7}$$

For example $\mathcal{D}_+X$, B and C have weights $n=1/2, 3/2, -1$ respectively. The super-Virasoro algebra is similarly expressed. Denote $s \equiv z-w-\theta\zeta$

$$T(z,\theta)T(w,\zeta) \sim \frac{\frac{1}{4}\hat{c}}{s^3}+\frac{3}{2}\frac{\theta-\zeta}{s^2}T(w,\zeta)+\tfrac{1}{2}\frac{1}{s}\mathcal{D}_+T(w,\zeta)+\frac{\theta-\zeta}{s}\mathcal{D}_+^2T(w,\zeta) \tag{9.C.8}$$

where $\hat{c}=\frac{2}{3}c$ represents the central charge.

D) BRST **Quantization**

The combined action S_X+S_{BC} is BRST invariant. For completeness, we record the BRST transformations on the fields

$$\begin{aligned} \delta X^\mu &= \lambda C\mathcal{D}_+^2X^\mu - \tfrac{1}{2}\lambda\,\mathcal{D}_+C\mathcal{D}_+X^\mu + \text{ complex conjugate} \\ \delta C &= \lambda C\mathcal{D}_+^2 C - \frac{1}{4}\lambda\mathcal{D}_+C\mathcal{D}_+C \\ \delta B &= -\lambda T \end{aligned} \tag{9.D.1}$$

where λ is a constant Grassmann-valued parameter and T is the stress tensor superfield, defined in C). The associated BRST supercurrent is

$$J_{\mathrm{BRST}} = C(T^{(X)}+\tfrac{1}{2}T^{(BC)})-\frac{3}{4}\mathcal{D}_+(C\mathcal{D}_+CB) \tag{9.D.2}$$

The BRST charge

$$Q_{\mathrm{BRST}} = \oint dz\,d\theta\,J_{\mathrm{BRST}} \tag{9.D.3}$$

is conserved and nilpotent precisely in $D=10$, in analogy with the bosonic string. Vertex operators for physical states commute with Q_{BRST}, while vertex operators of spurious states are Q_{BRST}-commutators of some operator. We shall not carry out BRST quantization any further here.

E) Vertex operators for physical states

We distinguish two kinds of vertex operators: those for the NS sector, creating space-time bosons, and those for the R sector, creating space-time fermions (also sometimes called the fermion vertex operator). For closed strings, this produces four different kinds of vertex operators: NS-NS and R-R creating bosons, and NS-R and R-NS creating fermions. The R-vertex cannot be constructed directly out of the fields X^μ, since it must produce a branch cut starting at its insertion points.

(1) The NS vertex operator

For simplicity, we specialize to the closed RNS string, and consider NS-NS vertex operators first; the left-moving part may be constructed using the chiral splitting theorem for the RNS string, to be derived in the next section, §F. We begin by constructing RNS vertex operators for NS-NS states without ghost dependence.

Vertex operators in the NS-NS sector are given in a fashion completely analogous to the bosonic string

$$V(\varepsilon, k) = \int_\Sigma d\mu_E P_n(\varepsilon, \mathcal{D}_+ X, \mathcal{D}_- X, \mathcal{D}_+^2 X, \dots) e^{ik\cdot X} \tag{9.E.1}$$

Here, P_n is a polynomial involving a sum over terms, each of which has a total number n of $\mathcal{D}_+$ derivatives. By $sU(1)$ invariance of V, P_n must then also have a total number n of $\mathcal{D}_-$ derivatives. (On non-flat $N=1$ supergeometries, the curvature superfield R_{+-} may also enter, just as the Gaussian curvature R_g entered for the bosonic string in the dilaton vertex. R_{+-} effectively counts for one $\mathcal{D}_+$ and for one $\mathcal{D}_-$.) By construction, $V(\varepsilon, k)$ is invariant under sDiff transformations. It remains to guarantee that $V(\varepsilon, k)$ is sWeyl invariant.

Under constant sWeyl transformations, $\mathcal{D}_+$ and $\mathcal{D}_-$ have weight $1/2$, $d\mu_E$ has weight -1 and the exponential has weight k^2. Given that $\#\mathcal{D}_+ = n_{\mathcal{D}} = \#\mathcal{D}_-$, we have

$$M^2 = -k^2 = n_{\mathcal{D}} - 1 \tag{9.E.2}$$

The GSO projection forces n to be odd only, since the number of NS fields applied to the ground state in each sector (left and right moving) must be odd. Thus, we have $n_{\mathcal{D}} = 2n+1$, with $n = 0, 1, 2, \ldots$ and the masses of NS-NS states are given by

$$M^2 = 2n \qquad\qquad n = 0, 1, 2, \ldots \tag{9.E.3}$$

Non-constant sWeyl transformations will require in addition that certain conditions be fulfilled on the polarization tensors ε, such as transversality.

The massless string NS-NS vertex operator is

$$V(\varepsilon, k) = \varepsilon_{\mu\nu}(k) \int_\Sigma d\mu_E \mathcal{D}_+ X^\mu \mathcal{D}_- X^\nu e^{ik\cdot X} \tag{9.E.4}$$

sWeyl invariance requires $\varepsilon_{\mu\nu}(k)k^\mu = \varepsilon_{\mu\nu}(k)k^\nu = 0$, and we recover the vertex operator for dilaton, graviton and $B_{\mu\nu}$ fields. It is useful to work out this vertex operator in components, dropping the auxiliary field F, and other terms that vanish by the field equations, such as $\partial_{\bar z}\psi_+$. (The full justification for dropping terms that vanish by the field equations may be given with the help of the Chiral Splitting Theorem in the next section, §F.) Restricting to left-movers only, we use the fact that

$$X^\mu = X_L{}^\mu + X_R{}^\mu \tag{9.E.5}$$

and obtain the left-moving NS part of the vertex operator (9.E.?) as follows:

$$\int d\theta\, \mathcal{D}_+ X_L^\mu e^{ik\cdot X_L} = (\partial_z x_L^\mu - i\psi_+^\mu \psi_+^\nu k_\nu) e^{ik\cdot x_L} \tag{9.E.6}$$

This vertex contains the bosonic vertex operator, as well as an operator that allows for an application of an even number of NS fermions only.

(2) The R vertex operator

Ramond states transform under spinor representations of the (double cover of the) Lorentz group, Spin$(1,9)$. This group has two inequivalent Majorana-Weyl spinors S_α and S'^α, each of dimension 16 so that we let $\alpha = 1, \ldots, 16$. To describe the representations S_α and S'^α, we introduce the Clifford algebra of Spin$(1,9)$

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} I_{(32)} \tag{9.E.7}$$

where $I_{(32)}$ is the identity matrix of dimension 32. The chirality matrix $\bar{\Gamma}$ is defined by

$$\bar{\Gamma} \equiv \Gamma^0 \Gamma^1 \cdots \Gamma^9$$

and satisfies

$$\bar{\Gamma}^2 = I_{(32)} \quad \text{and} \quad \{\Gamma^\mu, \bar{\Gamma}\} = 0 \tag{9.E.8}$$

Weyl spinors of left (resp. right) chirality are eigenvectors of $\bar{\Gamma}$ with eigenvalues +1 (resp. −1). The charge conjugation matrix C is defined by the relation

$$(\Gamma^\mu)^T = -C\Gamma^\mu C^{-1} \tag{9.E.9}$$

and anti-commutes with $\bar{\Gamma}$:

$$\{\bar{\Gamma}, C\} = 0 \; . \tag{9.E.10}$$

The charge conjugate ψ^c of ψ is defined by

$$\psi^c \equiv C\bar{\psi}^T \qquad\qquad \bar{\psi} = \psi^\dagger \Gamma^0 \tag{9.E.11}$$

A Majorana spinor is a spinor whose charge conjugate is equal to itself, $\psi^c = \psi$. It is consistent (and convenient) to choose $C = \Gamma^0$, so that a Majorana spinor is *real*: $\psi^c = \psi^* = \psi$. We shall choose a basis in which

$$\bar{\Gamma} = \begin{pmatrix} I_{(16)} & 0 \\ 0 & -I_{(16)} \end{pmatrix} \; , \qquad C = \Gamma^0 = \begin{pmatrix} 0 & I_{(16)} \\ -I_{(16)} & 0 \end{pmatrix} \tag{9.E.12}$$

A 32 component Majorana spinor ψ now decomposes into two 16 component Majorana-Weyl spinors S_α and S'^α. It will be convenient to work with Γ-matrices in this basis:

$$\psi = \begin{pmatrix} S_\alpha \\ S'^\alpha \end{pmatrix} \qquad\qquad \Gamma^\mu = \begin{pmatrix} 0 & (\gamma^\mu)_{\alpha\beta} \\ (\gamma^\mu)^{\alpha\beta} & 0 \end{pmatrix} \tag{9.E.13}$$

with $\alpha, \beta = 1, \ldots, 16$. It then follows from (9.E.9), (9.E.12) that $\gamma^\mu_{\alpha\beta} = \gamma^\mu_{\beta\alpha}$ and $\gamma^{\mu\alpha\beta} = \gamma^{\mu\beta\alpha}$. The generator of Spin(1, 9) transformations in this representation is $\frac{1}{2}\Gamma^{\mu\nu}$, with

$$\begin{aligned} \Gamma^{\mu\nu} &\equiv \tfrac{1}{2}[\Gamma^\mu, \Gamma^\nu] = \begin{pmatrix} \gamma^{\mu\nu} & 0 \\ 0 & \gamma^{\mu\nu} \end{pmatrix} \\ \gamma^{\mu\nu} &\equiv \frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right] \; . \end{aligned} \tag{9.E.14}$$

Clearly, $\Gamma^{\mu\nu}$ commutes with $\bar{\Gamma}$, and $\frac{1}{2}\gamma^{\mu\nu}$ are the representation generators of Spin$(1,9)$ in one of the Weyl representations.

The SO$(1,9)$ transformation properties in the Ramond sector are governed by the SO$(1,9)$ currents (we omit the contribution from the x-field)

$$j^{\mu\nu}(z) = \psi_+^\mu\, \psi_+^\nu(z) \tag{9.E.15}$$

Clearly, the field ψ_+^μ itself transforms under the vector representation of SO$(1,9)$:

$$j^{\mu\nu}(z)\psi_+^\rho(w) \sim \frac{1}{z-w}(\eta^{\mu\rho}\psi_+^\nu(w) - \eta^{\nu\rho}\psi_+^\mu(w)) \ . \tag{9.E.16}$$

We seek to construct fields $S_\alpha(z)$ and $S'^\alpha(z)$ such that transform as spinors under Spin$(1,9)$ with representation matrices $\gamma^{\mu\nu}$:

$$j^{\mu\nu}(z)S_\alpha(w) \sim \frac{1}{z-w}\left(-\tfrac{1}{2}\right)(\gamma^{\mu\nu})_\alpha{}^\beta S_\beta(w) \tag{9.E.17a}$$

$$j^{\mu\nu}(z)S'^\alpha(w) \sim \frac{1}{z-w}\left(-\tfrac{1}{2}\right)(\gamma^{\mu\nu})^\alpha{}_\beta S'^\beta(w) \ . \tag{9.E.17b}$$

From these OPE's, and from the fact that $j^{\mu\nu}$ is bilinear in ψ_+^μ, we may infer the OPE of ψ_+ with S:

$$\psi_+^\mu(z)S_\alpha(w) \sim \frac{1}{(z-w)^{\frac{1}{2}}}\frac{1}{\sqrt{2}}(\gamma^\mu)_{\alpha\beta}S'^\alpha(w) \tag{9.E.18a}$$

$$\psi_+^\mu(z)S'^\alpha(w) \sim \frac{1}{(z-w)^{\frac{1}{2}}}\frac{1}{\sqrt{2}}(\gamma^\mu)^{\alpha\beta}S_\beta(w) \ . \tag{9.E.18b}$$

The relative normalization between S and S' is arbitrary, and has been fixed above on symmetry grounds. The conformal weights of $S_\alpha(z)$ and $S'^\alpha(z)$ are computed from their OPE with the stress tensor $T^{(\psi)} = -\frac{1}{2}\partial_z\psi_+ \cdot \psi_+$:

$$\begin{aligned} T^{(\psi)}(z)S_\alpha(w) &\sim \left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\frac{1}{\sqrt{2}}\frac{1}{(z-w)^{3/2}}\gamma^\mu_{\alpha\beta}\,\psi_+^\mu(z)S'^\beta(w) \\ &\sim \frac{1}{16}(\gamma^\mu\gamma_\mu)_\alpha^\beta S_\beta(w)\frac{1}{(z-w)^2} \ . \end{aligned} \tag{9.E.19}$$

Since $\gamma^\mu \gamma_\mu = 10\, I_{(16)}$, we find that the conformal weight of S_α (and similarly of S'^α) is 5/8.

To obtain the spin fields S_α and S'^α, we make use first of the action of the currents $j^{\mu\nu}$ in the Cartan subalgebra of Spin$(1,9)$ only. These components mutually commute and may be bosonized in terms of 5 independent free scalar Bose fields:

$$j^{2a,2a+1}(z) = -i\partial_z \phi^a(z) \qquad a = 0,1,2,3,4\ . \tag{9.E.20}$$

The R-fields $\psi_+{}^\mu$ are exponentials of ϕ^a with ± 1 weights:

$$\frac{1}{\sqrt{2}}\,(\psi_+^{2a} \pm i\psi_+^{2a+1}) = e^{\pm i\phi^a}\ . \tag{9.E.21}$$

The spin fields are exponentials of ϕ^a with $\pm\frac{1}{2}$ weights:

$$S_\alpha(z) = e^{i\alpha\cdot\phi}\ . \tag{9.E.22}$$

where we identify the space-time indices with the spinor representation weights, given by $\alpha = (\pm\frac{1}{2},\, \pm\frac{1}{2},\, \pm\frac{1}{2}, \pm\frac{1}{2},\, \pm\frac{1}{2})$. The spinors S_α are obtained for α with an even number of $-$ signs, while S'^α is obtained with an odd number of $-$ signs. The conformal dimension of this field is easily verified to be $\frac{1}{2}\alpha^2 = 5/8$, with the help of the standard formula for the conformal weight of an exponential of a $c=1$ conformal field.

One may complete the analysis by showing that the currents $j^{\mu\nu}$ corresponding to roots may also be constructed in terms of exponentials (multiplied by certain cocycle factors) and that $j^{\mu\nu}$ indeed satisfies the correct OPE with S_α and S'^α.

With the above normalization, one also obtains the OPE's of S with S and with S':

$$\begin{aligned} S'^\alpha(z) S_\beta(w) &\sim \frac{\delta_\beta{}^\alpha}{(z-w)^{5/4}} + \frac{1}{(z-w)^{1/4}}\frac{1}{2}\gamma^{\mu\nu} j_{\mu\nu}(w) \\ S_\alpha(z) S_\beta(w) &\sim \frac{1}{(z-w)^{3/4}}\,\gamma^\mu_{\alpha\beta}\,\psi_\mu \end{aligned} \tag{9.E.23}$$

(Notice that the construction of spinor fields that we have just given for Spin$(1,9)$ may be carried over in a straightforward way to Spin$(2n)$. In particular, this type of construction is needed to obtain the spinor of SO(16) necessary to build the full current algebra of

$E_8 \times E_8$ in the heterotic string. There, the spin fields have precisely conformal dimension $(1,0)$, since the weight vectors all have length $\alpha^2 = 2$.)

It remains to construct the Ramond vertex operator. The spin field by itself cannot produce a vertex operator for massless fermions, since its conformal weight is $5/8 \neq 1$. The resolution of this issue is found in a contribution to the R vertex operator from the (β,γ) and possibly (b,c) ghost system. The relevant conformal primary field is obtained as follows. (We consider only the left-moving part of the vertex operator.)

$$W_-(z,u,k) = u^\alpha(k)\mathcal{O}(z)S_\alpha(z)e^{ik\cdot x_L(z)} \tag{9.E.24}$$

Here, u^α is a spinor analogue of the polarization tensors ε that entered the construction of NS states and vertex operators. The operator $\mathcal{O}(z)$ depends on the β and γ ghosts, and is a primary field of weight $3/8$, so that W_- is of weight 1 for massless states (with $k^2 = 0$). $\mathcal{O}(z)$ is constructed in such a way that W_- is BRST invariant; it is an analogue of the spin field, but now for the superghost system.

The (β,γ) ghost system may be represented in terms of two free boson fields ϕ and χ, such that

$$\gamma = e^{\phi-\chi} \qquad\qquad \beta = e^{-\phi+\chi}\partial_z\chi \tag{9.E.25}$$

with

$$\begin{aligned} \phi(z)\,\phi(w) \quad &\sim \quad -\ln(z-w) \quad c_\phi = 13 \\ \chi(z)\chi(w) \quad &\sim \quad \ln(z-w) \quad c_\chi = -2 \end{aligned} \tag{9.E.26}$$

The ghost number current is $j_z = -\partial_z\phi$. The operator $\mathcal{O}(z)$ is again an exponential with weight $\frac{1}{2}$:

$$\mathcal{O}(z) = e^{-\frac{1}{2}\phi(z)} \ . \tag{9.E.27}$$

Because the central charge of the ϕ system is 13, the dimension of $\mathcal{O}(z)$ is $3/8$, instead of $1/8$, as would be the case for a field ϕ of central charge 1.

The above construction of W_- implies, though, that W_- has ghost number $-1/2$! But overall ghost number is given in terms of the index theorem, and this would suggest that the number of Ramond vertex operators that can be inserted on any Riemann surface is determined in terms of the topology of the surface, which is clearly absurd.

The resolution of this problem lies in the subtleties of the quantization of the (β,γ) system, which we shall not discuss here. The final result is that the quantization of the (β,γ) system yields vertex operators in different possible pictures, labeled by their ghost number. One shows, for example, that there is an equivalenT vertex operator for W_-, which had ghost number $-1/2$. This operator is W_+ with ghost number $+\frac{1}{2}$ and is given by

$$W_+(z,u,k) = e^{\phi/2} u^\alpha (\gamma_\mu)_{\alpha\beta} S'^\beta(z)(\partial_z x^\mu - i\psi_+^\mu k\cdot\psi_+)e^{ik\cdot x} \qquad (9.E.28)$$

Vertex operators for R states may now be inserted on any worldsheets in any arbitrary (even) numbers.

F) The Chiral Splitting Theorem

In critical RNS string theory, we have powerful results on the superholomorphicity properties of correlation functions for (factorized) vertex operators, at fixed internal momenta. This result is contained in the *Chiral Splitting Theorem* for the RNS string.

We begin by defining *factorized vertex operators.* A vertex operator

$$V_i = \int d^2 z_i\, d^2\theta_i\, W_i(z_i,\theta_i,\bar z_i,\bar\theta_i)$$

is factorized provided W_i is the product of holomorphic and anti-holomorphic factors

$$W_i(z_i,\theta_i,\bar z_i,\bar\theta_i) = W_i(z_i,\theta_i)\widetilde{W}_i(\bar z_i,\bar\theta_i)$$

Any vertex operator may be written as a linear combination of factorized vertex operators. For example, for NS-NS massless states, we have

$$W_i(z_i,\theta_i;\bar z_i,\bar\theta_i) = \left(\xi_{i\mu}\mathcal{D}_+ X^\mu e^{ik\cdot X_L}\right)\left(\bar\xi_{i\bar\mu}\mathcal{D}_- X^{\bar\mu} e^{ik\cdot X_R}\right) \qquad (9.F.1)$$

Next, let A_I and B_I, $I=1,\ldots,h$ be any canonical basis of homology 1-cycles with canonical intersection form. We define h independent internal loop momenta $p_I\in\mathbb{R}^{10}$, associated with each A_I-cycle, by

$$p_I = \oint_{A_I} dz\, d\theta\, \mathcal{D}_+ X \qquad (9.F.2)$$

Let the spin structures for left and right chirality spinors be equal, and equal to ν.

The correlation function of the unintegrated vertex operators W_i at fixed internal loop momenta p_I and fixed supermoduli are defined by*

$$\begin{aligned} &\langle W_1 \dots W_N \rangle_E (p_I) \\ &\quad \equiv \int DX \int DBDC \; W_1 \dots W_N \prod_{I=1}^{h} \delta \left(p_I^\mu - \int_{A_I} dz\, d\theta \; \mathcal{D}_+ X^\mu \right) \\ &\quad\qquad \prod_{K=1}^{\dim \, s\mathcal{M}_h} | \langle \mu_K , B \rangle |^2 e^{-S_X - S_{BC}} \end{aligned} \tag{9.F.3}$$

(I) The first part of the *Chiral Splitting Theorem* states that the RNS amplitudes factorize at fixed internal loop momenta:

$$\langle W_1 \dots W_N \rangle_E (p_I) = \delta(k) \mathcal{C}_\nu^F \bar{\mathcal{C}}_\nu^F \tag{9.F.4}$$

Here, ${\mathcal{C}_\nu}^F = {\mathcal{C}_\nu}^{\bar{F}}(z_i, \theta_i; \zeta_i, ; m_i; p_I, k_i)$ is a complex analytic function of supermoduli m_K, $K = 1, \dots, \dim s\mathcal{M}_h$, of the insertion points (z_i, θ_i), and of the left-moving factors $\zeta_{i\mu}$ of the polarization tensors. $\bar{\mathcal{C}}_\nu^F$ is the complex conjugate of ${\mathcal{C}_\nu}^F$, with the same spin structure

$$\bar{\mathcal{C}}_\nu^F(\bar z_i,\, \bar\theta_i; \bar\zeta_i;\, \bar m_K;\, p_I,\, k_i) = {\mathcal{C}_\nu}^F(z_i,\, \theta_i; \zeta_i;\, m_K \;\, p_I,\, k_i)^* \tag{9.F.5}$$

The functions C_ν^F may be evaluated explicitly using Green functions and determinants, using the prime form and the Szegö kernel introduced in §V.

(II) The second part of the Chiral Splitting Theorem prescribes the transition amplitudes for Type II and heterotic strings. We shall denote the analogous complex functions $\mathcal{C}^B = \mathcal{C}^B(z_i; m_k; p_I, k_i)$ for the bosonic string, compactified on the 16-dim lattices of $E_8 \times E_8$ or $\mathrm{Spin}(32)/Z_2$.

Type II A, B:

$$A_h = \delta(k) \sum_{\nu, \bar\nu} Q_{\nu\bar\nu} \int_{\mathbb{R}^{10h}} d^{10} p_I \int_{s\mathcal{M}_h} dm_K \, d\bar m_K \prod_{i=1}^{N} \int_\Sigma d^2 z_i \, d^2\theta_i \, C_\nu^F \bar C_{\bar\nu}^F \tag{9.F.6}$$

* It is understood that for genus $h = 0$ (resp. $h = 1$) three (resp. 1) of the vertex operators should not be integrated over, and should be multiplied by a factor of $\delta(\langle C, \psi \rangle)$, where ψ is a superconformal Killing vector.

Heterotic $E_8 \times E_8$, Spin$(32)/Z_2$

$$A_h = \delta(k) \sum_\nu Q_\nu \int_{\mathbb{R}^{10\,h}} d^{10} p_I \int_{s\mathcal{M}_h} dm_K \int_{\mathcal{M}_h} d\bar{m}_k \prod_{i=1}^N \int_\Sigma dz_i \, d\theta_i \int_\Sigma d\bar{z}_i \, C_\nu^F \bar{C}^B \qquad (9.F.7)$$

Here $Q_{\nu\bar\nu}$ and Q_ν are spin structure dependent weight factors which realize the GSO projection. If carried out in the supermoduli picture, ν and $\bar\nu$ should run only over even or odd. But, if the integrals are understood in components, then one should sum over all individual spin structures in each class as well. In general $Q_{\nu\bar\nu} = \pm 1$, and $Q_\nu = \pm 1$.

G) Tree-Level Amplitudes for NS-NS states

To tree level, the worldsheet has the topology of the sphere, or stereographically projected plane, and there are neither moduli, nor odd moduli. Thus, we let $g = 2|dz|^2$ and $\chi = 0$. We concentrate on the scattering of massless NS-NS states, which can be represented by the vertex operator

$$V(\varepsilon, k) = \varepsilon_{\mu\nu}(k) \int_\Sigma d^2 z \, d^2\theta \; \mathcal{D}_+ X^\mu \mathcal{D}_- X^\nu e^{ik\cdot x} \; . \qquad (9.G.1)$$

By factorizing $\varepsilon_{\mu\nu}(k) = \zeta_\mu(k)\bar\zeta_\nu(k)$, we may start from a more convenient vertex operator

$$V(\zeta, \bar\zeta; k) = \int_\Sigma d^2 z \, d^2\theta \, e^{ik\cdot X + \xi\cdot\mathcal{D}_+ X + \bar\xi\cdot\mathcal{D}_- X} \qquad (9.G.2)$$

and restrict to its contributions linear in ζ and linear in $\bar\zeta$. It is understood that ζ and $\bar\zeta$ are Grassmann parameters. The X-propagator is easily calculated:

$$\langle X^\mu(z,\theta) X^\nu(z',\theta')\rangle = -\ln |z - z' - \theta\theta'|^2 \, \eta^{\mu\nu} \qquad (9.G.3)$$

The determinants of x, ψ, b, c and β, γ fields all combine into an overall constant, which we denote by $\mathfrak{z}$.

The chiral splitting theorem is easily used here: there are no internal loop momenta, so that the correlation function of the unintegrated vertex operators

$$W(z_i, \theta_i; \bar z_i, \bar\theta_i) = W_L(z_i, \theta_i) W_R(\bar z_i, \bar\theta_i) \qquad (9.G.4)$$

will yield a product of a function C^F, complex analytic in z_i, θ_i times its complex conjugate. Here, the chiral vertex operators are

$$W_L(z_i,\theta_i) = e^{ik_i\cdot X_L(z_i,\theta_i)+\zeta_i\cdot\mathcal{D}_+X_L(z_i,\theta_i)} \tag{9.G.5}$$

and analogously for W_R, where the chiral fields X_L are contracted with the effective propagator

$$\langle X_L^\mu(z,\theta)X_L^\nu(z',\theta')\rangle = -\ln(z-z'-\theta\theta')\eta^{\mu\nu}\ . \tag{9.G.6}$$

One readily finds

$$\begin{aligned}
&\langle W_L(z_1,\theta_1)\cdots W_L(z_N,\theta_N)\rangle = \mathcal{C}^F(z_i,\theta_i)\\
&\mathcal{C}^F = \exp\sum_{i\neq j}^{N}\left\{ik_i\cdot\zeta_j\,\frac{\theta_{ij}}{z_{ij}}+\tfrac{1}{2}\zeta_i\cdot\zeta_j\,\frac{1}{z_{ij}}+\frac{1}{2}k_i\cdot k_j\ln z_{ij}\right\}
\end{aligned} \tag{9.G.7}$$

where we have used the standard notation

$$\begin{cases}\theta_{ij} = \theta_i-\theta_j\ ;\\[4pt] z_{ij} = z_i-z_j-\theta_i\theta_j\ .\end{cases} \tag{9.G.8}$$

At tree level, we have invariance under superconformal automorphisms in $O\mathrm{Sp}(1,2)$; the integration measure must be properly treated to account for this invariance. This group of transformations is defiend as follows; consider the matrices

$$K=\begin{pmatrix}0&+1&0\\-1&0&0\\0&0&1\end{pmatrix}\qquad\qquad T=\begin{pmatrix}a&b&\alpha\\c&d&\beta\\\gamma&\delta&A\end{pmatrix} \tag{9.G.9}$$

$$z\longrightarrow\frac{az+b+\alpha\theta}{cz+d+\beta\theta}\qquad\qquad\theta\longrightarrow\frac{\gamma z+\delta+A\theta}{cz+d+\beta\theta} \tag{9.G.10}$$

where Latin entries are commuting and Greek entries are anticommuting. To obtain a *superconformal* transformation T, the line element $dz = dz+\theta\,d\theta$ must be transformed into itself up to a conformal scaling. Equivalently the form

$$z_{12}=z_1-z_2-\theta_1\theta_2=\frac{v_1w_2-v_2w_1-\psi_1\psi_2}{w_1w_2} \tag{9.G.11}$$

should transform into itself up to a conformal scaling. This is uniquely achieved when the orthosymplectic form is left invariant under T:

$$T^T K T = D \tag{9.G.12}$$

The weight under which the difference transforms is easily derived, and we have

$$T\colon\ z_{12} \to \bar{z}_{12} = \frac{z_{12}}{(cz_1 + d + \beta\theta_1)(cz_2 + d + \beta\theta_2)} \tag{9.G.13}$$

Similarly the line element transforms as

$$dz \to d\bar{z} = \frac{dz}{(cz + d + \beta\theta)^2} \tag{9.G.14}$$

and the volume element as

$$dz \wedge d\theta \to \frac{dz\wedge d\theta}{(cz + d + \beta\theta)}\ . \tag{9.G.15}$$

Elements in $O\mathrm{Sp}(1,2)$ are in unique correspondence with a triplet of points in the superplane (z_1, θ_1), (z_2, θ_2), (z_3, θ_3) obeying one single (Grassmann valued) constraint. The counting works out because $O\mathrm{Sp}(1,2)$ has 3 commuting and 2 anticommuting parameters. The constraint is an $O\mathrm{Sp}(1,2)$ invariant Grassmann valued function dependent on three points, given by

$$\Delta = \frac{z_{12}\theta_3 + z_{31}\theta_2 + z_{23}\theta_1 + \theta_1\theta_2\theta_3}{(z_{12}z_{23}z_{31})^{1/2}} \tag{9.G.16}$$

The natural value for Δ is of course 0, which implies that one θ is dependent. With this value for Δ, it is easy to see that there is a unique correspondence between triplets of points satisfying $\Delta = 0$, and elements of $O\mathrm{Sp}(1,2)$, so that the latter may be accordingly parametrized. The invariant volume element induced on $O\mathrm{Sp}(1,2)$ is then

$$d\mu = \frac{dz_1 dz_2 dz_3 d\theta_1 d\theta_2 d\theta_3}{(z_{12}z_{23}z_{31})^{1/2}}\,\Delta \tag{9.G.17}$$

As for the bosonic string, the volume of the superconformal transformation group must be extracted from the amplitudes. As a result of $O\mathrm{Sp}(1,2)$ invariance, the 0, 1, 2 point functions vanish.

We now work out the three point amplitude. In evaluating $\mathcal{C}^F$, one retains terms proportional to $\zeta_1\zeta_2\zeta_3$; however, the term with three θ's vanishes because $\theta_{12}\theta_{23}\theta_{31} = 0$. Defining the tensors

$$K_{\mu_1\mu_2\mu_3} = \eta_{\mu_1\mu_2}k_{1\mu_3} + \eta_{\mu_2\mu_3}k_{2\mu_1} + \eta_{\mu_3\mu_1}k_{3\mu_2} \tag{9.G.18}$$

we find for the three point function

$$\langle V(\varepsilon_1,k_1)V(\varepsilon_2k_2)V(\varepsilon_3k_3)\rangle = 4(2\pi)^{10}\delta(k)\varepsilon_1^{\mu_1\bar\mu_1}\varepsilon_2^{\mu_2\bar\mu_2}\varepsilon_3^{\mu_3\bar\mu_3}K_{\mu_1\mu_2\mu_3}K_{\bar\mu_1\bar\mu_2\bar\mu_3} \tag{9.G.19}$$

As a result of transversality of the polarization tensors, the 3 point function for massless external particles also vanishes.

To compute the four point amplitude, we shall fix superconformal invariance as follows: $z=z_1$, $z_2=0$, $z_3=1$, $z_4=\infty$, $\theta_1,\theta_2,\theta_3=\theta_4=0$, and then have

$$\begin{aligned}\exp\mathcal{G}_4^\zeta = &\left\{\zeta_1\cdot\zeta_2\zeta_3\cdot\zeta_4\frac{1}{z_{12}z_{34}} + \zeta_1\cdot\zeta_3\zeta_2\cdot\zeta_4\frac{1}{z_{13}z_{24}} + \zeta_1\cdot\zeta_4\zeta_2\cdot\zeta_3\frac{1}{z_{14}z_{23}}\right\}\\ &+\left\{\zeta_1\cdot\zeta_2\left(k_1\cdot\zeta_3k_2\cdot\zeta_4\frac{\theta_2\theta_1}{z_{12}z_{13}z_{24}} + k_1\cdot\zeta_4k_2\cdot\zeta_3\frac{\theta_2\theta_1}{z_{12}z_{14}z_{23}}\right) + \text{perm.}\right\}\end{aligned} \tag{9.G.20}$$

In principle, one should now multiply this whole expression by the one involving the $\bar\zeta$'s, perform the integrals over z and θ and regroup terms, clearly a feudal task. The calculation is enormously simplified by the factorization properties of the Veneziano-integrals. Recall that we have the ordinary integrals

$$\int\frac{d^2z}{\pi}z^A\bar z^{\tilde A}(1-z)^B(1-\bar z)^{\widetilde B} = \frac{\Gamma(-1-\tilde A-\widetilde B}{\Gamma(-\tilde A)\Gamma(-\widetilde B)}\quad\frac{\Gamma(1+A)\Gamma(1+B)}{\Gamma(A+B+2)} \tag{9.G.21}$$

provided $A-\tilde A$ and $B-\widetilde B$ are integers, which is always the case in string theory. Using the reciprocity formula for Γ-functions and the fact that $A-\tilde A$ and $B-\widetilde B$ are integers, this expression is actually symmetric under $(A,B)\leftrightarrow(\tilde A,\widetilde B)$, as one might expect from complex conjugation. More importantly, the answer factorizes into a product of factors each only dependent either on the parameters for the z or $\bar z$ coordinates. This product property implies that one must only consider say the z coordinates to find the full amplitude, which by the same token will also completely factorize as a function of ζ's and $\bar\zeta$'s. An analogous formula is derived for the super-integrals we need:

$$\int \frac{d^2 z_1}{\pi} d^2\theta_2 [\theta_1\theta_2]^a [\bar\theta_1\bar\theta_2]^{\tilde a} z_{12}^A \, \bar z_{12}^{\tilde A} (1-z_1)^B (1-\bar z_1)^{\widetilde B}$$
$$= (-2i)^{1-a}(+2i)^{1-\tilde a} \frac{\Gamma(-\tilde a - \tilde A - \widetilde B)}{\Gamma(-\tilde A)\Gamma(-\widetilde B)} \frac{\Gamma(1+A)\Gamma(1+B)}{\Gamma(A+B+1+a)} \tag{9.G.22}$$

Here a and $\tilde a$ are either 0 or 1, and the integrals are symmetric under $(aAB) \leftrightarrow (\tilde a \tilde A \widetilde B)$. With the help of (9.G.22), it is now straightforward to evaluate the four point function

$$\begin{aligned} &\big\langle V(\varepsilon_1,k_1)V(\varepsilon_2,k_2)V(\varepsilon_3,k_3)V(\varepsilon_4,k_4)\big\rangle \\ &\qquad = (2\pi)^{10}\delta(k) g^4 \int d^2 z_1 d^2\theta_2 |z_{12}|^{-s} |z_1 - 1|^{-u} e^{\mathcal{G}_4^\zeta + \mathcal{G}_4^{\bar\zeta}} \\ &\qquad = \pi (2\pi)^{10}\delta(k) g^4 \frac{\Gamma(-s/2)\Gamma(-t/2)\Gamma(-u/2)}{\Gamma\left(1+\frac{s}{2}\right)\Gamma\left(1+\frac{t}{2}\right)\gamma\left(+\frac{u}{2}\right)} \varepsilon^{1\bar 1}\varepsilon^{2\bar 2}\varepsilon^{3\bar 3}\varepsilon^{4\bar 4} K_{1234} K_{\bar 1\bar 2\bar 3\bar 4} \end{aligned} \tag{9.G.23}$$

Using the abbreviation i for μ_i to save some writing, we have

$$\begin{aligned} K_{1234} &= (st\eta_{13}\eta_{24} - su\eta_{14}\eta_{23} - tu\eta_{12}\eta_{34}) \\ &\quad - s(k_1^4 k_3^2 \eta_{24} + k_2^3 k_4^1 \eta_{13} - k_1^3 k_4^2 \eta_{23} - k_2^4 k_3^1 \eta_{14}) \\ &\quad + t(k_2^1 k_4^3 \eta_{13} + k_3^4 k_1^2 \eta_{24} - k_2^4 k_1^3 \eta_{34} - k_3^1 k_4^2 \eta_{12}) \\ &\quad - u(k_1^2 k_4^3 \eta_{23} + k_3^4 k_2^1 \eta_{14} - k_1^4 k_2^3 \eta_{34} - k_3^2 k_4^1 \eta_{12}) \end{aligned} \tag{9.G.24}$$

H) One loop Amplitudes for NS-NS states in Type II A, B

We begin by using part (I) of the Chiral Splitting Theorem, and we compute the Chiral Amplitudes $\mathcal{C}_\nu^F$. The spin structure for left and right movers is the same here.

For *even spin structures* ν, there are no odd moduli, and there are no Dirac zero modes. The metric on the torus is taken to be $g = 2|dz|^2$, and the torus is represented as usual by a parallelogram with sides 0, 1, τ, $1+\tau$, with $\tau \in H$ in the upper half complex plane H. The S_X and S_{BC} actions are free actions in components:

$$\begin{aligned} S_X + S_{BC} = \frac{1}{4\pi}\int_\Sigma d^2 z \Big(&\partial_z x \cdot \partial_{\bar z} x - \psi_+ \cdot \partial_{\bar z}\psi_+ - \psi_- \cdot \partial_z \psi_- \\ &+ 2b\partial_{\bar z} c + 2\bar b \partial_z \bar c + 2\beta\partial_{\bar z}\gamma + 2\bar\beta\partial_z\bar\gamma \Big)\ . \end{aligned} \tag{9.H.1}$$

The ghost and anti-ghost insertions are reduced to

$$b\bar{b}\, c\bar{c}\ . \tag{9.H.2}$$

No anti-superghost insertions occur for even spin structures, since there are no zero modes there. In the NS-NS sector, vertex operators do not involve ghosts, so we may integrate out b, c, $\bar{b}$, $\bar{c}$, β, γ, $\bar{\beta}$ and $\bar{\gamma}$. We shall also retain the determinants of x and $\psi_\pm$ integrations. Thus

$$\delta(k)\mathcal{C}_\nu \mathcal{C}_\nu = M_\nu \bar{M}_\nu\, \mathcal{F}_\nu \bar{\mathcal{F}}_\nu \tag{9.H.3}$$

where $M_\nu \bar{M}_\nu$ arises from determinants and $\mathcal{F}_\nu \bar{\mathcal{F}}_\nu$ arises from correlation functions of vertex operators. We find

$$\begin{aligned} M_\nu \bar{M}_\nu = &\left(\frac{\det{}' \Delta_0}{(\mathrm{Im}\,\tau)^2}\right)^{-5} \left(\det \not{D}_+\right)^5_\nu \left(\det \not{D}_-\right)^5_\nu \\ &\left(\frac{\det{}' \Delta_{-1}}{(\mathrm{Im}\,\tau)^2}\right)^{+1} (\det\ \Delta^-_{-1/2})^{-1}_\nu \end{aligned} \tag{9.H.4}$$

As usual, on the torus, these determinants are related to one another, and we find

$$\begin{aligned} \frac{\det{}' \Delta_0}{(\mathrm{Im}\,\tau)^2} &= \frac{\det{}' \Delta_{-1}}{(\mathrm{Im}\,\tau)^2} = |\eta(\tau)|^4 \\ (\det \not{D}_+ \not{D}_-)_\nu = (\det\ \Delta^-_{1/2})_\nu &= \left|\frac{\vartheta[\nu](0|\tau)}{\eta(\tau)}\right|^2 \end{aligned} \tag{9.H.5}$$

Putting all together:

$$M_\nu = \frac{\vartheta[\nu](0|\tau)^4}{\eta(\tau)^{12}} \qquad\qquad \bar{M}_\nu = \frac{\overline{\vartheta[\nu](0|\tau)^4}}{\overline{\eta(\tau)}^{12}} \tag{9.H.6}$$

up to an overall phase which is independent of τ, but which may depend upon ν. Clearly such a phase can be absorbed into the spin structure weights $Q_{\nu\bar{\nu}}$. The vertex contributions $\mathcal{F}_\nu$ are easily evaluated:

$$\begin{aligned} \mathcal{F}_\nu(z_i, \theta_i, \zeta, k, p) = \exp\{&i\pi p^2\tau + 2\pi p \cdot \sum_i(-\zeta_i\theta_i + ik_i z_i) \\ &-\tfrac{1}{2}\sum_{i\neq j}(k_i\cdot k_j G_\nu(z_i,\theta_i;z_j,\theta_j) + \zeta_i\cdot\zeta_j \mathcal{D}^i_+\mathcal{D}^j_+ G_\nu \\ &+ 2ik_i\cdot\zeta_j\mathcal{D}^j_+ G_\nu)\}\ . \end{aligned} \tag{9.H.7}$$

Here, the correlation function G_ν is given by

$$\begin{aligned} G_\nu(z_i,\theta_i;z_j,\theta_j) &= \langle X_L(z_i,\theta_i)X_L(z_j,\theta_j)\rangle \\ &= -\ln\, E(z_i,z_j) + \theta_i\theta_j S_\nu(z_i,z_j) \end{aligned} \tag{9.H.8}$$

where $E(z,w)$ is the prime form, defined in §V, and S_ν is the Szegö kernel. For the torus, these functions are particularly simple:

$$\begin{aligned} E(z,w) &= \frac{\vartheta_1(z-w|\tau)}{\vartheta_1'(0|\tau)}\ ; \\ S_\nu(z-w) &= \frac{\vartheta[\nu](z-w|\tau)\vartheta_1'(0|\tau)}{\vartheta[\nu](0|\tau)\vartheta_1(z-w|\tau)}\ . \end{aligned} \tag{9.H.9}$$

For *odd spin structures*, ν, there is now one complex odd modulus χ and there are 10 Dirac zero modes, one for each space-time direction. The action $S_X + S_{BC}$ is the one given on in (9.H.1) but there is now also an insertion of the worldsheet supercurrent involving

$$\chi S_{z+}\ \ S_{z+} = \Big(\psi_+\cdot\partial_z x - \tfrac{1}{2}b\gamma + \frac{3}{2}\,\beta\partial_z c + \partial_z\beta\, c\Big) + c.c\ . \tag{9.H.10}$$

Integrating out χ will bring down an insertion of $\psi_+\cdot\partial_z x$ and of $\psi_-\cdot\partial_{\bar z} x$. Combining this with the ghost and anti-ghost insertions, we have

$$b\bar b\, c\bar c\, \delta(\beta)\delta(\bar\beta)\delta(\gamma)\delta(\bar\gamma)\int S_{z+}\int S_{\bar z-}\ . \tag{9.H.11}$$

Now, we specialize to correlation functions of massless NS-NS states. As shown in (9.E.5), these operators involve at most 2 factors of ψ_+. To obtain a non-zero contribution from odd spin structures, we need to have at least 5 external vertex operators. For example, the anomaly diagrams with 6 external vertex operators receives contributions solely from odd spin structure. We shall concentrate on amplitudes with 4 or fewer external vertex operators, and odd spin structures will never contribute to those. Notice that this means that for these amplitudes, there is no difference between Type II A, B. For completeness, we record that for ν odd, we have $M_\nu = M_{\bar\nu} = 1$.

It remains to work out the amplitudes with 0, 1, 2, 3, 4 external vertex operators. We begin with determining the factors $Q_{\nu\bar\nu}$ of spin structure weights. Following Mumford, we label ϑ-functions as follows:

$$\begin{aligned}
\vartheta_{00}(z,\tau) &= \vartheta_3(z,\tau) = \vartheta(z,\tau) \\
\vartheta_{01}(z,\tau) &= \vartheta_4(z,\tau) = \vartheta\left(z+\tfrac{1}{2},\tau\right) \\
\vartheta_{10}(z,\tau) &= \vartheta_2(z,\tau) = e^{i\pi\tau/4+\pi i z}\vartheta\left(z+\frac{\tau}{2},\tau\right) \\
\vartheta_{11}(z,\tau) &= \vartheta_1(z,\tau) = e^{i\pi\tau/4+\pi i z}\vartheta\left(z+\tfrac{1}{2}+\frac{\tau}{2},\tau\right) \\
\vartheta(z,\tau) &\equiv \sum_{n=-\infty}^{\infty} e^{i\pi n^2\tau+2\pi i n z}
\end{aligned} \tag{9.H.12}$$

Modular transformations of ϑ_2, ϑ_3, ϑ_4 obey

$$\begin{aligned}
\vartheta_{ab}(0,\tau+1) &= e^{\frac{i\pi}{4}a}\vartheta_{a(b+a)}(0,\tau) \\
\vartheta_{ab}\left(0,-\frac{1}{\tau}\right) &= (-)^{ab}\sqrt{-i\,\tau}\,\vartheta_{ba}(0,\tau)
\end{aligned} \tag{9.H.13}$$

The Dedekind η-function satisfies

$$\begin{aligned}
\eta(\tau+1)^{12} &= -\eta(\tau)^{12} \\
\eta\left(-\frac{1}{\tau}\right)^{12} &= \tau^6\eta(\tau)^{12}
\end{aligned} \tag{9.H.14}$$

Thus

$$\begin{aligned}
M_{ab}(\tau+1) &= -e^{i\pi a}M_{a(b+a+1)}(\tau) \\
M_{ab}\left(-\frac{1}{\tau}\right) &= -\frac{1}{\tau^4}M_{ba}(\tau)\ .
\end{aligned} \tag{9.H.15}$$

We must now require modular invariance of the amplitude:

$$\sum_{(a,b)} Q_{(ab)\bar{\nu}}M_{(ab)}(\tau+1) = \sum_{(a,b)} Q_{(a,b)\bar{\nu}}M_{(ab)}(\tau) \tag{9.H.16}$$

$$\sum_{(a,b)} Q_{(ab)\bar{\nu}}M_{(ab)}(-1/\tau) = -\frac{1}{\tau^4}\sum_{(a,b)} Q_{(a,b)\bar{\nu}}M_{ab}(\tau)\ .$$

(The factor of $-1/\tau^4$ is compensated for by the p-integral.) Clearly from $\tau\to-1/\tau$, we get

$$Q_{(01)\bar{\nu}} = Q_{(10)\bar{\nu}} \tag{9.H.17}$$

while from $\tau \to \tau + 1$, we get

$$Q_{(01)\bar{\nu}} = -Q_{(00)\bar{\nu}} \tag{9.H.18}$$

Setting

$$Q_{(01)\bar{\nu}} = Q_{(10)\bar{\nu}} = -Q_{(00)\bar{\nu}} \tag{9.H.19}$$

we fix the overall phase of the amplitude, which is unobservable anyway. One proceeds similarly for $\bar{\nu}$.

It is now straightforward to compute the 0-*point function*. (Recall that in the bosonic string this object produced the first divergence we had encountered.) Since there are no vertex operator insertions, we simply have

$$A_1(0-pt) = \mathrm{Vol}(M) \int_{\mathcal{M}_1} \frac{d^2\tau}{\tau_2^4} \sum_{\nu\bar{\nu}} Q_{\nu\bar{\nu}} M_\nu M_{\bar{\nu}} \tag{9.H.20}$$

It suffices to examine the combination

$$\sum_\nu Q_{\nu\bar{\nu}} M_\nu = Q_{(01)\bar{\nu}} M_{(01)} + Q_{(10)\bar{\nu}} M_{(10)} + Q_{(00)\bar{\nu}} M_{(00)} \tag{9.H.21}$$

Notice that since $M_{(11)} = 0$ the corresponding contribution has been omitted in (9.H.21). Now, using the expressions for M_ν, and the equality for Q's, we find

$$\sum_\nu Q_{\nu\bar{\nu}} = \frac{Q_{(01)\bar{\nu}}}{\eta(\tau)^{12}} \left[\vartheta_4(0|\tau)^4 + \vartheta_2(0|\tau)^4 - \vartheta_3(0|\tau)^4\right] = 0 \tag{9.H.22}$$

by a famous Jacobi ϑ-function identity. This identity arose once before in matching bosonic and fermionic multiplicities at all mass levels. In fact, this cancellation is the result of space-time supersymmetry. The 0-point function, and thus the 1-point functions vanish. Recall from the discussion of the bosonic string that the vanishing of the 1-point function means that Minkowski flat space-time is a one-loop solution to the superstring equations!

Notice that our argument for cancellation was effected only in the left-moving sector; thus, the same argument holds true in heterotic string theory as well.

The above summation identity on ϑ-functions is only one of a series, usually called the Riemann identities on theta functions. It is most useful here to express two more of these identities in terms of the Dirac propagator (i.e. the Szegö kernel):

$$\begin{aligned} &\sum_\nu Q_{\nu\bar\nu} M_\nu S_\nu(z_1-z_2)S_\nu(z_2-z_1)=0 \\ &\sum_\nu Q_{\nu\bar\nu} M_\nu S_\nu(z_1-z_2)S_\nu(z_2-z_3)S_\nu(z_3-z_1)=0 \end{aligned} \tag{9.H.23}$$

Without giving details here, one similarly shows with the help of these identities that the two- and three-point functions vanish. In physics terms, this means that to one loop order, there is neither mass renormalization (massless particles remain massless) nor coupling constant renormalization. These results again arise from space-time supersymmetry, and are usually referred to as *non-renormalization theorems*.

I) One-loop amplitudes in the heterotic string

Let us consider the free fermion representation of the heterotic strings, starting from 32 Majorana-Weyl fermions λ^α, represented by 32 Euclidean Weyl spinors $\lambda^\alpha_-(\bar z)$. For simplicity, consider amplitudes with external states that are neutral under the gauge group. Then the one loop contribution of the internal degree of freedom will result entirely from the determinantal factors arising from integrating out λ^α_-. These depend upon the spin structures, just as the ones for space-time degrees of freedom ψ^μ_-.

For the gauge group $\mathrm{Spin}(32)/Z_2$, all λ^α_- have the same spin structure, and the determinant for given spin structure contributes a factor

$$\bar M^4_{\bar\nu} = \frac{\overline{\vartheta[\bar\nu](0|\tau)}^{16}}{\overline{\eta(\tau)}^{48}} \tag{9.I.1}$$

The only modular invariant we can form is with all coefficients $Q_{\bar\nu}$ equal for all 3 even spin structures.

For the gauge group $E_8\times E_8$, λ^α_- are divided into two classes of 16. For each class, the determinant is

$$\bar M^2_{\bar\nu} = \frac{\overline{\vartheta[\nu](0|\tau)}^{8}}{\overline{\eta(\tau)}^{24}} \ . \tag{9.I.2}$$

The only modular invariant is

$$\sum_{\bar\nu} \bar M^2_\nu = \frac{\overline{\vartheta_2(0|\tau)}^8+\overline{\vartheta_3(0|\tau)}^8+\overline{\vartheta_4(0|\tau)}^8}{\overline{\eta(\tau)}^{24}} \ . \tag{9.I.3}$$

One can show, using relations on the number of representations of an integer as a sum of squares, that this function equals the ϑ-function for the E_8 lattice: $\vartheta_{E_8}(0|\tau)$. Furthermore, the partition functions for Spin(32)/Z_2 and $E_8 \times E_8$ are the same in view of

$$\sum_{\bar{\nu}} \bar{M}_{\bar{\nu}}^4 = \left(\sum_{\bar{\nu}} \bar{M}_{\bar{\nu}}^2\right)^2 . \tag{9.I.4}$$

We see that no other modular invariant combinations are allowed. You may think that $\sum\limits_{\nu} Q_\nu M_\nu = 0$ is allowed, but this would mean that some states enter with negative weight, and this is impossible since there are no fermions here! This argument shows now that $E_8 \times E_8$ and Spin(32)Z_2 are indeed the only possibilities, as we had promised.

J) The NS-NS 4 point function

It remains to discuss the 4-point functions, again for massless NS-NS states. Clearly, the 4-point function cannot vanish, lest the theory be non-interacting. It is evaluated by making use of a Riemann identity on ϑ-functions with 4 Szegö kernels:

$$\sum_{\nu} Q_{\nu\bar{\nu}} M_\nu S_\nu(z_1 - z_{\sigma(1)}) S_\nu(z_2 - z_{\sigma(2)}) S_\nu(z_3 - z_{\sigma(3)}) S_\nu(z_1 - z_{\sigma(4)}) = 16\pi^4 Q_{(00)\bar{\nu}} \tag{9.J.1}$$

where $\sigma(i)$ is a permutation of i, *without fixed points.*

The final result for the 4-point function can be conveniently expressed as follows:

$$A_1(k_i, \varepsilon_i) = \delta(k) \varepsilon^{1\bar{1}} \varepsilon^{2\bar{2}} \varepsilon^{3\bar{3}} \varepsilon^{4\bar{4}} K^{1234} K^{\bar{1}\bar{2}\bar{3}\bar{4}} A_1(s,t) \tag{9.J.2}$$

where the single Lorentz-invariant function $A_1(s,t)$ is given by

$$A_1(s,t) = \frac{1}{2} \int_{\mathcal{M}_1} \frac{d^2\tau}{\tau_2^2} \int_\Sigma \frac{d^2 z_1}{\tau_2} \int_\Sigma \frac{d^2 z_2}{\tau_2} \int_\Sigma \frac{d^2 z_3}{\tau_2} e^{\frac{s}{2}(G_{12}+G_{34}-G_{13}-G_{24})} e^{+\frac{t}{2}(G_{23}+G_{14}-G_{13}-G_{24})} \tag{9.J.3}$$

where G_{ij} stands for the 1-loop propagator of the x-field:

$$G_{ij} = G(z_i - z_j)|\tau) \equiv -\ln \left| \frac{\vartheta_1(z_i - z_j|\tau)}{\theta_1'(0|\tau)} \right|^2 + \frac{2\pi}{\tau_2} \operatorname{Im}(z_i - z_j)^2 \tag{9.J.4}$$

and K is the kinematical factor that also arose in the tree-level amplitudes $A_0(k_i, \epsilon_i)$ for NS-NS transition amplitudes. Recall that K is polynomial in moments. By translation invariance on the torus, the point z_4 may be chosen at will, say at $z_4 = 0$.

It remains to study the function $A_1(s,t)$! We begin by examining the convergence structure of the above functional representation as a function of s and t.

(1) For fixed τ, and $z_i \to z_j$, $i \neq j$, we have

$$G(z_i - z_j|\tau) \to -\ln |z_i - z_j|^2 \to +\infty \tag{9.J.5}$$

(2) while for z_i's well separated, but $\mathrm{Im}\,\tau \to \infty$, we have

$$G(z_i - z_j|\tau) \to -\tau_2 \qquad\qquad \to -\infty \ . \tag{9.J.6}$$

The latter limit imposes the strictest convergence condition:

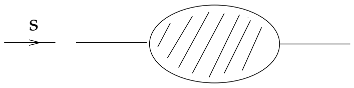
$$\mathrm{Re}(s),\ \mathrm{Re}(t),\ \mathrm{Re}(u) \geq 0 \tag{9.J.7}$$

But, since we also have $s + t + u = 0$ for on-shell massless external states, the convergence condition (9.J.7) requires $\mathrm{Re}(s) = \mathrm{Re}(t) = \mathrm{Re}(u) = 0$. It is easily seen that this condition is sufficient for convergence, since the integral is bounded by the value at $s = t = 0$. This means that the integral is absolutely convergent for at least some values of the external momenta, in contrast with the case of the bosonic string, where the integral representation is never convergent for any momenta. Yet, the integral representation is defined only for unphysical momentum configurations where s, t, u are purely imaginary.

A second shortcoming of the integral representation is that formally, when s, t, u are close to the physical region corresponding to real values of s and t, $A_1(s,t)$ would be real. This would be inconsistent in an interacting theory, where the 1-loop 4-point function cannot be real, since its imaginary part is related by unitarity to the square of the tree level amplitude, which cannot vanish in any intracting theory.

Our task is thus to properly define the amplitude throughout $(s,t) \in \mathbb{C} \times \mathbb{C}$, including around the physical region where $(s,t) \in \mathbb{R} \times \mathbb{R}$. Naturally, one seeks an analytic continuation of the amplitude in both s and t. (Recall that even at tree level, the integral representation $A_0(s,t)$ was absolutely convergent only in a small region, and the full amplitude had to be analytically continued in s and t as well.)

To appreciate the physical meaning of such an analytic continuation, we compare with the situation in quantum field theory. There, the physical principle of causality is equivalent to the requirement of locality of observable fields, and this property, combined with Lorentz invariance, implies analytic dependence of the Green functions on the external momenta. Often, this analytic behavior is conveniently summarized in the form of a spectral representation. For example, the 2-point function (off-shell) may be expressed as



$$H(s) = \int_0^\infty dM^2 \, \frac{\rho(M^2)}{s - M^2 + i\varepsilon} \qquad \varepsilon > 0 \tag{9.J.8}$$

where the real and positive function $\rho(M^2)$ is the density of states at mass M, also called the spectral density. Provided the integral over M^2 converges, it follows that $H(s)$ is analytic throughout $\mathbb{C}$, except possibly on the positive real axis, where it may have poles and branch cuts.

In string theory, we do not have available at present a formulation of the amplitudes in terms of local field observables (i.e. commuting at space-like separations). It is thus not known à priori whether analyticity of the amplitudes can be derived solely from Lorentz invariance and causality. (Probably, in a good second quantized formulation, one should be able to derive this.) Thus, analyticity of string amplitudes must be established, and the questions that must be answered are

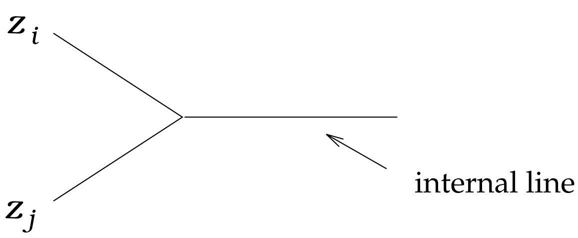
(1) does an analytic continuation exist?

(2) is the analytic continuation unique?

(3) is the analytic continuation physically acceptable?

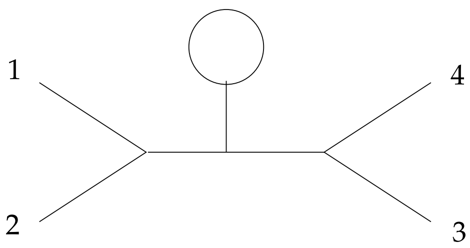
The present problem of analytic continuation of $A_1(s,t)$ is a difficult matter of pure analysis. Yet, all three questions above have been answered in the affirmative, so that one has a full analytically continued amplitude $A_1(s,t)$ available, defined throughout $(s,t) \in \mathbb{C} \times \mathbb{C}$, with cuts and poles on $\mathbb{R} \times \mathbb{R}$, but nowhere else.

What we have achieved here is the construction of a *finite* and *unitary* amplitude, at least to one loop order, that defines NS-NS string scattering! We shall not reproduce the details of the analytic continuatin procedure here. We shall limit ourselves to describing the physical singularities that appear in $A_1(s,t)$ upon analytic continuation. To do so it is most convenient to represent the different singularities that appear as a function of s and t in terms of QFT Feynman diagrams. The Feynman diagrams indicate the region in moduli space (including the positions of vertex operators z_i) that produce the singularities. The singularities essentially arise from combinations of the singular limits of G_{ij} in (9.J.5) and (9.J.6), and fall into five categories, schematically indicated by five different types of Feynman diagrams below. Whenever two external z_i and z_j approach one another the Feynman diagram representation will show that their external lines merge

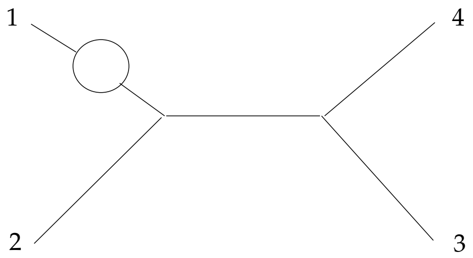


This singularity may be treated at the operator level by considering the operator product expansion (OPE) of the two external vertex operators. The internal line then represents all the operators that arise in the OPE. This type of singularity produces a pole in the total momentum traversing the internal line, i.e. in $-(k_i+k_j)^2$, and the pole turns out to be at the value of the mass^2 of the physical string states, i.e. $2n = M^2$, $n \in \mathbb{Z}^+$. The five different categories represented by diagrams (a), (b), (c), (d) and (e) arise, respectively, from 4, 3, 2, 1 and 0 external lines coming close together. Within each category, there are additional singularities (represented by a dotted line aross the loop) that arise from the asymptotic behavior in τ, given by (9.J.6).

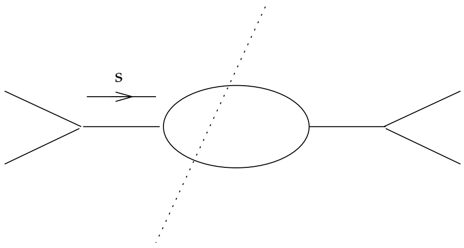
(a)



(b)



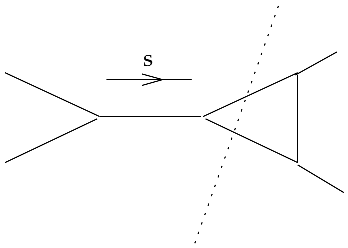
(c)



- double poles in s at $s = 2n$
- branch cuts in s, starting at

$$s = 2n, \quad n \in \mathbb{Z}^+.$$

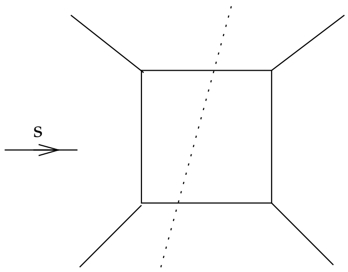
(d)



- single poles in s at $s = 2n$
- branch cuts in s, starting at

$$s = 2n, \quad n \in \mathbb{Z}^+.$$

(e)



- no poles
- branch cuts in s, starting at
  $$s = 2n, \quad n \in \mathbb{Z}^+$$
  with spectral density $\rho(2n, t)$
- $\rho(2n, t)$ has branch cuts in t, starting at $t = 2m$, $m \in \mathbb{N}^0$.

It is understood that within each category, we also include permutation of the diagrams shown, for example by adding diagrams on which the external legs have been permuted.

Now, for superstrings, diagrams in categories (a) and (b) vanish identically. This follows from the fact that the on-shell 1- and 2-point functions for massless states vanish identically. The singularities of the remaining diagrams are precisely the ones that one would expect on physical grounds, and their structure confirms the consistency of the string theory up to this order. A complete proof is in E. D'Hoker and D. H. Phong, "The Box Graph in Superstring Theory", Nucl. Phys. B440 (1995), 24.